SIGGRAPH ASIA 2018 TOKYO

max planck institut informatik

Sampling Analysis using Correlations for Monte Carlo Integration

Part 2: Error Analysis

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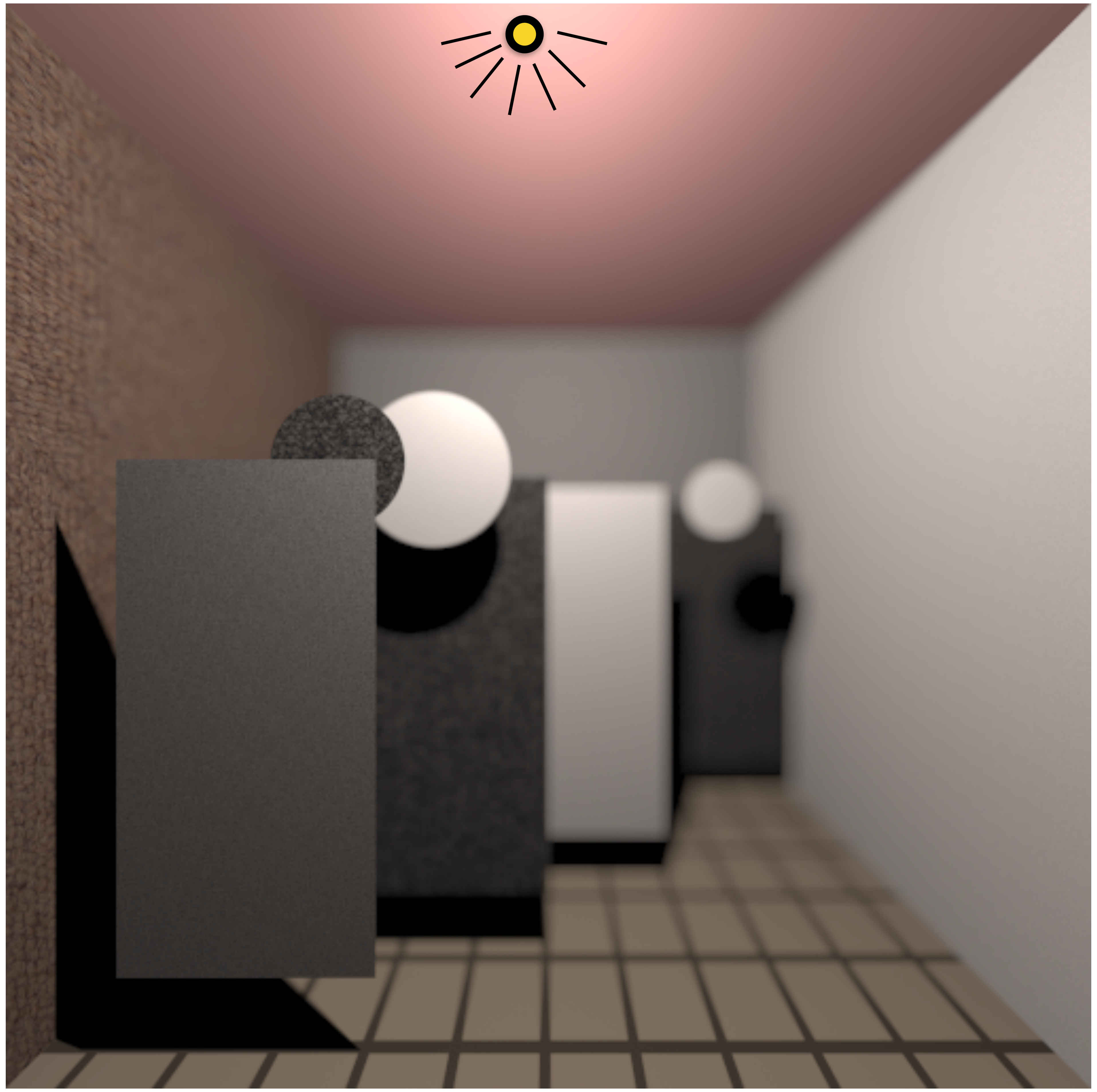
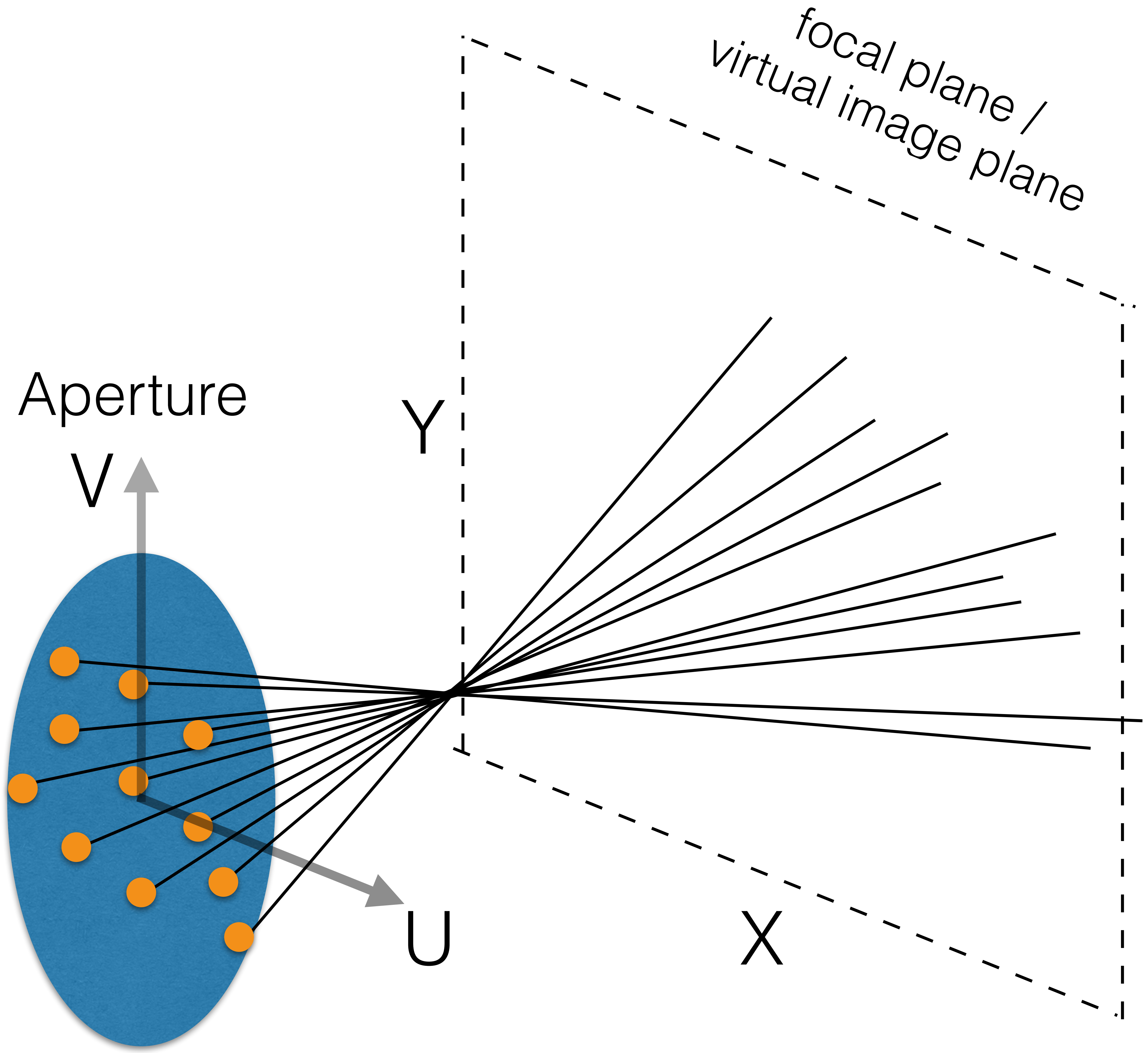
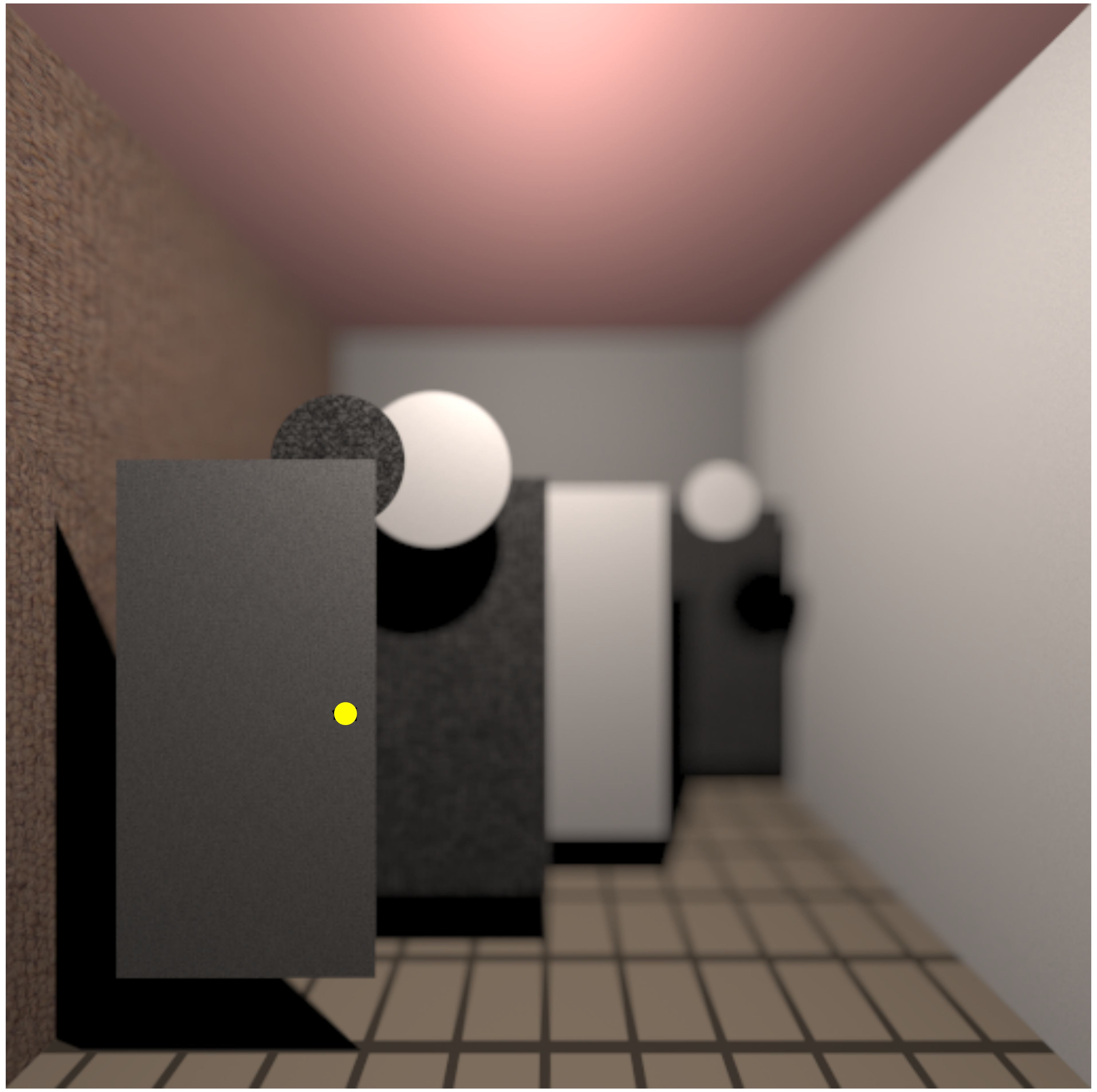
Monte Carlo Integration



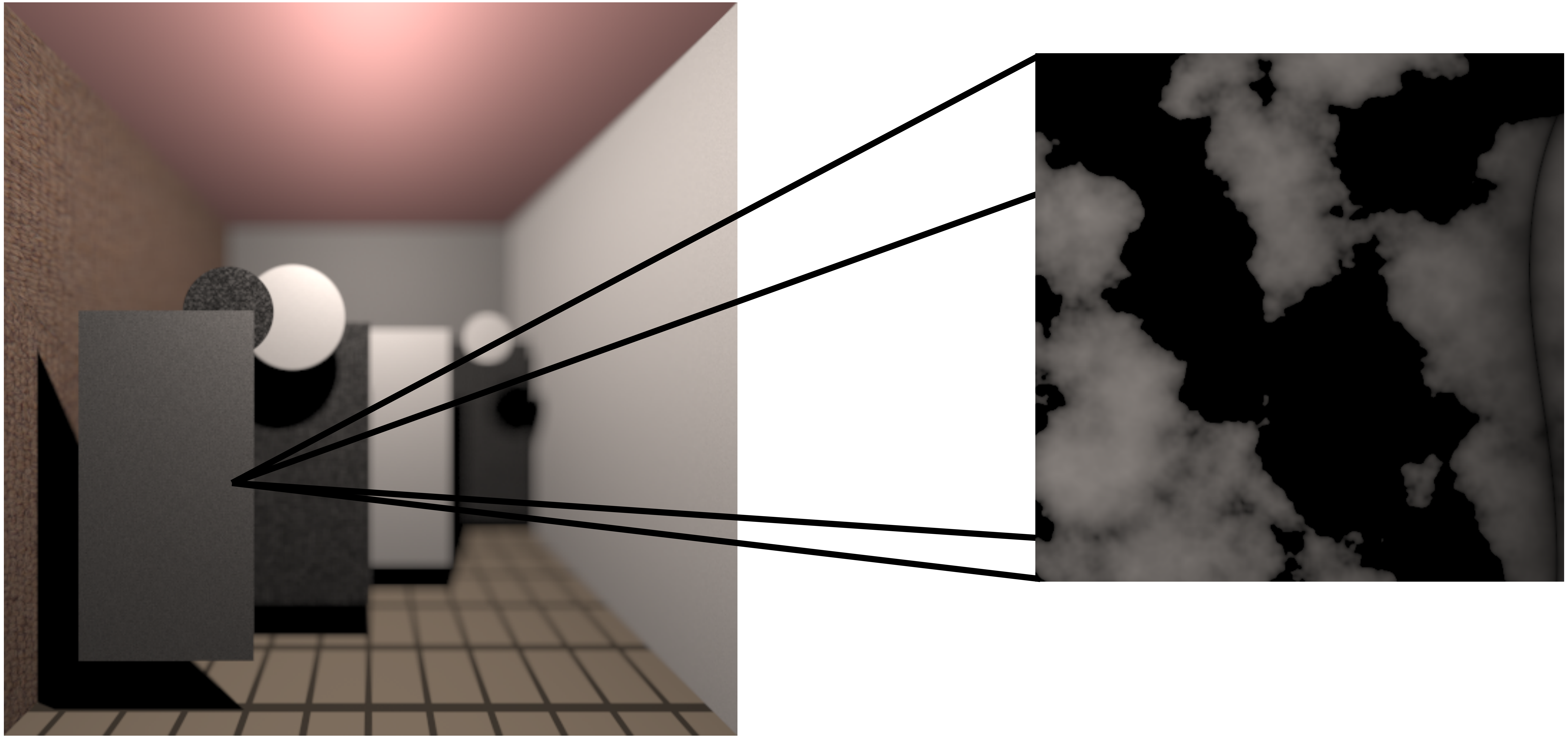


$$\int_x \int_y \int_u \int_v f(x, y, u, v)\, dv\, du\, dy\, dx$$

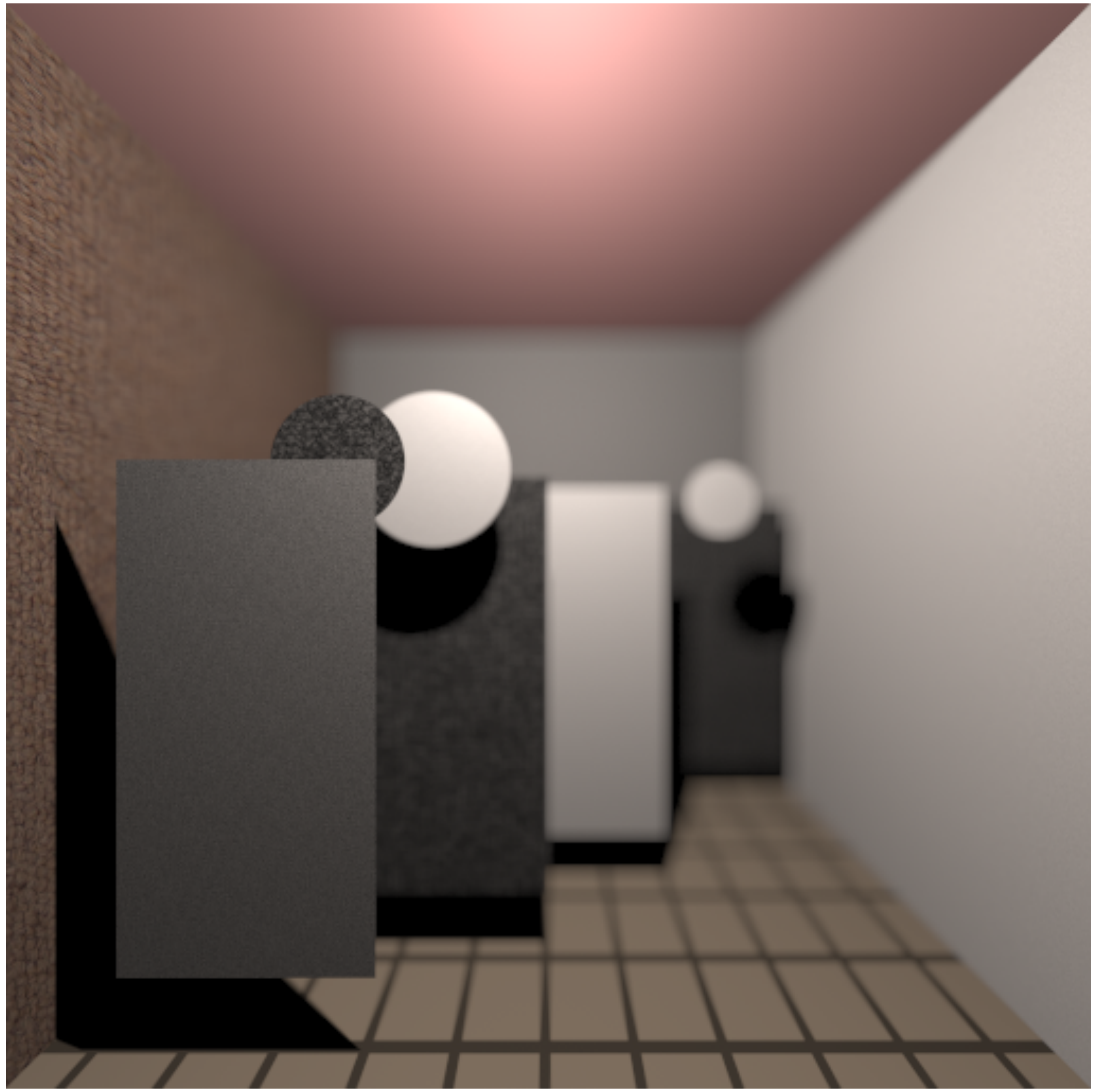
Monte Carlo Integration



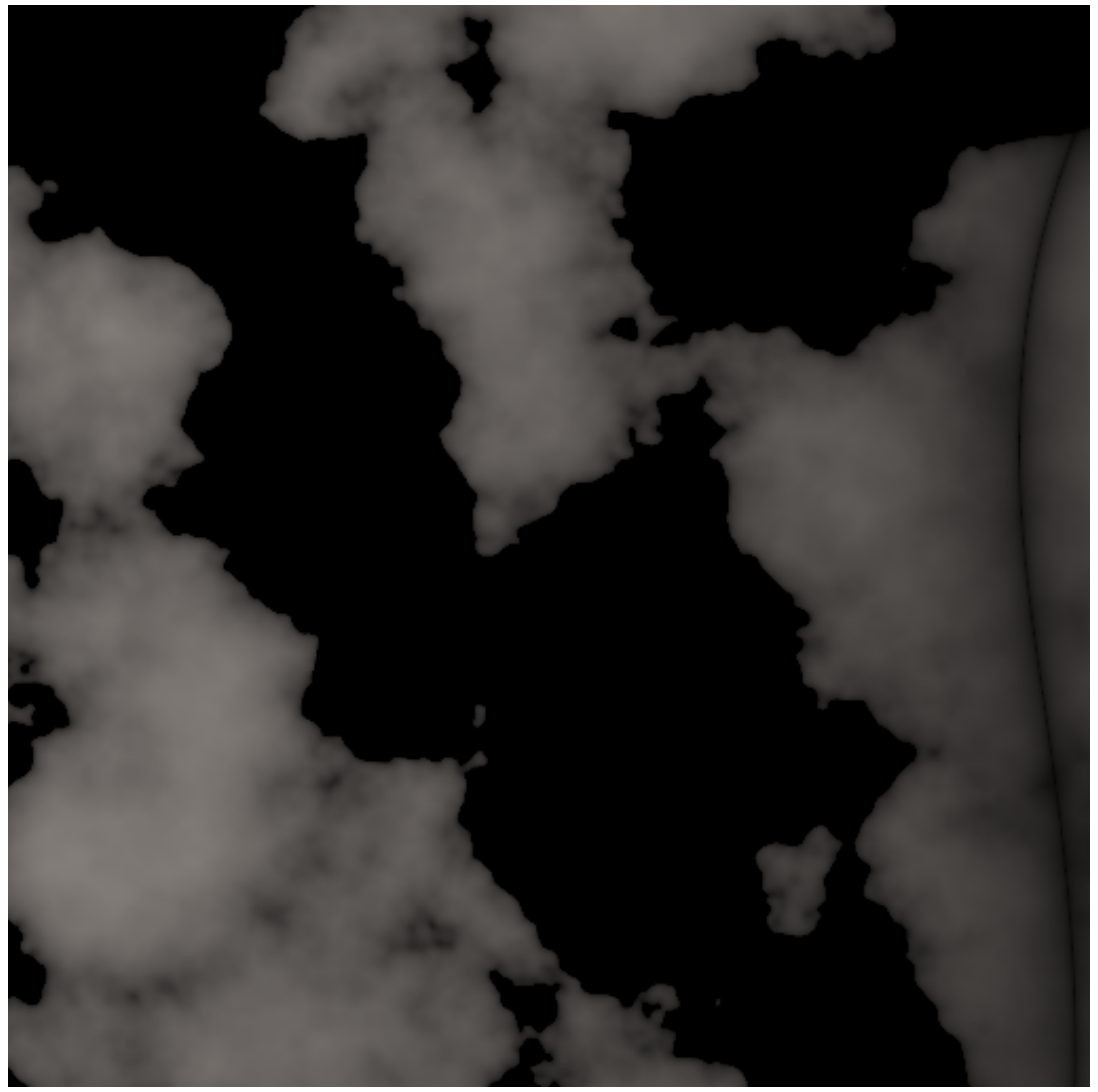
Monte Carlo Integration



Monte Carlo Integration

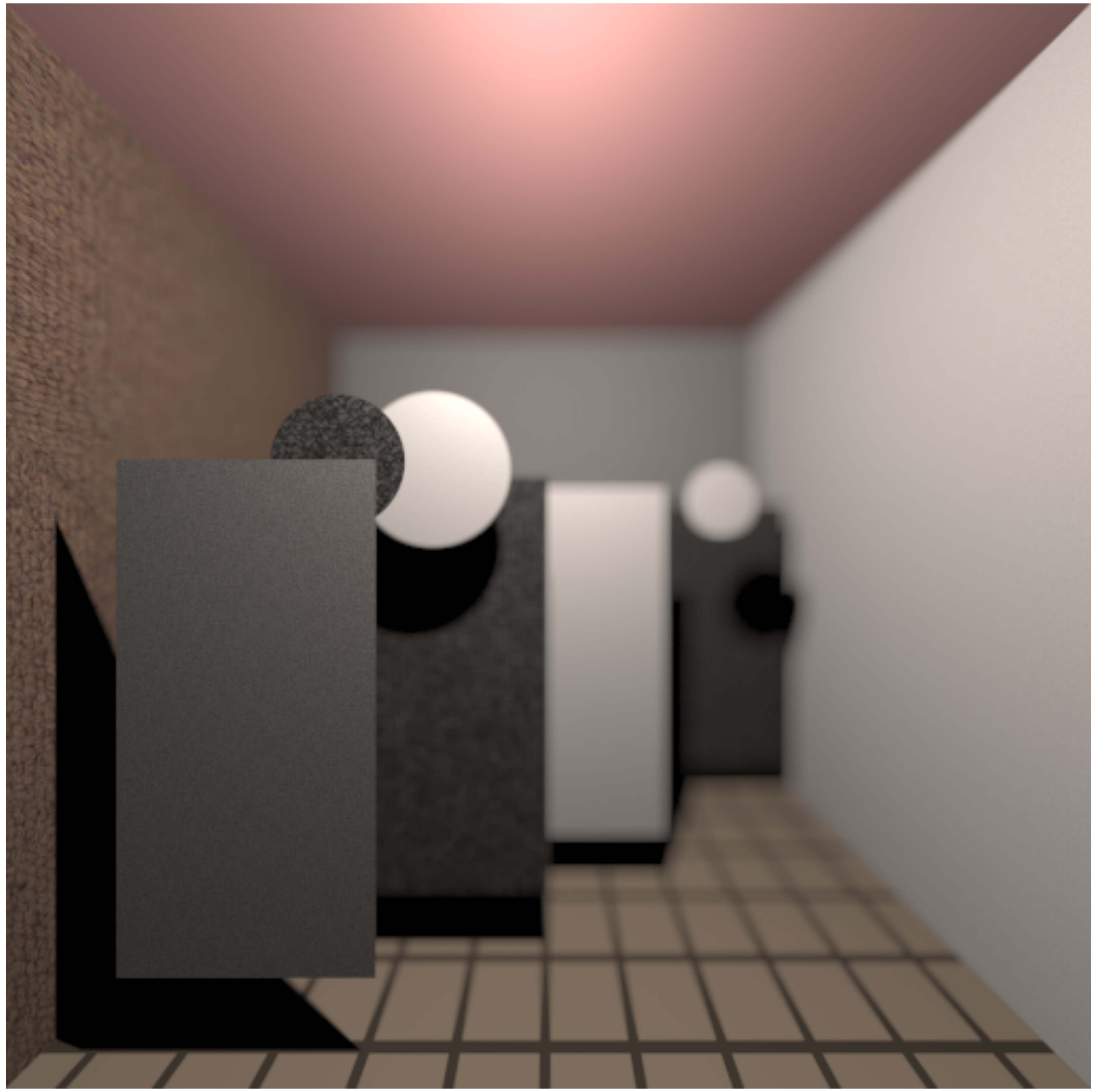


$$f(\vec{x})$$



$$I = \int_0^1 f(\vec{x}) d\vec{x}$$

Monte Carlo Integration



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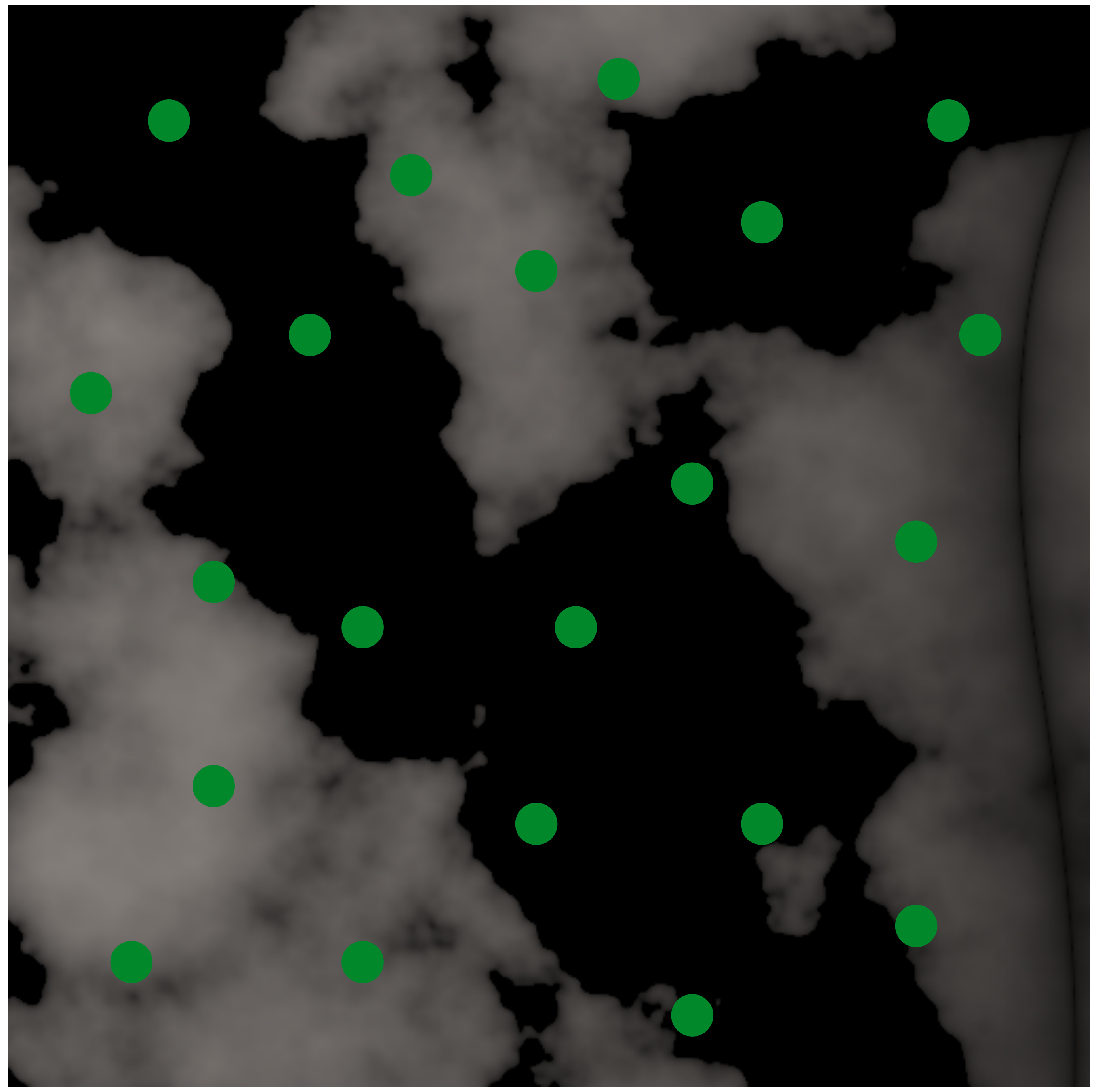

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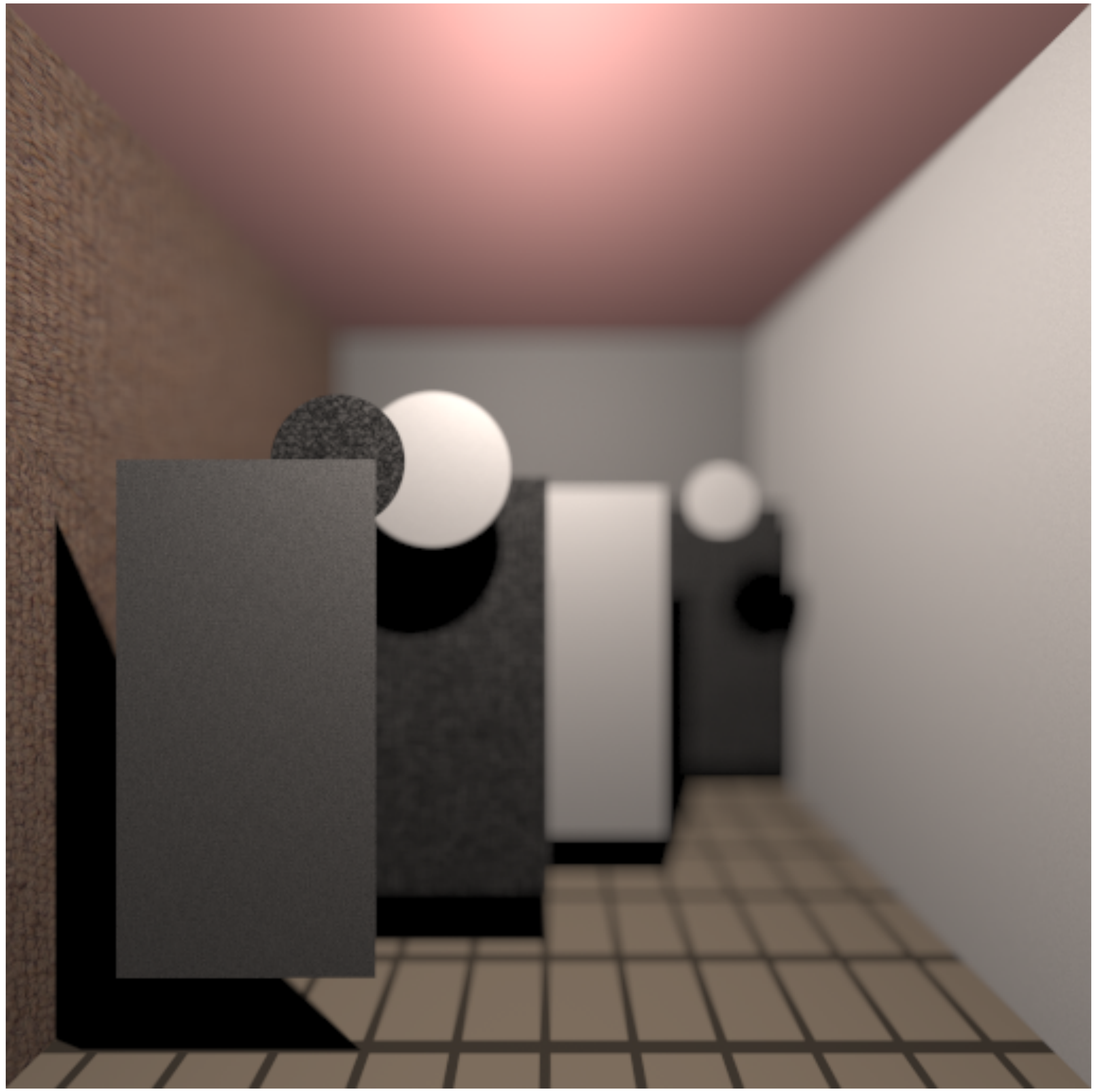
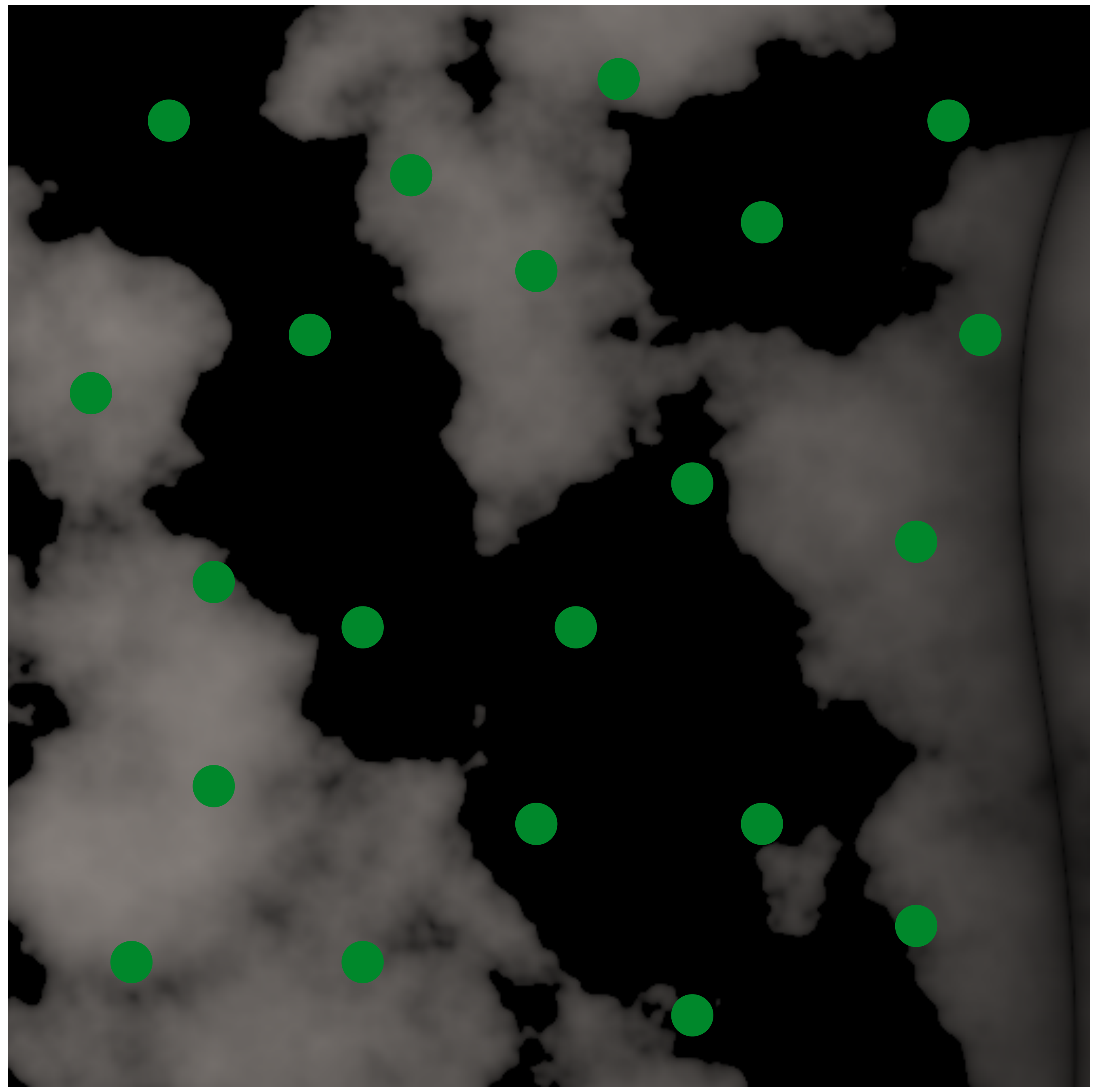
Monte Carlo Integration

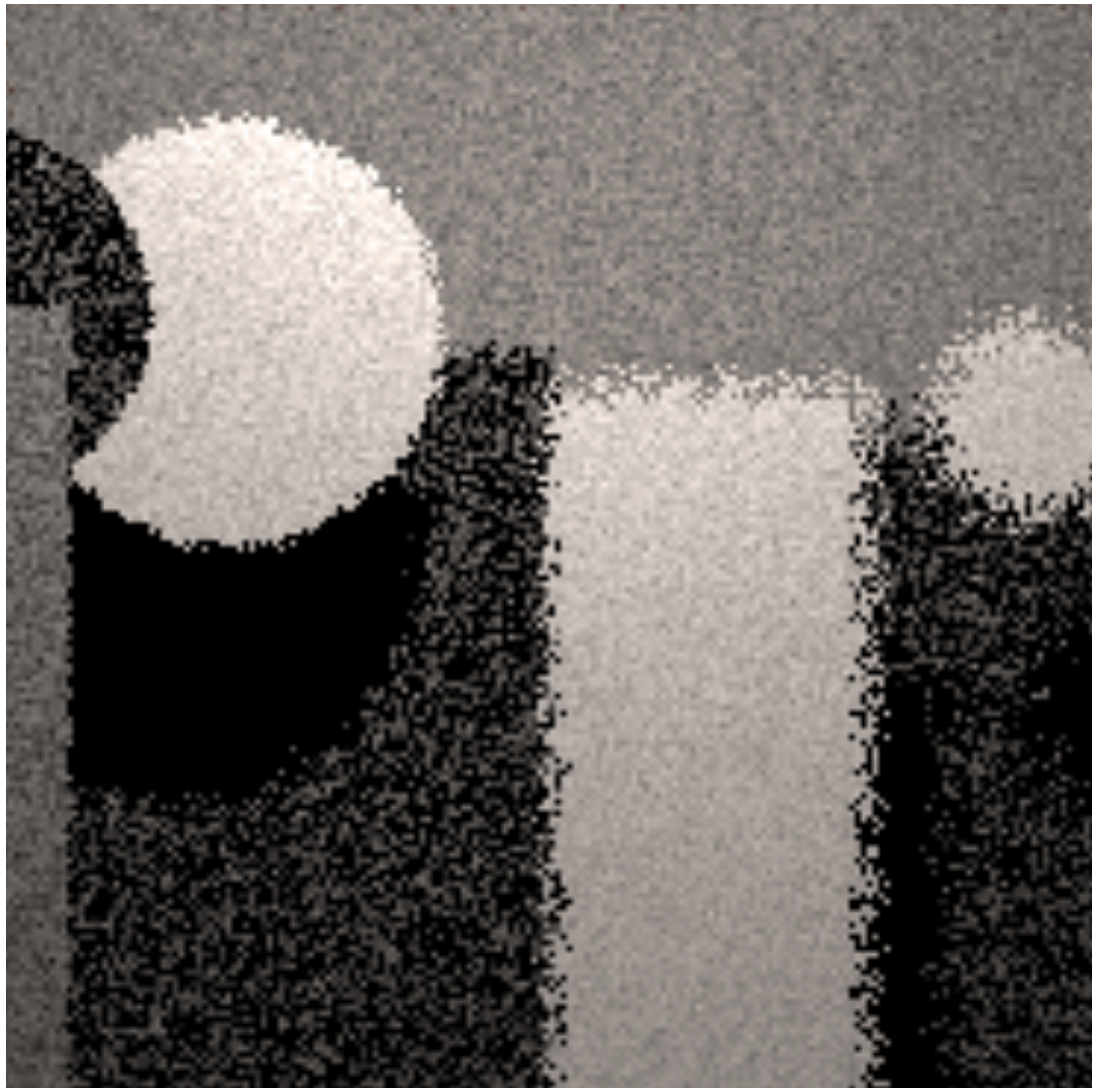


$f(\vec{x})$

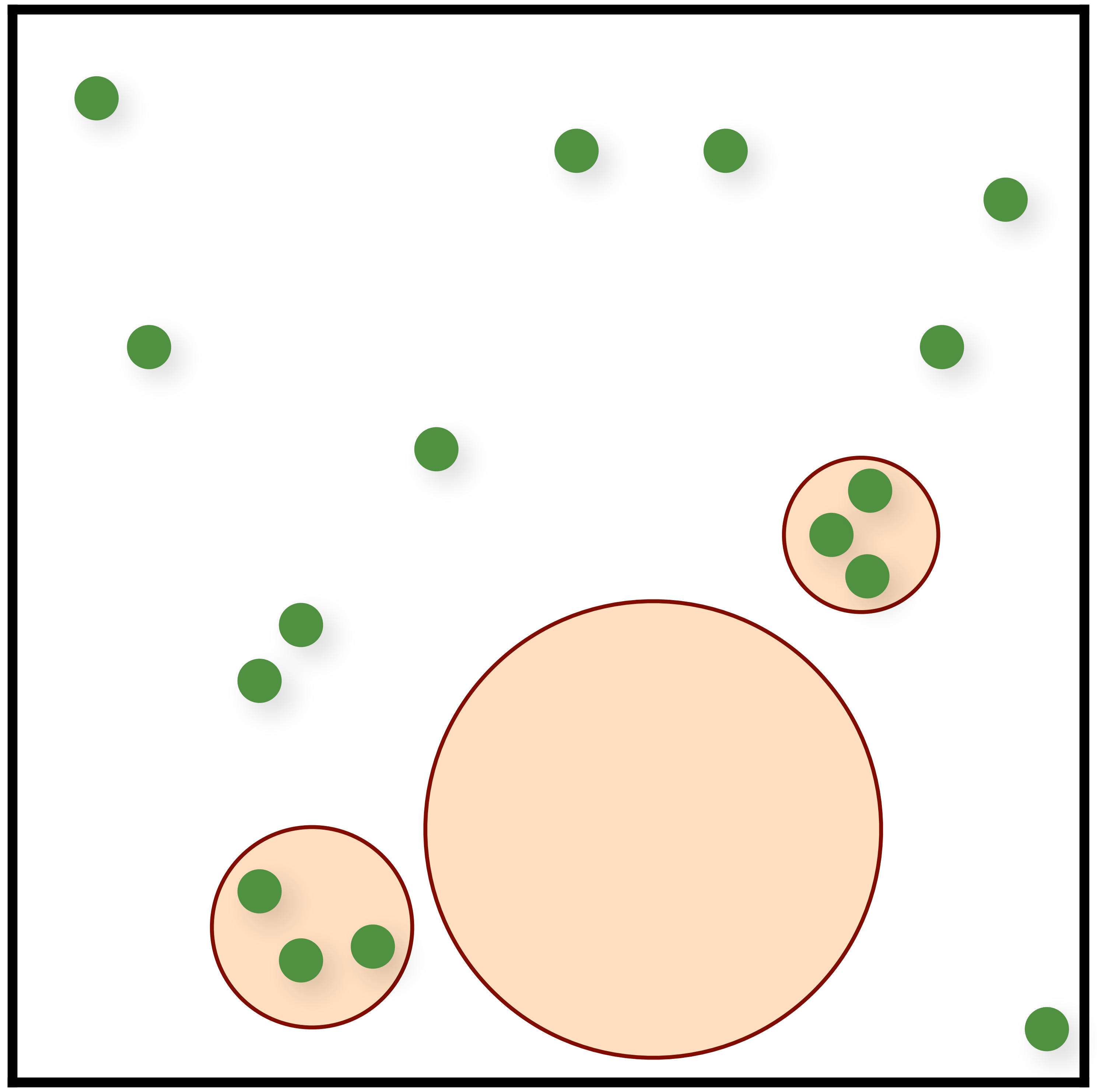


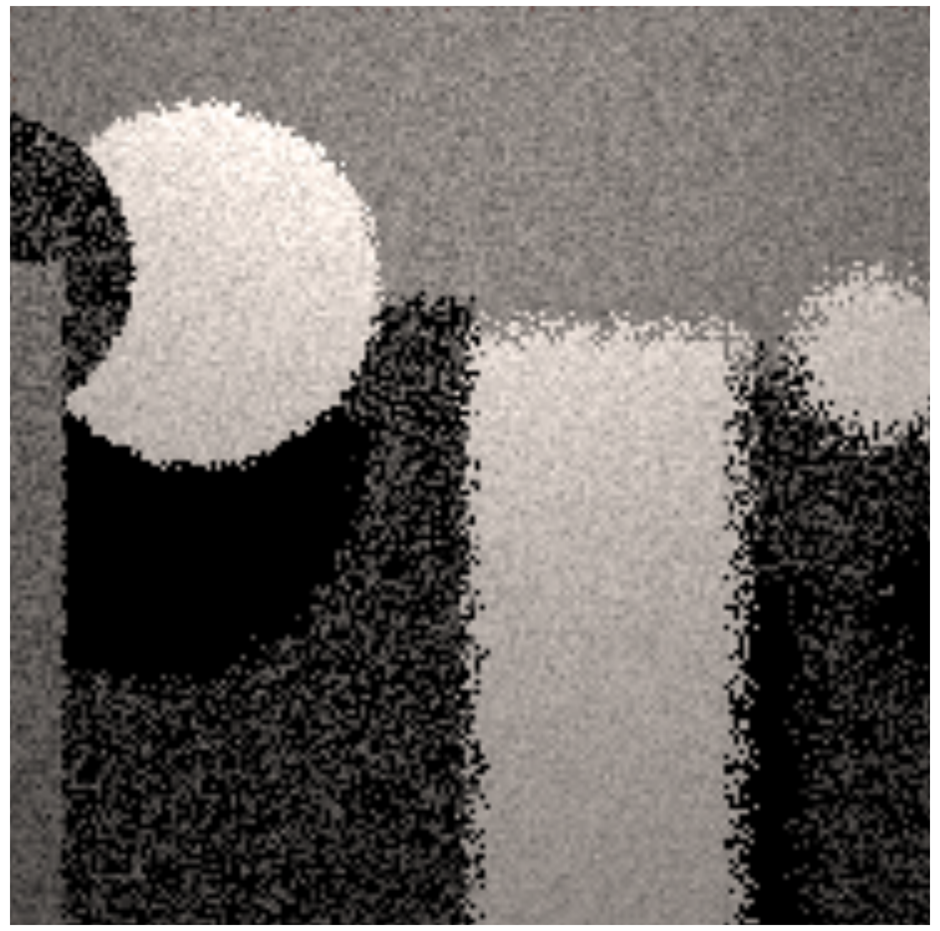
$$\hat{I} = \frac{1}{N} \sum_{k=1}^{N} \frac{f(\vec{x}_k)}{p(\vec{x}_k)}$$

Error as Noise during Monte Carlo Integration



Random





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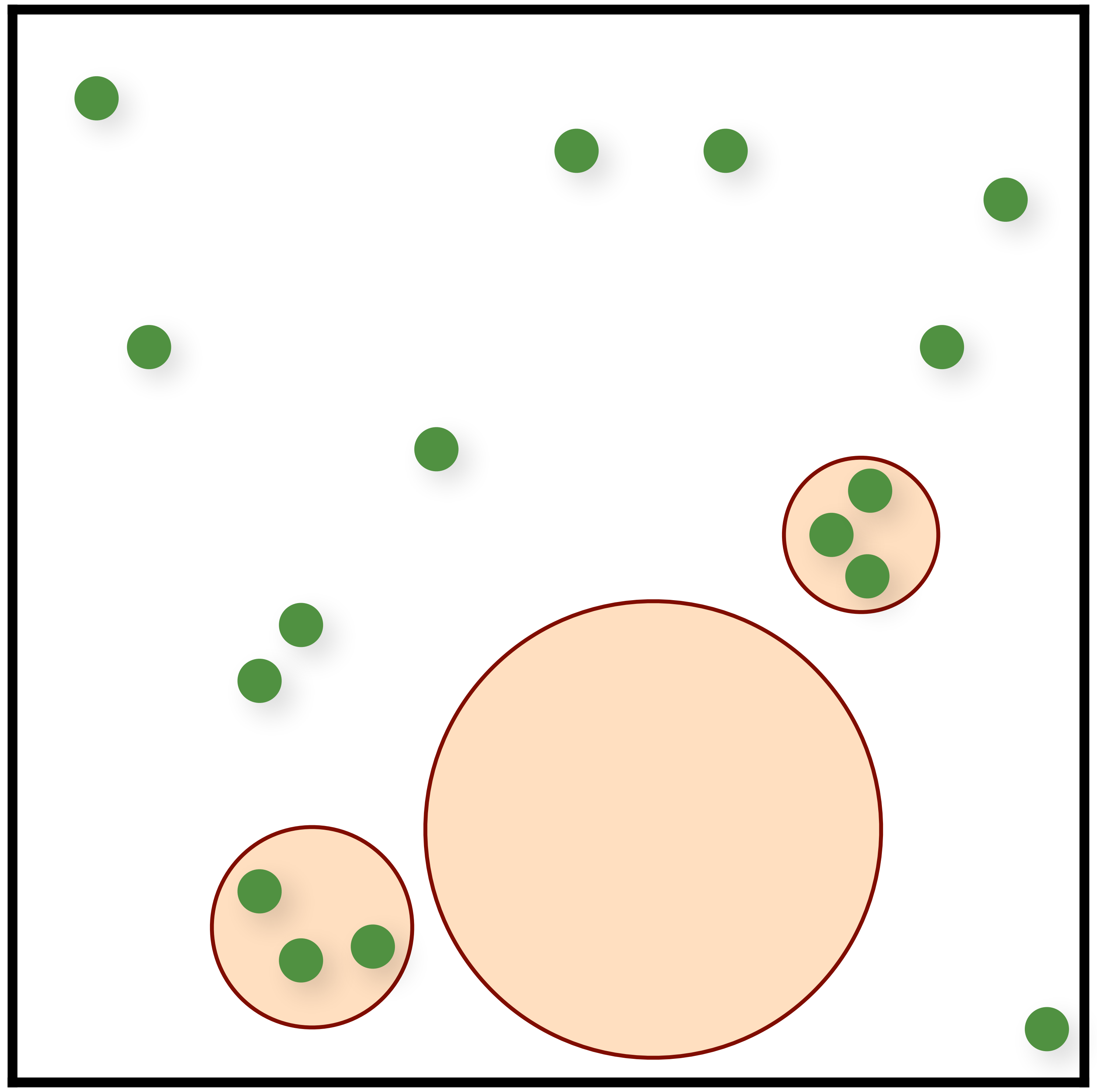

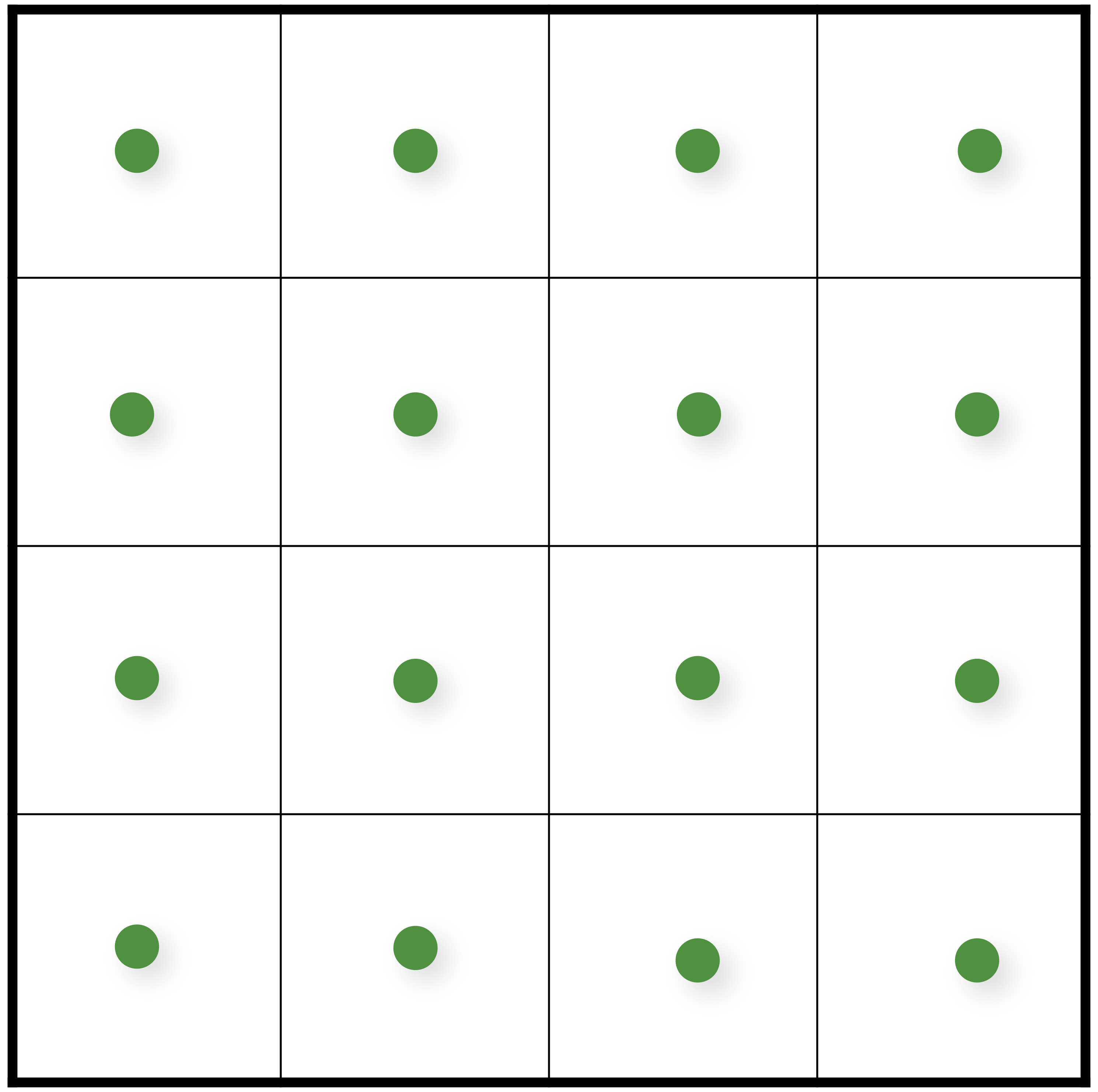

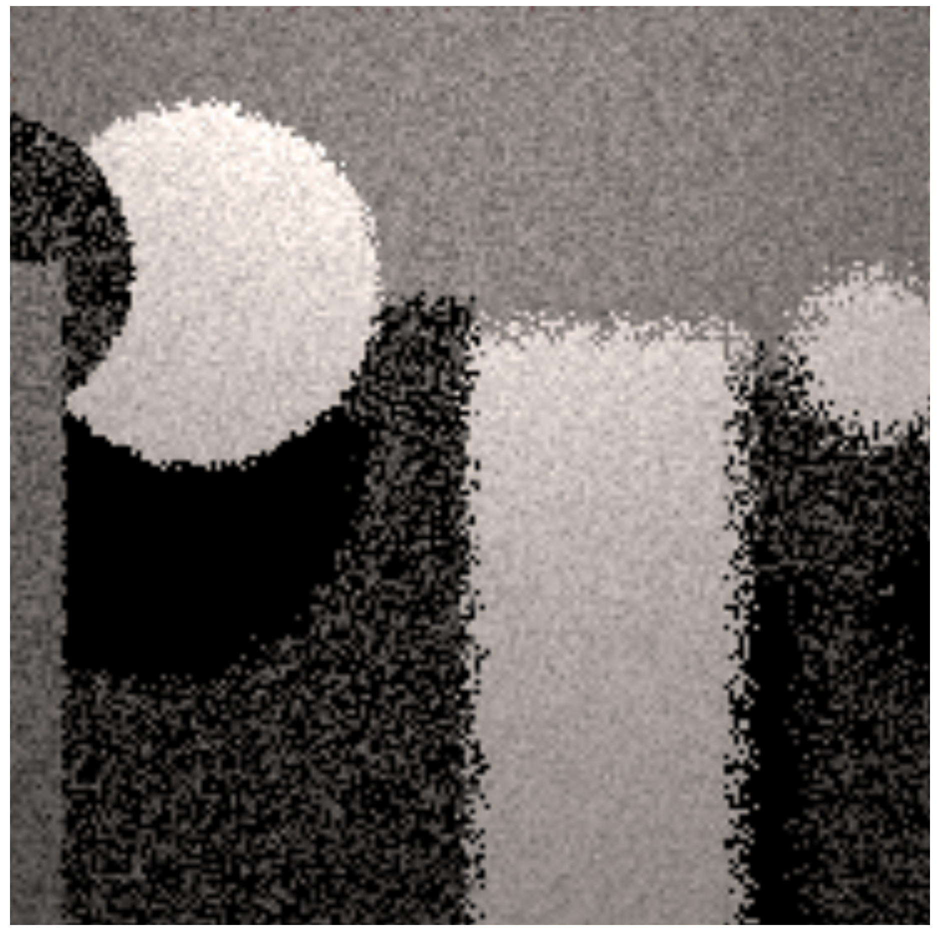

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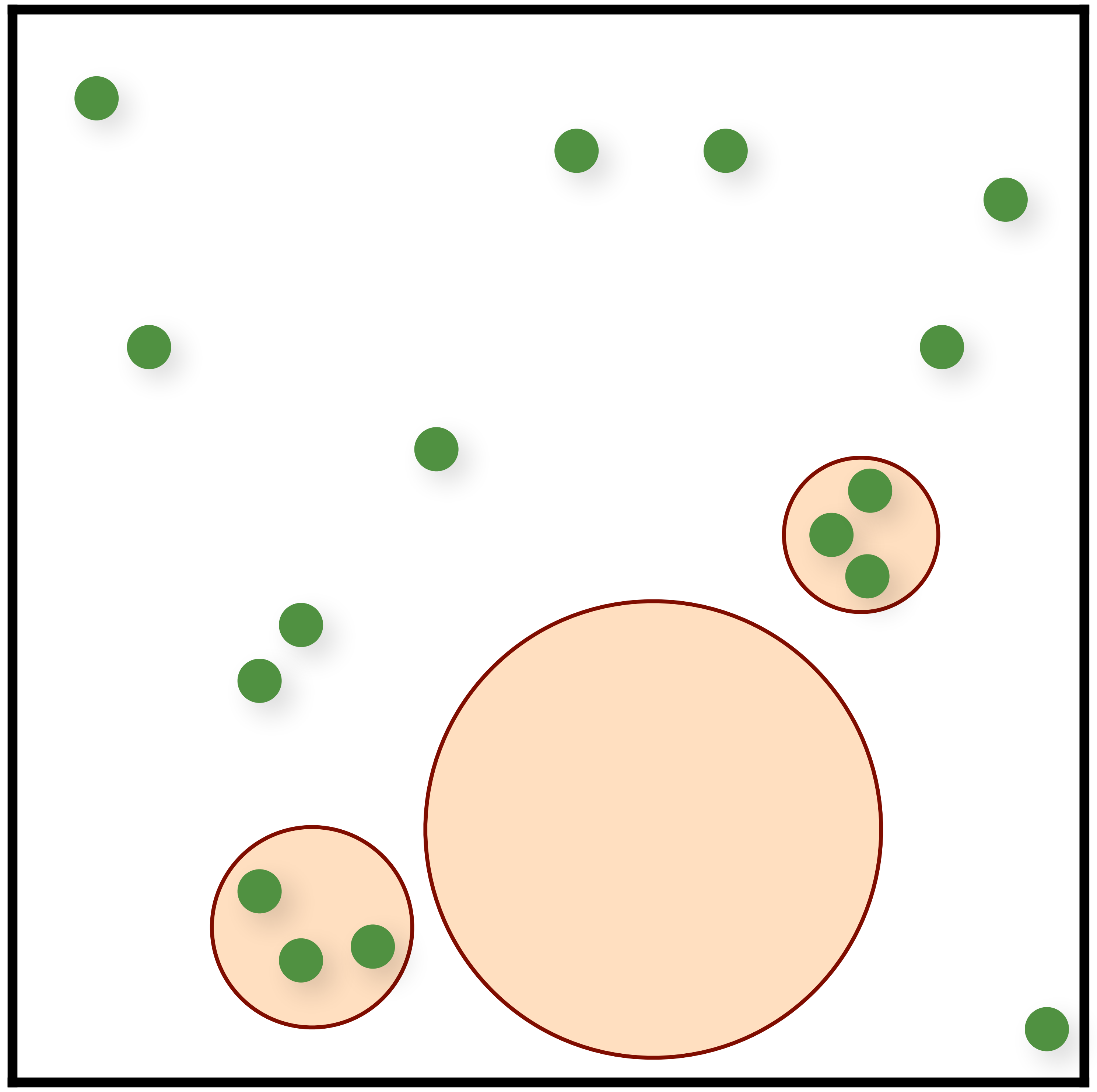
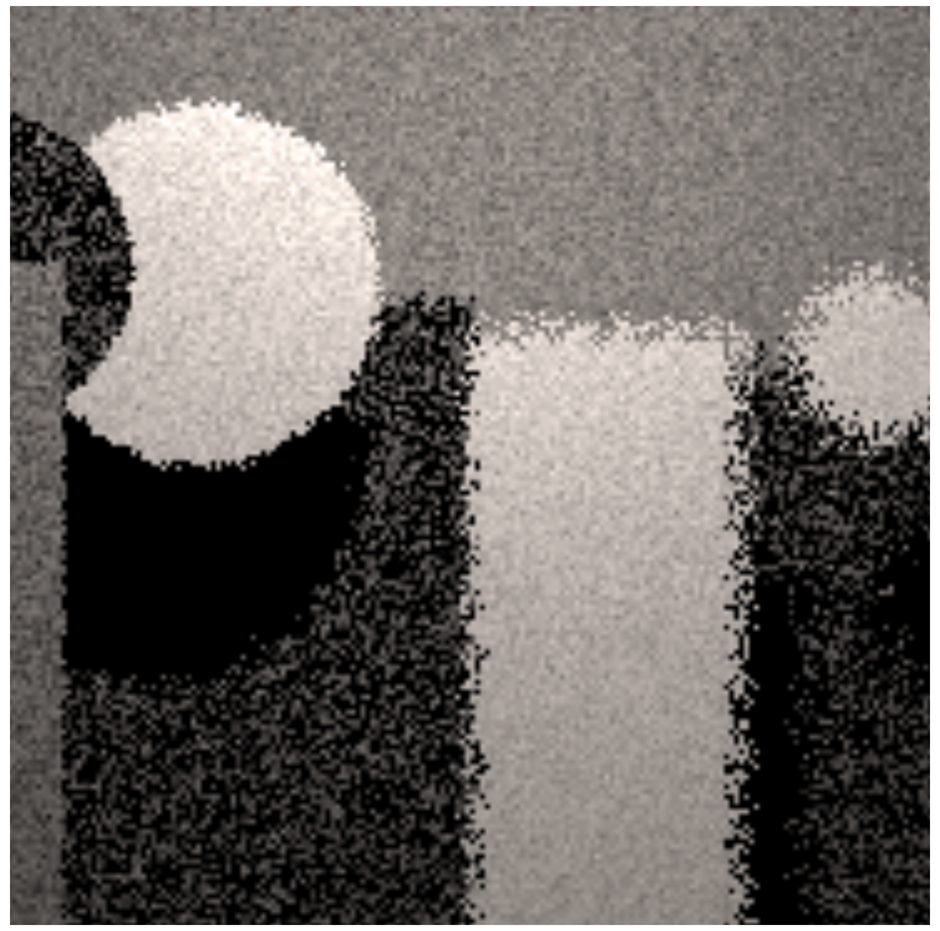
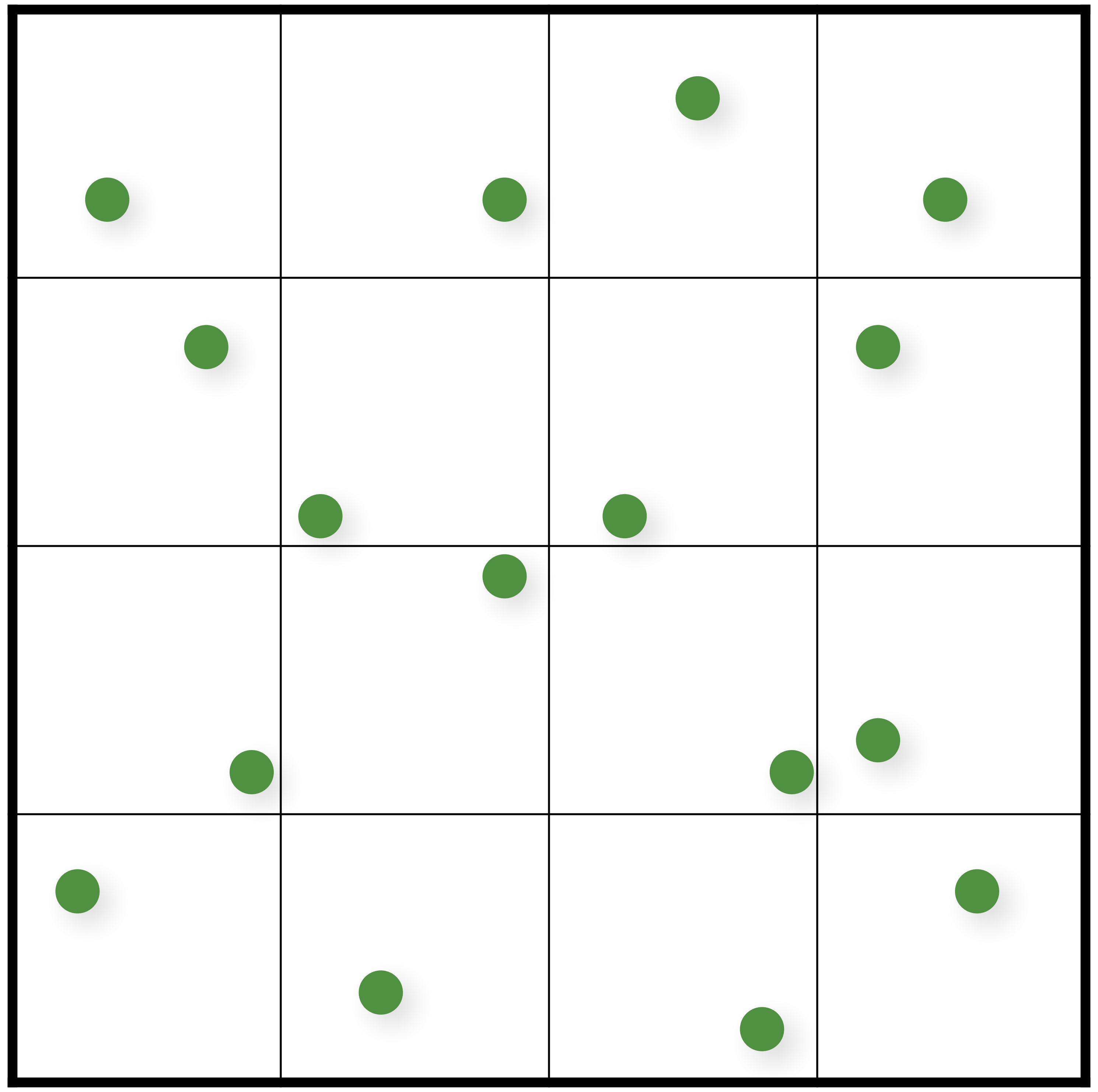
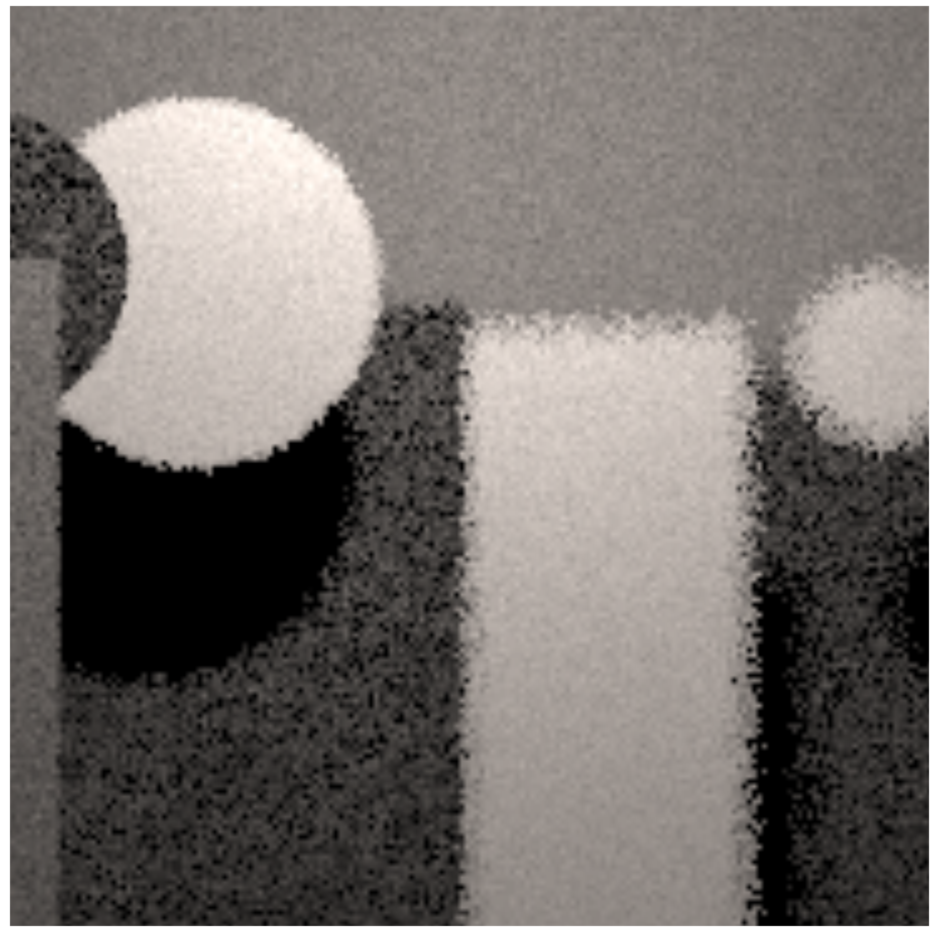


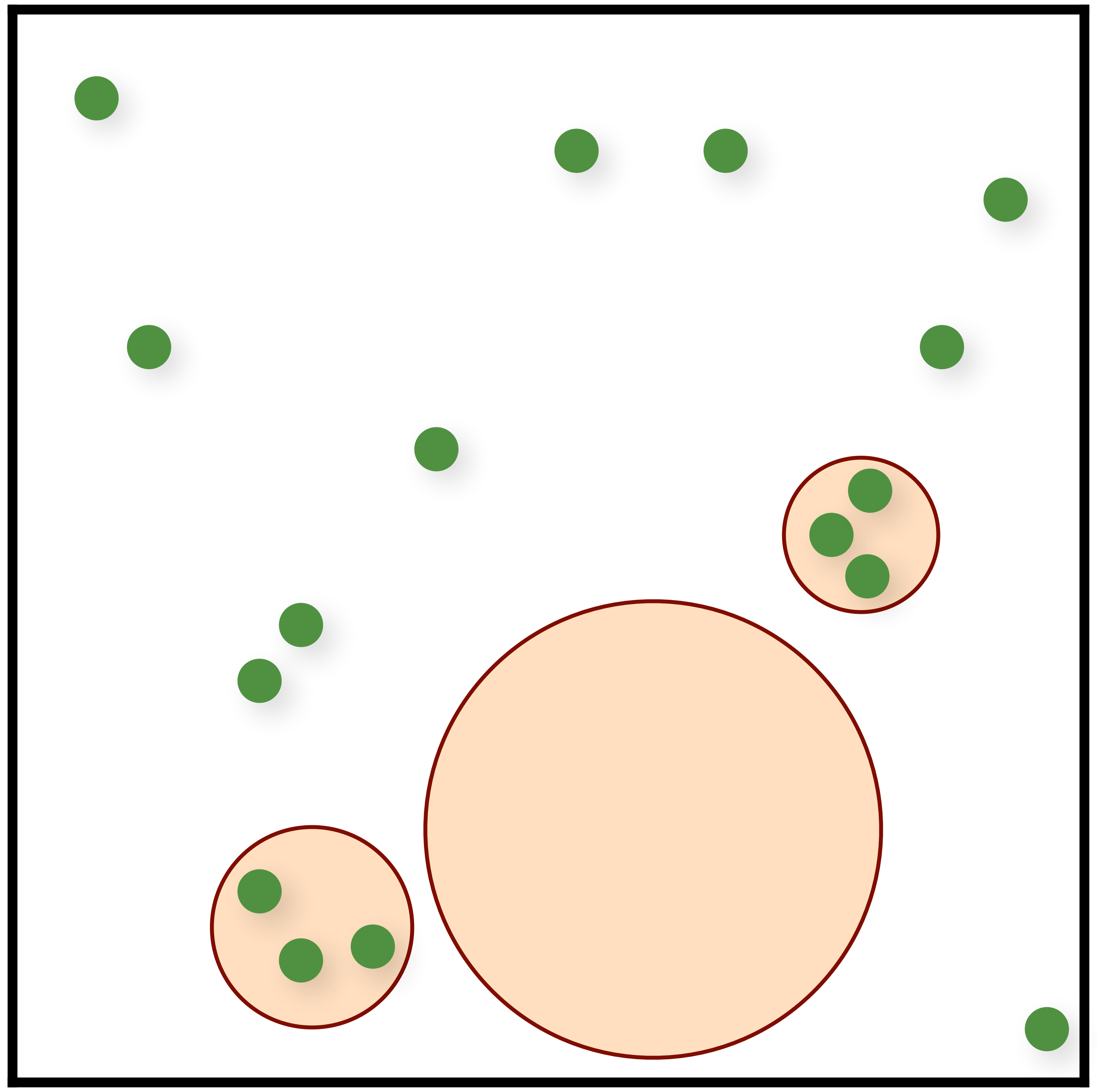


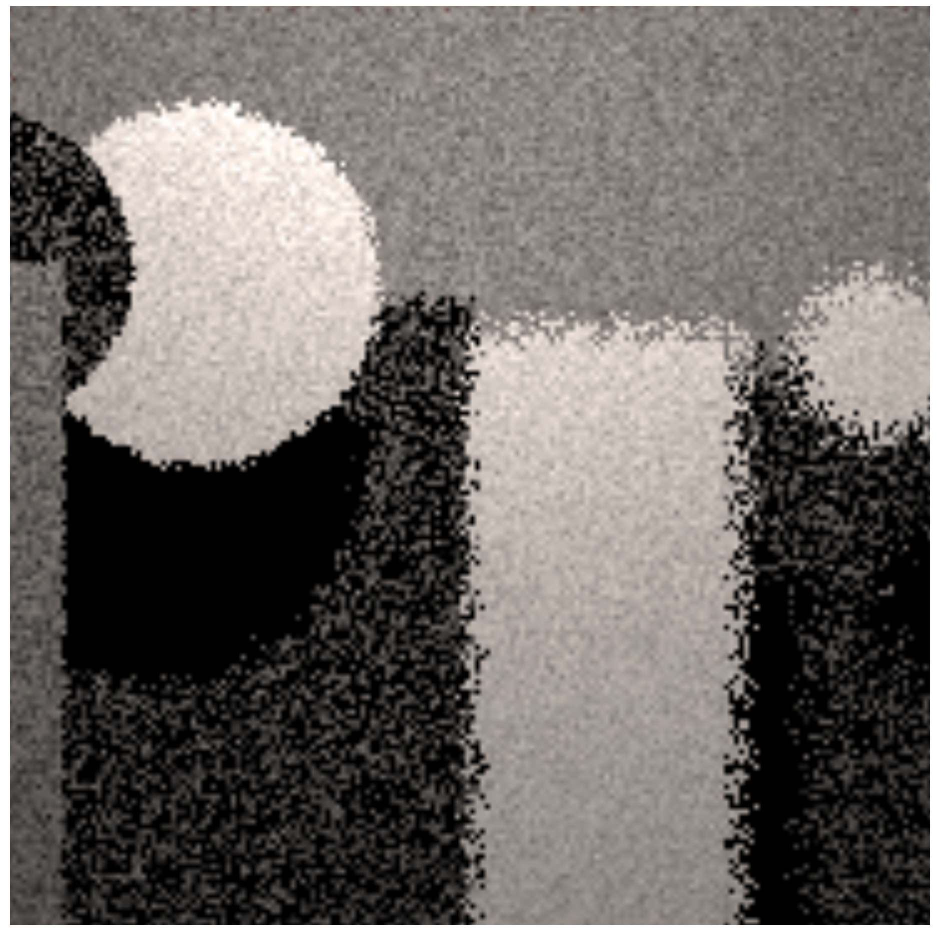
Randomly Jittered





Random





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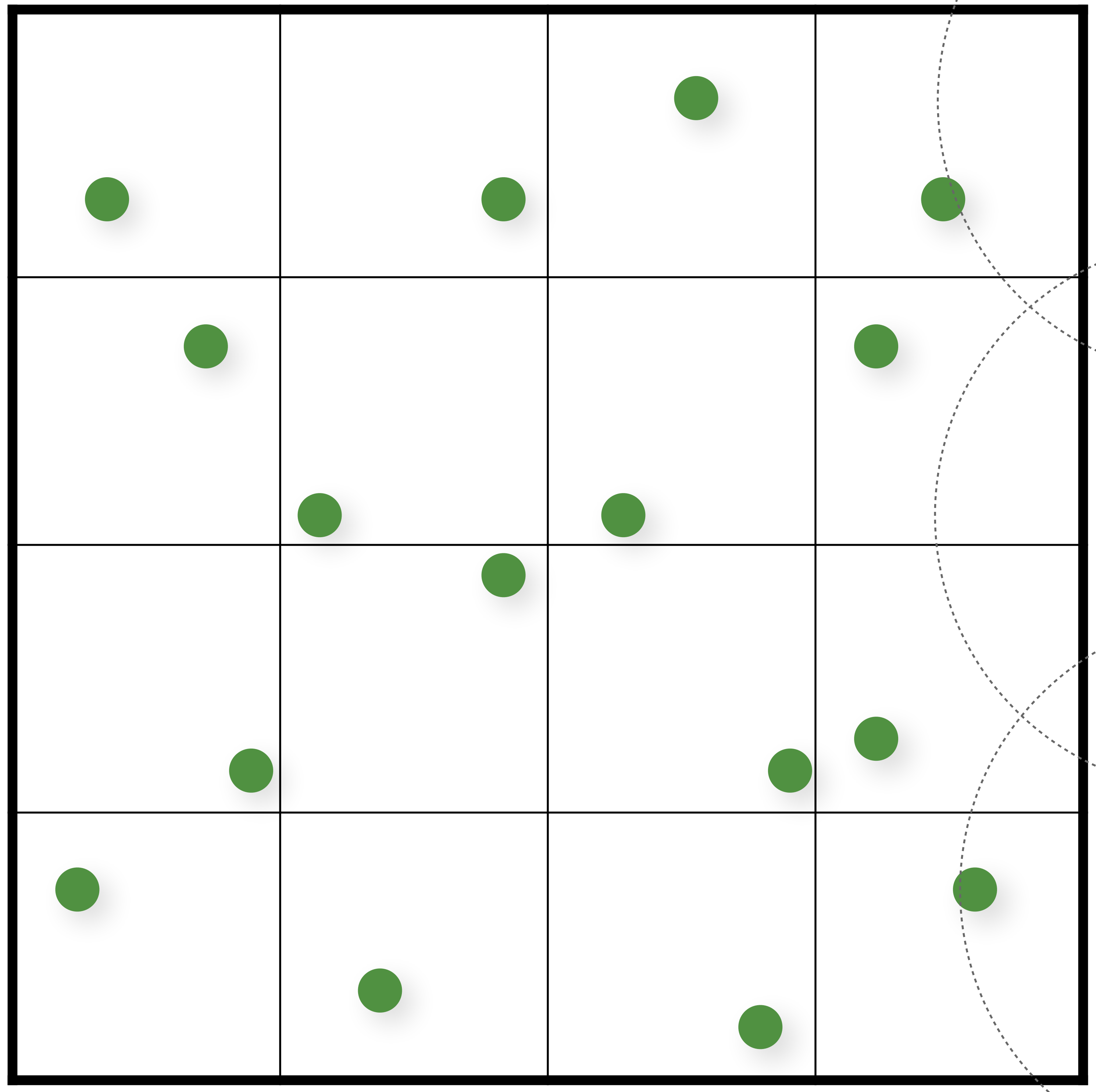
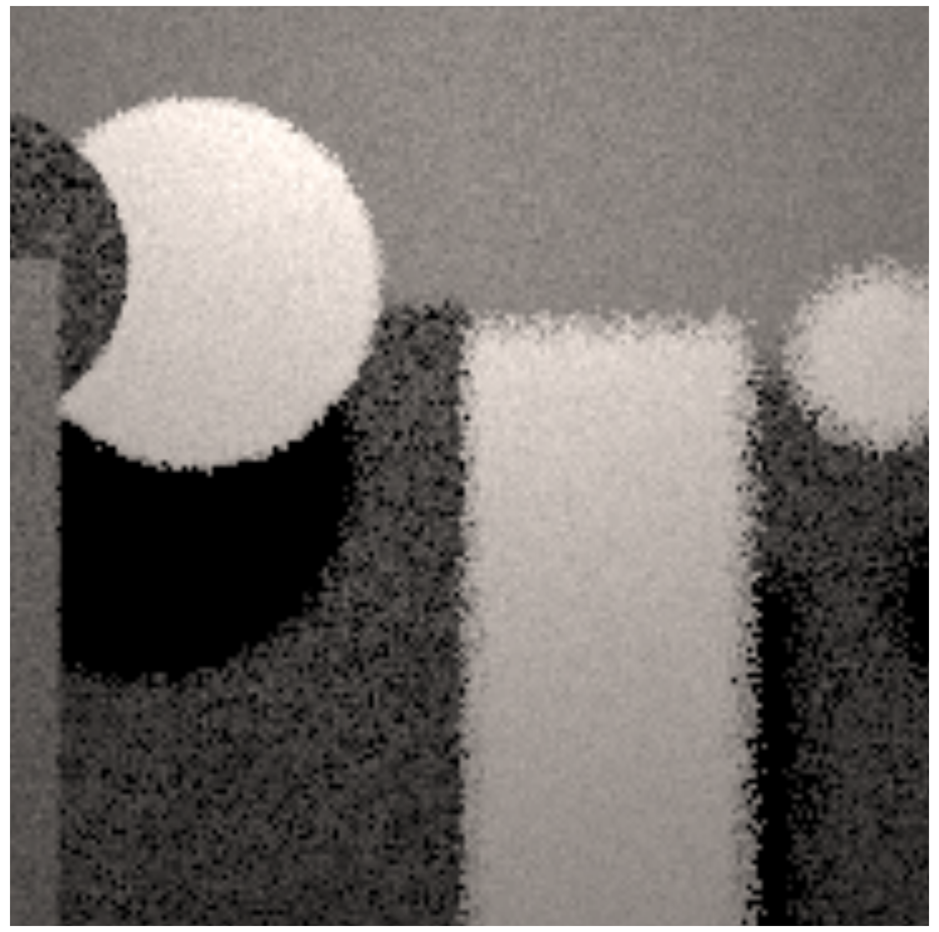
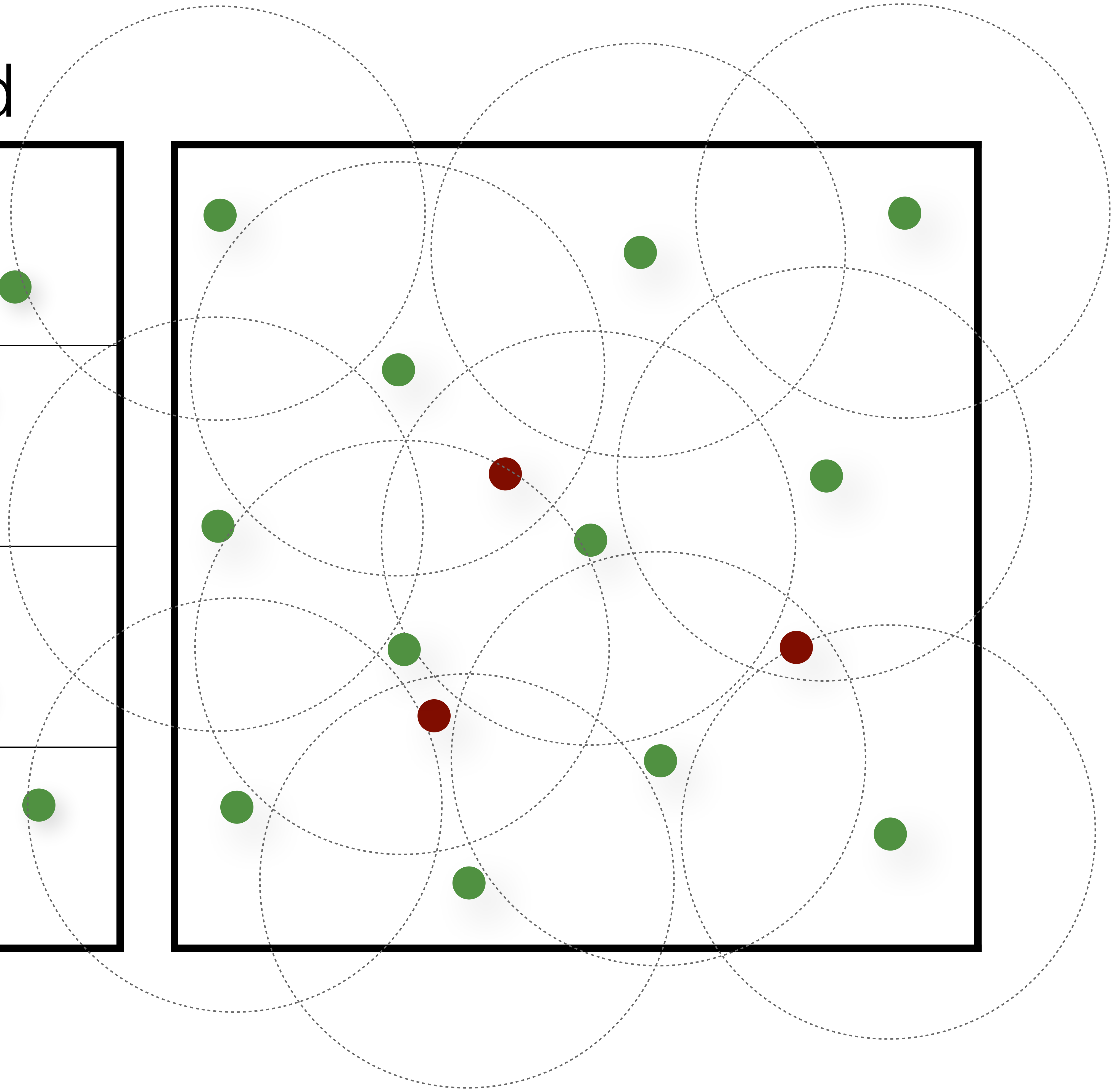
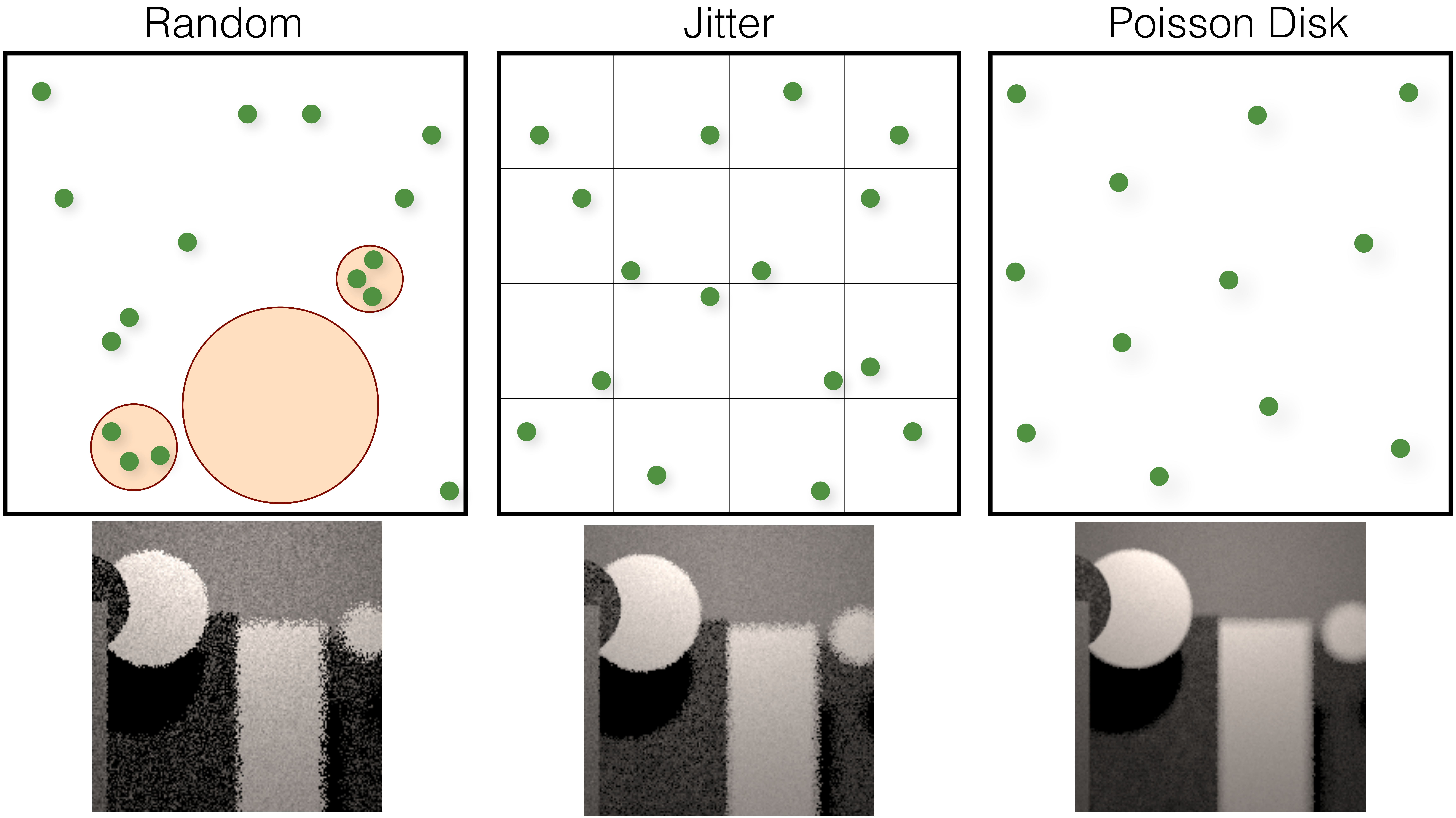
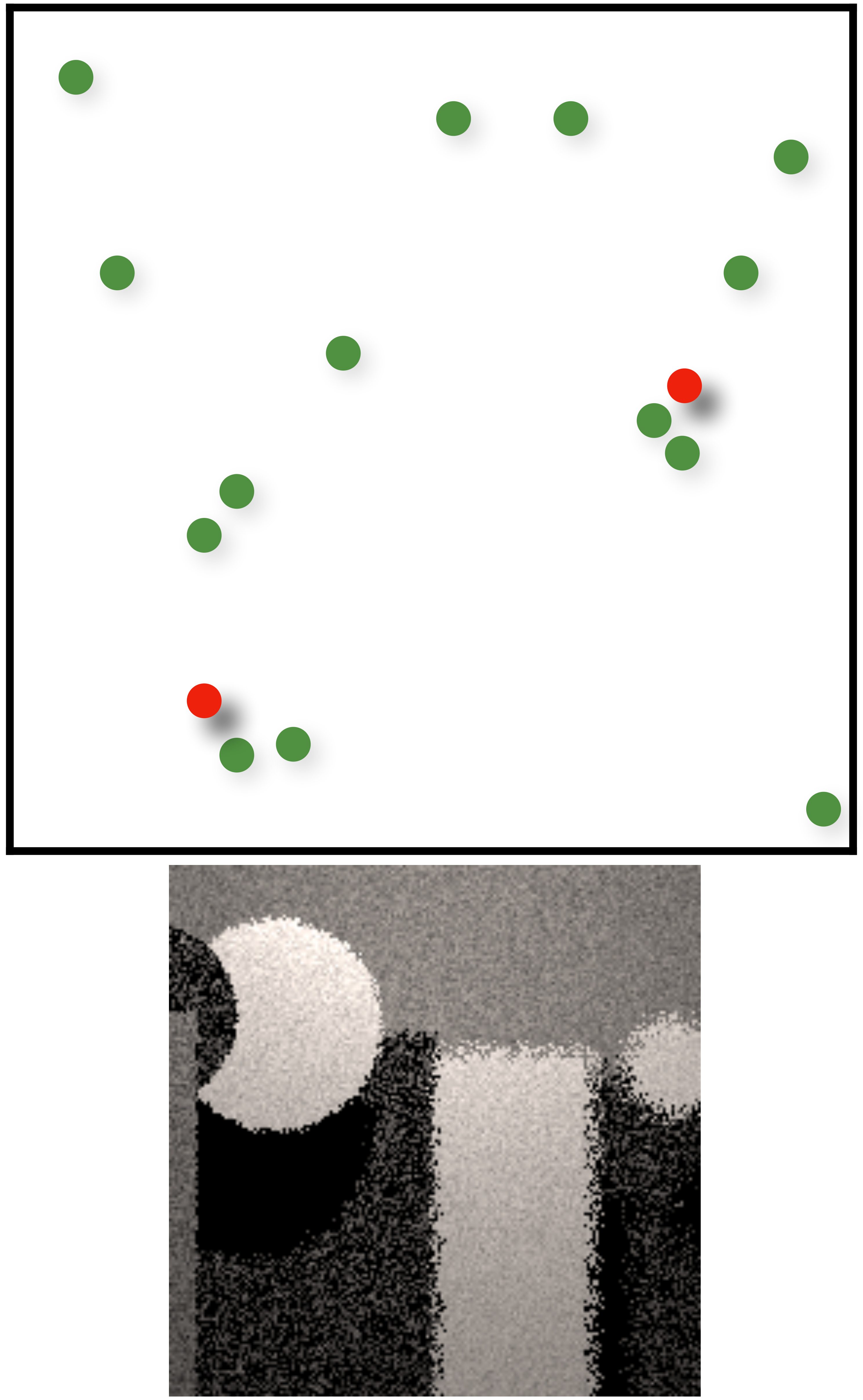




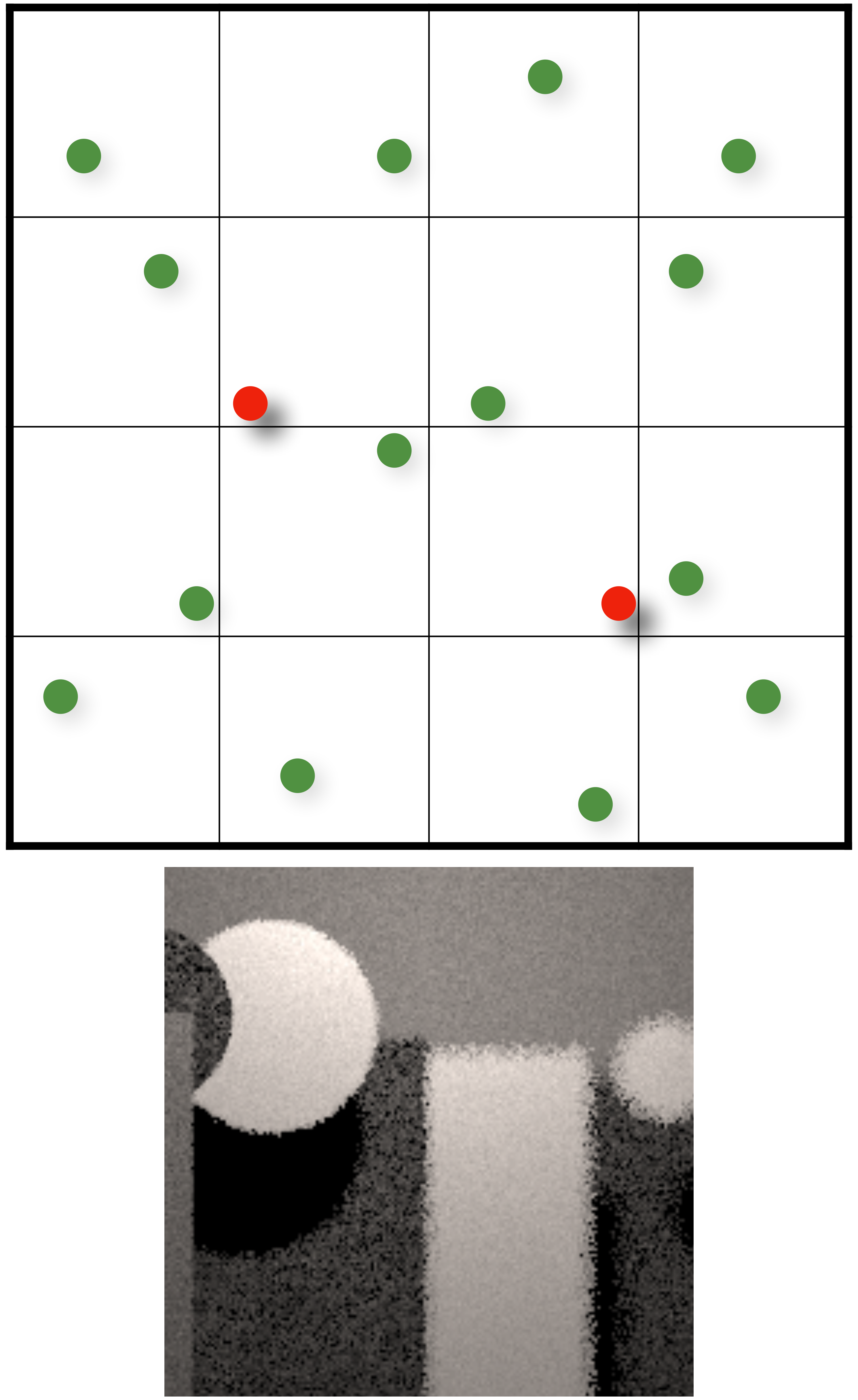



Random
Jitter
Poisson Disk

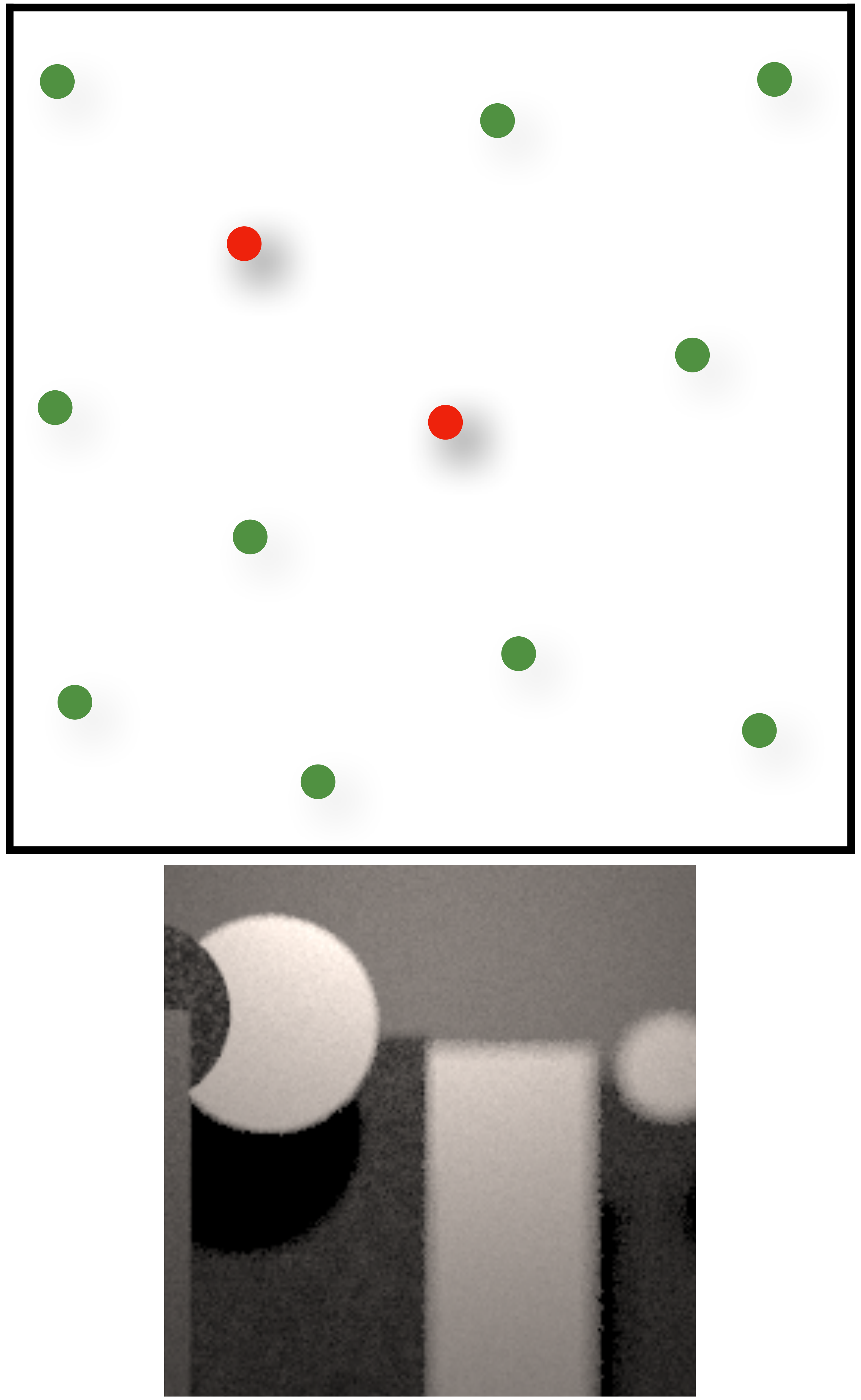
Random



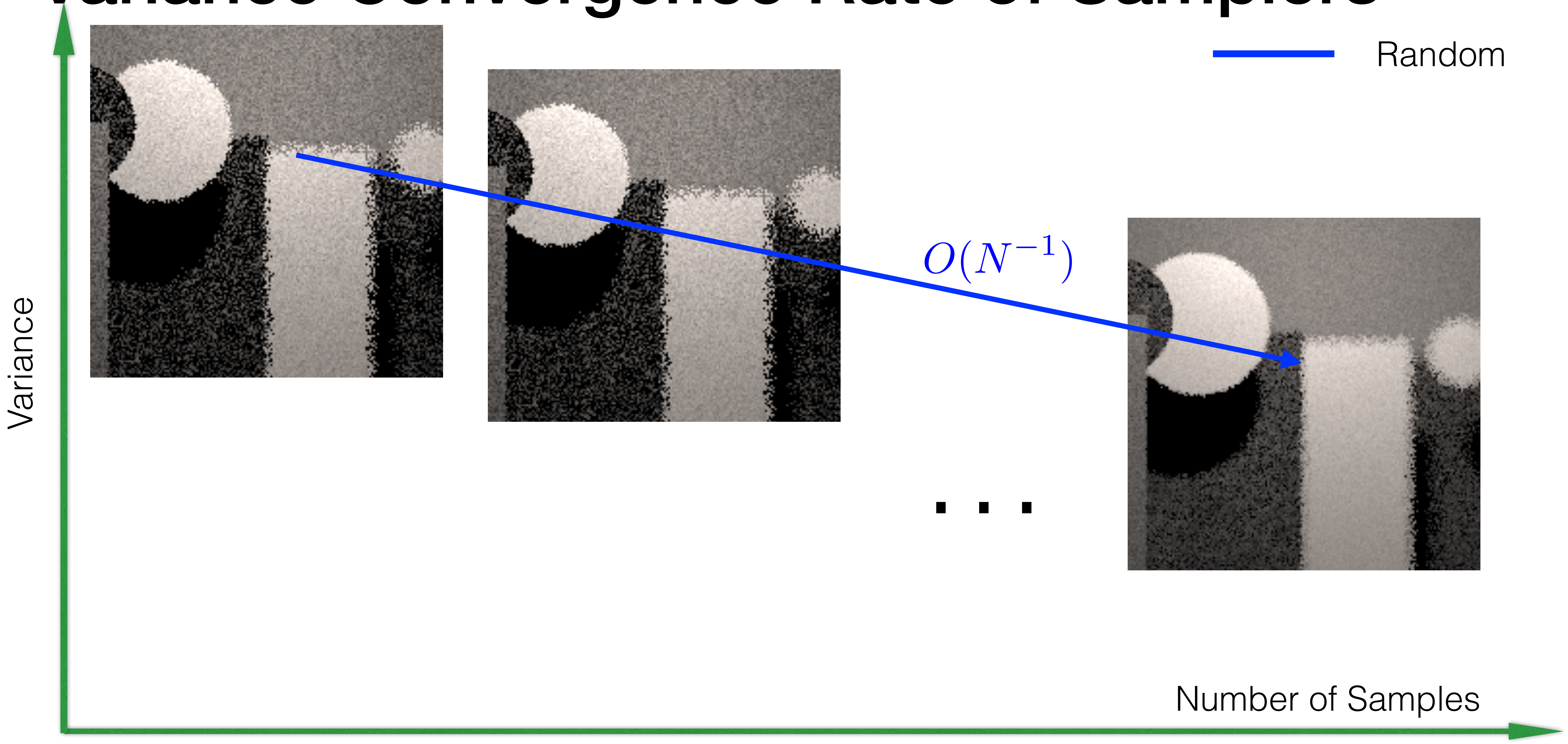
Jitter



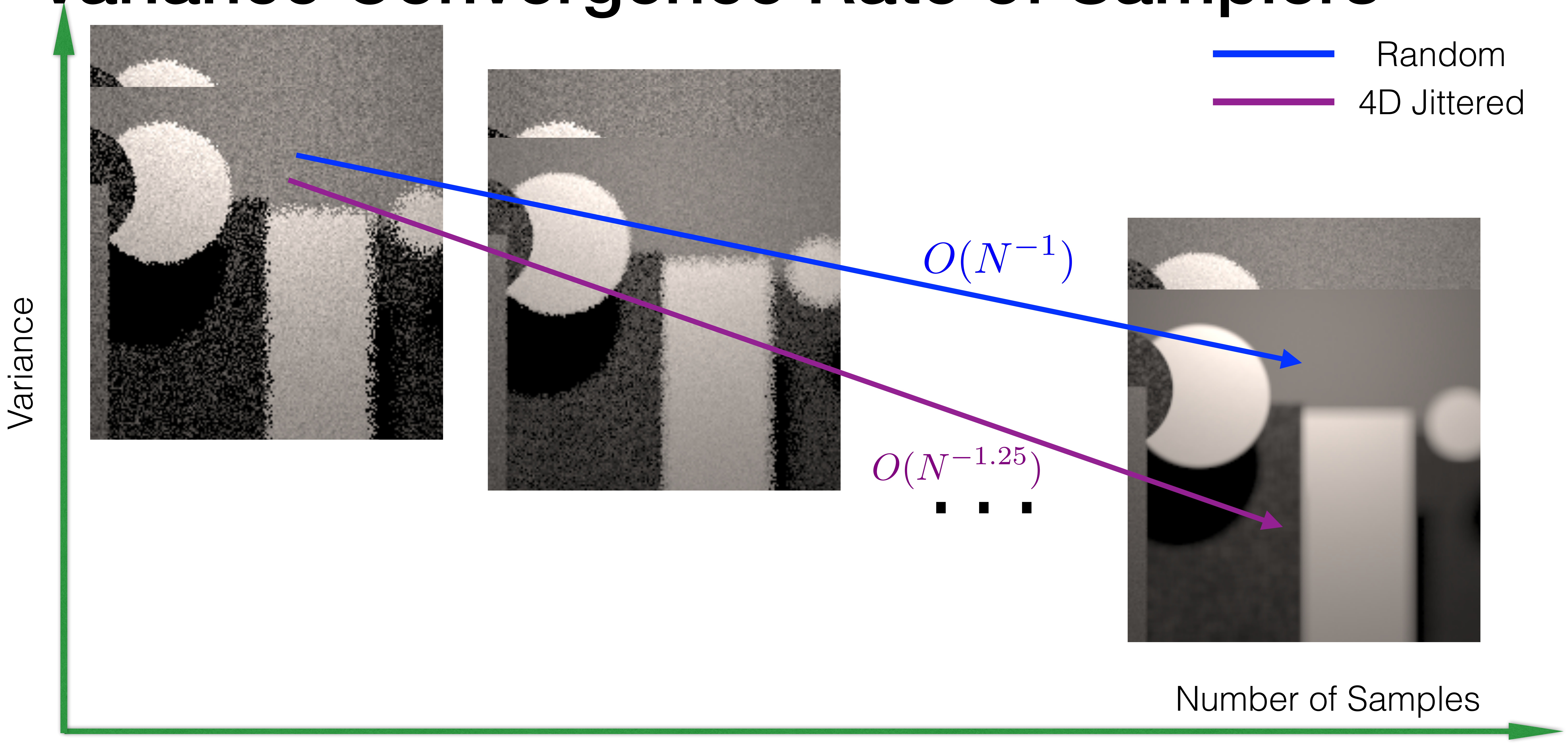
Poisson Disk



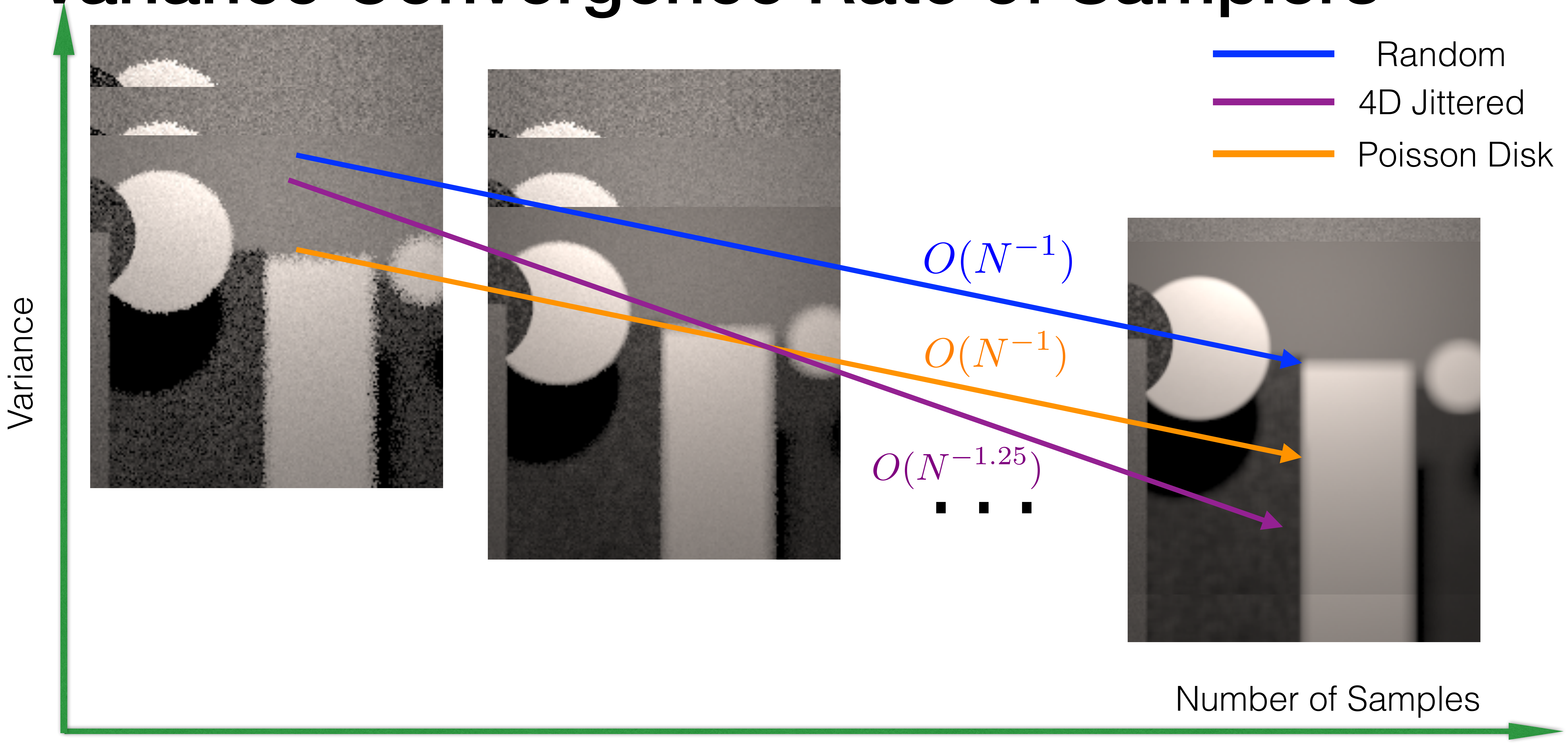
Variance Convergence Rate of Samplers



Variance Convergence Rate of Samplers



Variance Convergence Rate of Samplers



Overview

- Error Formulation in the Spatial Domain
- Error Formulation in the Fourier Domain
- Practical Results
- Conclusion: Design Principles

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- Error Formulation in the Fourier Domain
- Practical Results
- Conclusion: Design Principles

Error in Monte Carlo Integration

True Integral: $I = \int_0^1 f(x)dx$

Monte Carlo Estimation: $\hat{I} = \sum_{k=1}^{N} w_k(x_k)f(x_k) \qquad = \frac{1}{N}\sum_{k=1}^{N}\frac{f(x_k)}{p(x_k)}$

$$\text{Error: } \Delta = \hat{I} - I$$

Error in Monte Carlo Integration

$$\hat{I} = \sum_{k=1}^{N} w_k(x_k) f(x_k)$$

$$I = \int_0^1 f(x)dx$$

$$\text{Error: } \Delta = \hat{I} - I$$

Over multiple realizations:

$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

Error in Monte Carlo Integration

$$\hat{I} = \sum_{k=1}^{N} w_k(x_k) f(x_k)$$

$$I = \int_0^1 f(x)dx$$

$$\text{Error: } \Delta = \hat{I} - I$$

Over multiple realizations:

$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

$$\text{Bias:} \qquad \mathbb{E}[\Delta] = \mathbb{E}[\hat{I} - I] = \mathbb{E}[\hat{I}] - I$$

Error in Monte Carlo Integration

$$\hat{I} = \sum_{k=1}^{N} w_k(x_k) f(x_k)$$

$$I = \int_0^1 f(x)dx$$

$$\text{Error: } \Delta = \hat{I} - I$$

Over multiple realizations:

$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

$$\text{Bias:} \quad \mathbb{E}[\Delta] \quad = \mathbb{E}[\hat{I}] - I$$

Error in Monte Carlo Integration

$$\hat{I} = \sum_{k=1}^{N} w_k(x_k) f(x_k)$$

$$I = \int_0^1 f(x)dx$$

$$\text{Error: } \Delta = \hat{I} - I$$

Over multiple realizations:

$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

$$\text{Bias:} \quad \mathbb{E}[\Delta] = \mathbb{E}[\hat{I}] - I$$

$$\text{Variance:} \quad Var[\hat{I}] = \mathbb{E}[\hat{I}^2] - \mathbb{E}[\hat{I}]^2$$

Error: Bias and Variance

$$\hat{I} = \sum_{k=1}^{N} w_k(x_k) f(x_k)$$

$$I = \int_0^1 f(x)dx$$

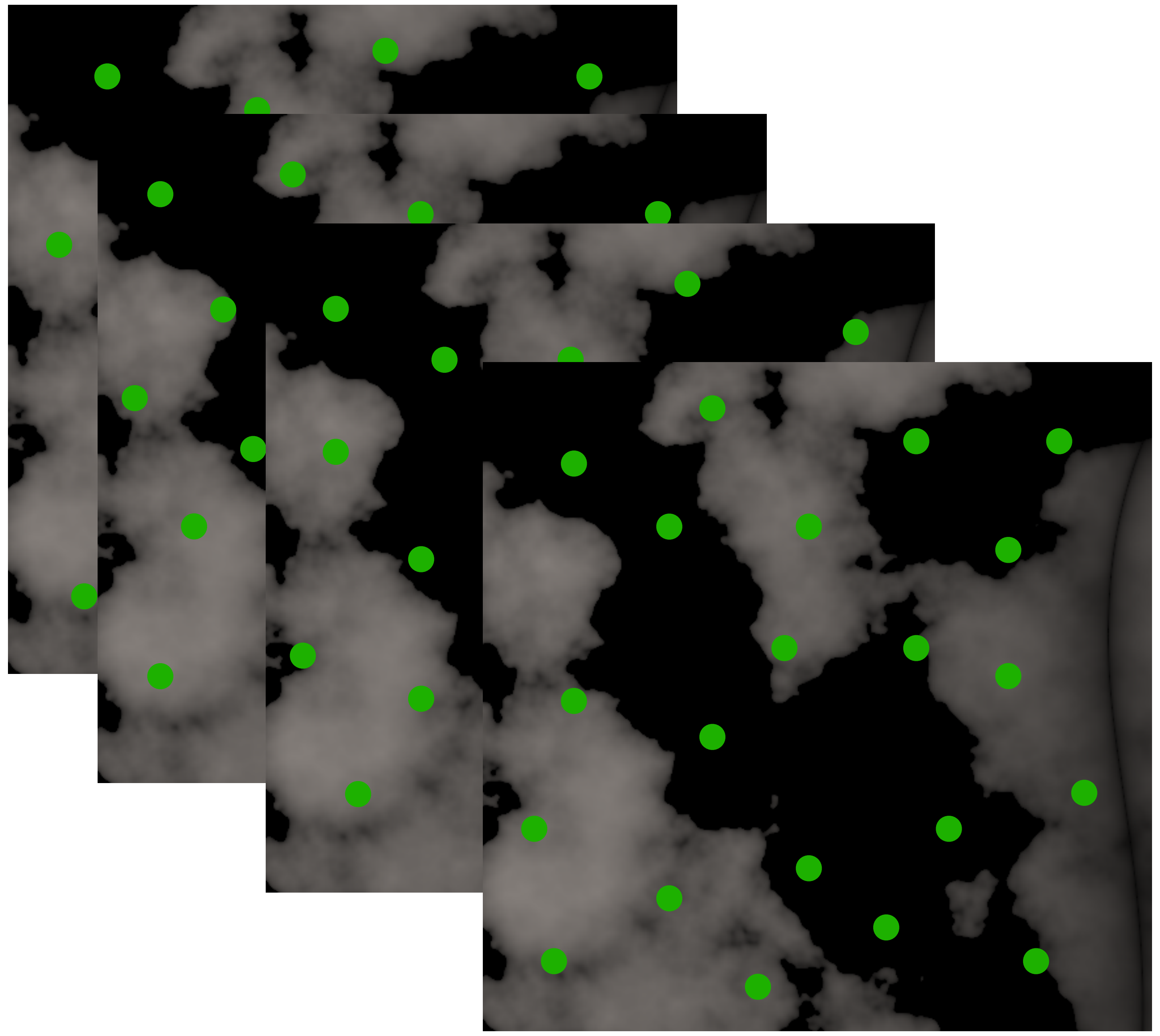
Bias: $\mathbb{E}[\Delta] = \mathbb{E}[\hat{I}] - I$

Variance: $Var[\hat{I}] = \mathbb{E}[\hat{I}^2] - \mathbb{E}[\hat{I}]^2$

$$\mathbb{E}[\hat{I}] = ?$$

Campbell's Theorem

$$\mathbb{E}\left[\sum_{k=1}^{N} f(x_k)\right] = \int_{\mathbb{R}^d} f(x)\lambda(x)dx$$

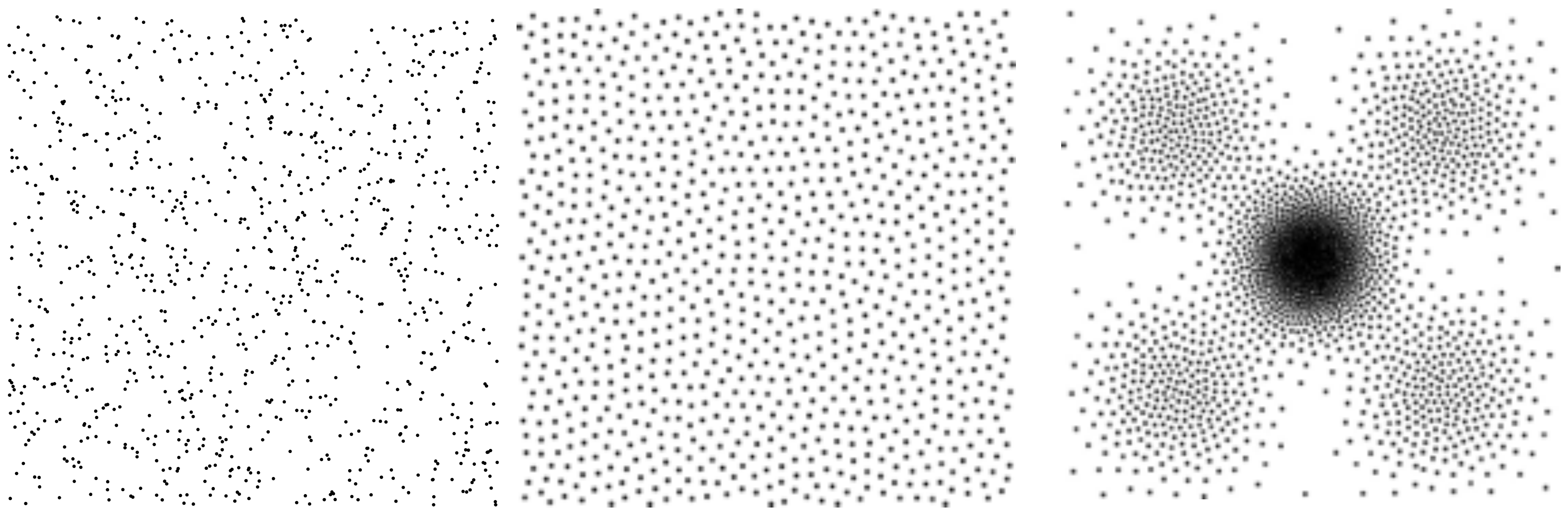


Campbell's Theorem

$$\mathbb{E}\left[\sum_{k=1}^{N} f(x_k)\right] = \int_{\mathbb{R}^d} f(x)\lambda(x)dx$$

$$\lambda(x)$$

First order product density



constant constant varying

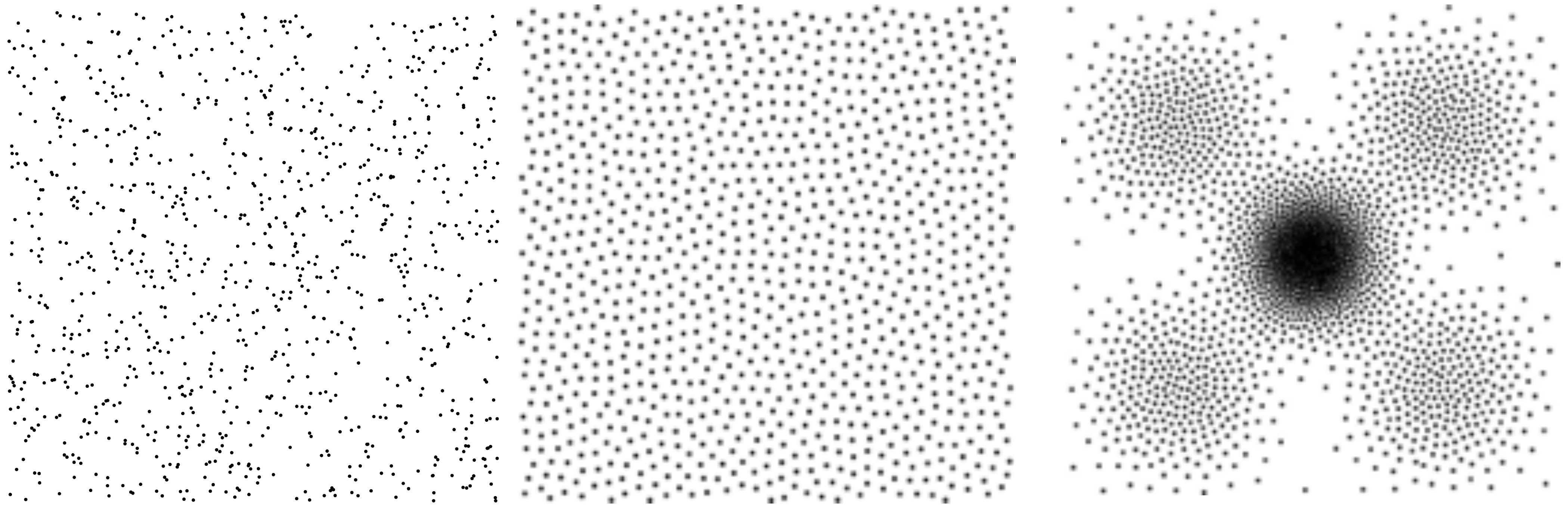
$$\mathbb{E}\left[\sum_{j,k} f(x_j, x_k)\right] = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x)f(y)\varrho(x,y)dxdy$$

Campbell's Theorem

$$\mathbb{E}\left[\sum_{k=1}^{N} f(x_k)\right] = \int_{\mathbb{R}^d} f(x)\lambda(x)dx$$

$\lambda(x)$

First order product density



constant constant varying

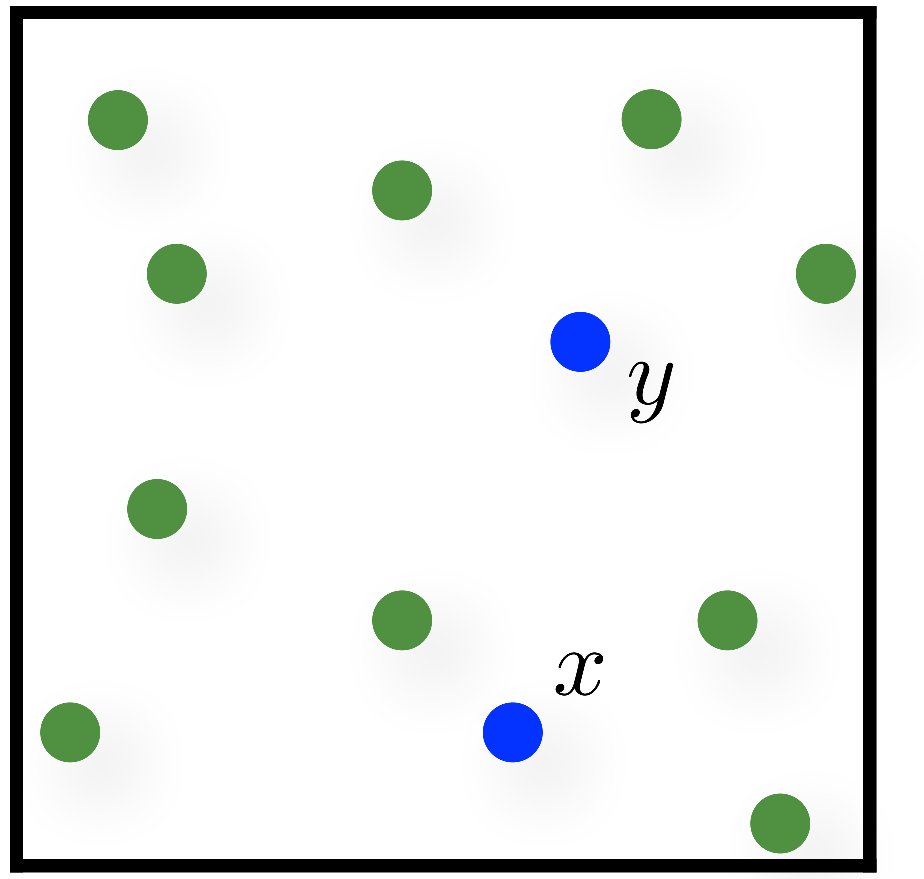
$$\mathbb{E}\left[\sum_{j,k} f(x_j, x_k)\right] = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x)f(y)\varrho(x,y)dxdy$$

$\varrho(x,y)$

Second order product density

Expected number of points around x & y

Measures the joint probability $p(x,y)$



Error: Bias and Variance

Bias: $\mathbb{E}[\Delta] = \mathbb{E}[\hat{I}] - I$

Variance: $Var[\hat{I}] = \mathbb{E}[\hat{I}^2] - \mathbb{E}[\hat{I}]^2$

$$\hat{I} = \sum_{k=1}^{N} w_k(x_k) f(x_k)$$

Error: Bias Term

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$$\mathbb{E}[\hat{I}] = \mathbb{E}\left[\sum_{k=1}^{N} w_k(x_k) f(x_k)\right] = \int_V w(x) f(x) \lambda(x) dx$$

Using Campbell's Theorem

Error: Bias Term

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$$\mathbb{E}[\hat{I}] = \int_V w(x) f(x) \lambda(x) dx$$

Error: Bias Term

$$\text{Bias:} \quad \mathbb{E}[\Delta] = \int_V w(x) f(x) \lambda(x) dx - I$$

$$w(x) = 1/\lambda(x) \longrightarrow \mathbb{E}[\Delta] = 0$$

Bias goes to zero

Error: Bias Term

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$w(x) = 1/\lambda(x) \longrightarrow \mathbb{E}[\Delta] = 0$

Bias goes to zero

For fixed sample count N

$$\lambda(x) = N p(x)$$

$$\hat{I} = \sum_{k=1}^{N} w_k(x_k) f(x_k) = \frac{1}{N} \sum_{k=1}^{N} \frac{f(x_k)}{p(x_k)}$$

Error: Bias Term

$$\text{Bias:} \quad \mathbb{E}[\Delta] = \int_V w(x) f(x) \lambda(x) dx - I$$

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$$\hat{I} = \frac{1}{N} \sum_{k=1}^{N} \frac{f(x_k)}{p(x_k)}$$

Bias goes to zero

Monte Carlo estimator is unbiased

Error: Variance Term

$$Var[\hat{I}] = \mathbb{E}[\hat{I}^2] - \mathbb{E}[\hat{I}]^2$$

$$\mathbb{E}\left[\hat{I}\right] = \int_V w(x)f(x)\lambda(x)dx$$

Error: Variance Term

$$\hat{I} = \sum_{k=1}^{N} w_k(x_k) f(x_k)$$

$$Var[\hat{I}] = \mathbb{E}[\hat{I}^2] - \mathbb{E}[\hat{I}]^2$$

$$\mathbb{E}[\hat{I}]^2 = \left(\int_V w(x) f(x) \lambda(x) dx \right)^2$$

$$\mathbb{E}[\hat{I}^2] = \mathbb{E}\left[\sum_{j \neq k} w(x_j) f(x_j) w(x_k) f(x_k) + \sum_k (w(x_k) f(x_k))^2 \right]$$

Error: Variance Term

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$$\mathbb{E}[\hat{I}^2] = \int_{V \times V} w(x) f(x) w(y) f(y) \varrho(x,y) dx dy + \int_V (w(x) f(x))^2 \lambda(x) dx$$

Using Campbell's Theorem

Error: Variance Term

$$Var[\hat{I}] = \underline{\mathbb{E}[\hat{I}^2]} - \underline{\mathbb{E}[\hat{I}]^2}$$

$$\underline{\mathbb{E}[\hat{I}]^2} = \left(\int_V w(x) f(x) \lambda(x) dx \right)^2$$

$$\underline{\mathbb{E}[\hat{I}^2]} = \int_{V \times V} w(x) f(x) w(y) f(y) \varrho(x, y) dx dy + \int_V (w(x) f(x))^2 \lambda(x) dx$$

Error: Variance Term

$$Var[\hat{I}] = \mathbb{E}[\hat{I}^2] - \mathbb{E}[\hat{I}]^2$$

$$\mathbb{E}\left[\hat{I}\right]^2 = \left(\int_V w(x)f(x)\lambda(x)dx\right)^2$$

$$\mathbb{E}[\hat{I}^2] = \int_{V\times V} w(x)f(x)w(y)f(y)\varrho(x,y)dxdy + \int_V (w(x)f(x))^2\lambda(x)dx$$

Error: Variance Term

$$Var[\hat{I}] = \underline{\mathbb{E}[\hat{I}^2]} - \underline{\mathbb{E}[\hat{I}]^2}$$

$$Var[\hat{I}] = \underline{\int_{V\times V} w(x)f(x)w(y)f(y)\varrho(x,y)dxdy + \int_V (w(x)f(x))^2\lambda(x)dx}$$

$$- \underline{\left(\int_V w(x)f(x)\lambda(x)dx\right)^2}$$

Oztireli [2016]

Error: Variance Term

For an unbiased
Monte Carlo Estimator

$$w(x) = 1/\lambda(x)$$

$$Var[\hat{I}] = \int_{V \times V} w(x)f(x)w(y)f(y)\varrho(x,y)dxdy + \int_V (w(x)f(x))^2 \lambda(x)dx$$

$$-\left(\int_V w(x)f(x)\lambda(x)dx\right)^2$$

Error: Variance Term

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Monte Carlo Estimator

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Monte Carlo Estimator

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$$- \left(\int_V f(x)dx\right)^2$$

Error: Variance Term

For an unbiased
Monte Carlo Estimator

$$w(x) = 1/\lambda(x)$$

$$Var[\hat{I}] = \int_{V\times V} w(x)f(x)w(y)f(y)\varrho(x,y)dxdy + \int_V (w(x)f(x))^2\lambda(x)dx$$

$$- \quad I^2$$

Error: Variance Term

For an unbiased
Monte Carlo Estimator

$$w(x) = 1/\lambda(x)$$

$$Var[\hat{I}] = \int_{V\times V} w(x)f(x)w(y)f(y)\varrho(x,y)dxdy + \int_{V} (w(x)f(x))^2\lambda(x)dx - I^2$$

Error: Variance Term

For an unbiased
Monte Carlo Estimator

$$w(x) = 1/\lambda(x)$$

$$Var[\hat{I}] = \int_{V \times V} f(x) f(y) \frac{\varrho(x,y)}{\lambda(x)\lambda(y)} dx dy + \int_V (w(x) f(x))^2 \lambda(x) dx - I^2$$

Error: Variance Term

For an unbiased
Monte Carlo Estimator

$$w(x) = 1/\lambda(x)$$

$$Var[\hat{I}] = \int_{V\times V} f(x)f(y)\frac{\varrho(x,y)}{\lambda(x)\lambda(y)}dxdy \quad + \int_V (w(x)f(x))^2\lambda(x)dx \; - \; I^2$$

Error: Variance Term

For an unbiased
Monte Carlo Estimator

$$Var[\hat{I}] = \int_{V \times V} f(x) f(y) \frac{\varrho(x,y)}{\lambda(x)\lambda(y)} dx dy \quad + \int_V \frac{f(x)^2}{\lambda(x)} dx \quad - \; I^2$$

Error: Variance Term

For an unbiased
Monte Carlo Estimator

$$Var[\hat{I}] = \int_{V \times V} f(x) f(y) \frac{\varrho(x,y)}{\lambda(x)\lambda(y)} dx dy + \int_V \frac{f(x)^2}{\lambda(x)} dx \; - \; I^2$$

Second order correlations

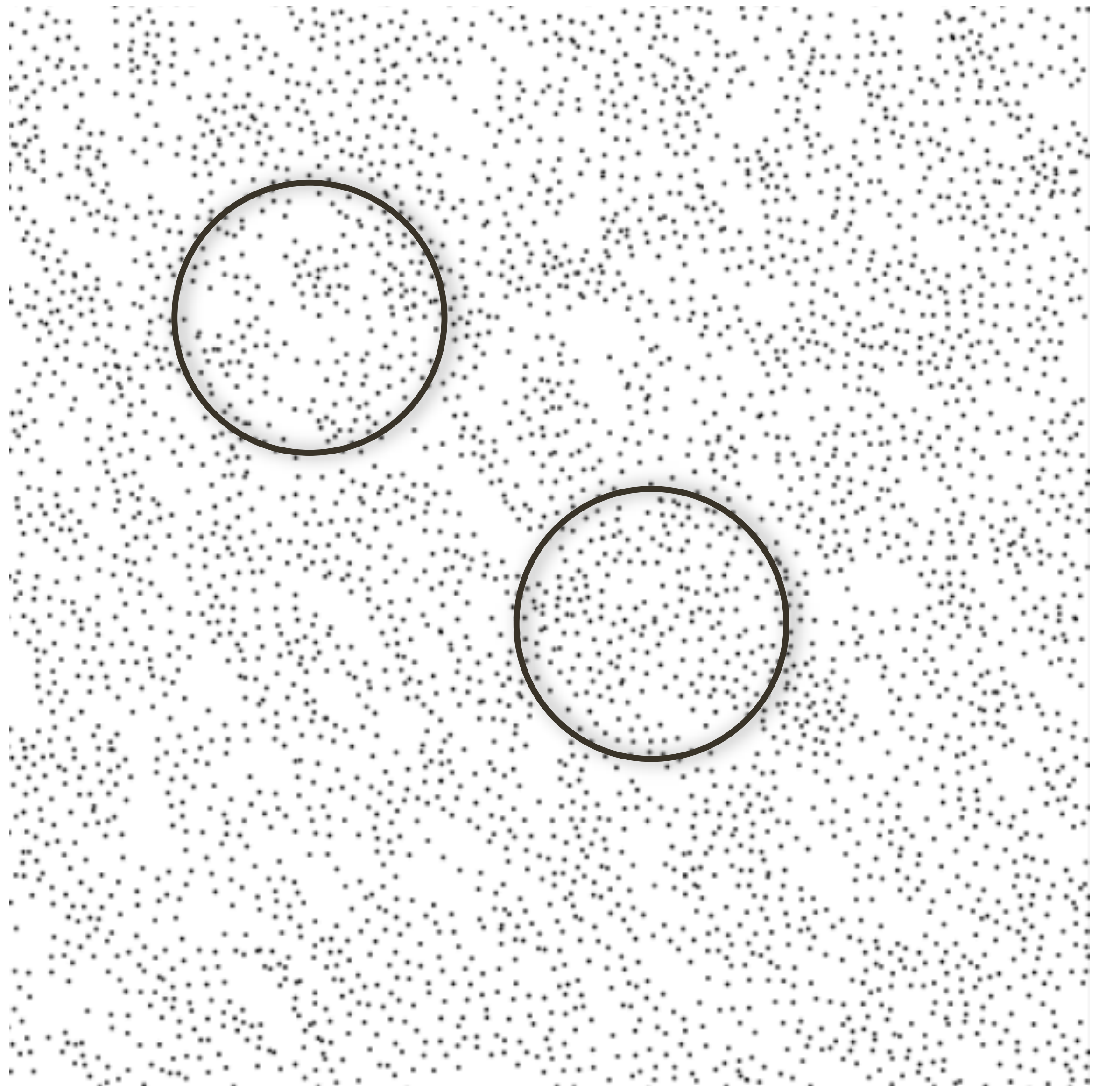
First order correlations

Variance only depends on the first and the second order correlations

Error: Variance Term

$$Var[\hat{I}] = \int_{V \times V} f(x) f(y) \frac{\varrho(x,y)}{\lambda(x)\lambda(y)} dx dy + \int_V \frac{f(x)^2}{\lambda(x)} dx \; - \; I^2$$

Stationary Point Processes



Stationary
(translation invariant)

$\lambda(x) = \lambda$ is a constant

$$\varrho(x, y) = \lambda^2 g(x - y)$$

Variance for Stationary Point Processes

$$\lambda(x) = \lambda$$

$$Var[\hat{I}] = \int_{V\times V} f(x)f(y)\frac{\varrho(x,y)}{\lambda(x)\lambda(y)}dxdy + \int_V \frac{f(x)^2}{\lambda(x)}dx \; - \; I^2$$

Variance for Stationary Point Processes

$$\lambda(x) = \lambda$$

$$Var[\hat{I}] = \int_{V \times V} f(x) f(y) \frac{\varrho(x,y)}{\lambda^2} dx dy \quad + \int_V \frac{f(x)^2}{\lambda} dx \; - \; I^2$$

Variance for Stationary Point Processes

$$\varrho(x,y) = \lambda^2 g(x-y)$$

$$Var[\hat{I}] = \frac{1}{\lambda^2}\int_{V\times V} f(x)f(y)\varrho(x,y)dxdy + \frac{1}{\lambda}\int_V f(x)^2 dx - I^2$$

Variance for Stationary Point Processes

$$\varrho(x, y) = \lambda^2 g(x - y)$$

$$Var[\hat{I}] = \frac{1}{\lambda^2} \int_{V \times V} f(x) f(y) \lambda^2 g(x - y) dx dy + \frac{1}{\lambda} \int_V f(x)^2 dx - I^2$$

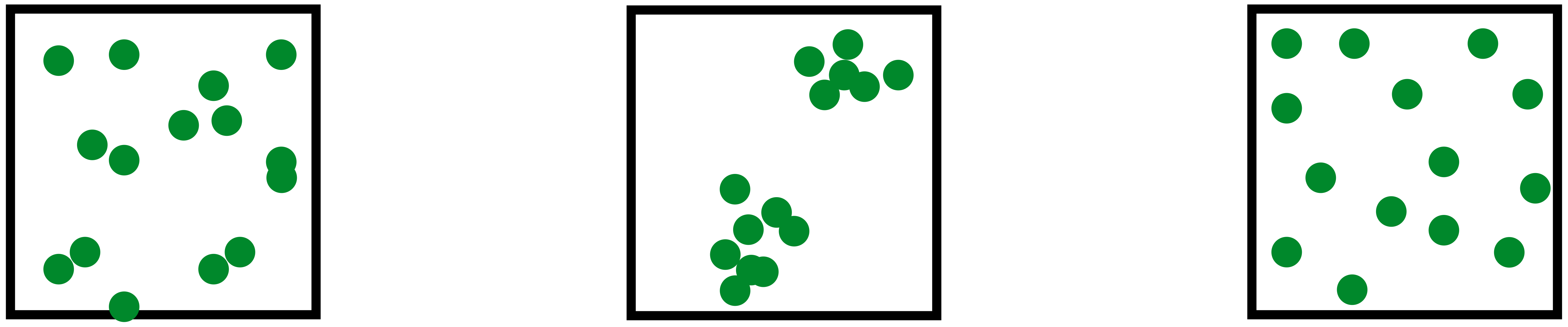
Variance for Stationary Point Processes

$$\varrho(x,y) = \lambda^2 g(x-y)$$

Arrangements **Density**

$$Var[\hat{I}] = \frac{1}{\lambda^2}\int_{V\times V} f(x)f(y)\lambda^2 g(x-y)dxdy + \frac{1}{\lambda}\int_V f(x)^2 dx - I^2$$

$g = 1$ $\quad$ $g > 1$ $\quad$ $g < 1$



Poisson Processes — Clusters — Well distributed

Variance for Stationary Point Processes

$$h = x - y$$

$$Var[\hat{I}] = \frac{1}{\lambda^2} \int_{V \times V} f(x) f(y) \lambda^2 g(x-y) dx dy + \frac{1}{\lambda} \int_V f(x)^2 dx - I^2$$

Variance for Stationary Point Processes

$$h = x - y$$

$$Var[\hat{I}] = \underbrace{\frac{1}{\lambda^2}\int_{V\times V} f(x)f(x-h)\lambda^2 g(h)\,dx\,dh + \frac{1}{\lambda}\int_V f(x)^2 dx} - \underbrace{I^2}$$

Variance for Stationary Point Processes

$$h = x - y$$

$$Var[\hat{I}] = \frac{1}{\lambda}\int_V f(x)^2 dx + \frac{1}{\lambda^2}\int_{V\times V} f(x)f(x-h)\lambda^2 g(h)dxdh \quad - \quad I^2$$

Variance for Stationary Point Processes

$$h = x - y$$

$$Var[\hat{I}] = \frac{1}{\lambda}\int_V f(x)^2 dx \quad + \quad \int_V \int_V f(x) f(x-h) g(h) dx dh \quad - \quad I^2$$

Autocorrelation: $a_f(h) = \int f(x) f(x-h) dh$

Variance for Stationary Point Processes

$$h = x - y$$

$$Var[\hat{I}] = \underline{\frac{1}{\lambda}\int_V f(x)^2 dx \quad + \quad \frac{1}{\lambda^2}\int_V a_f(h) g(h) dh} \qquad - \ \underline{I^2}$$

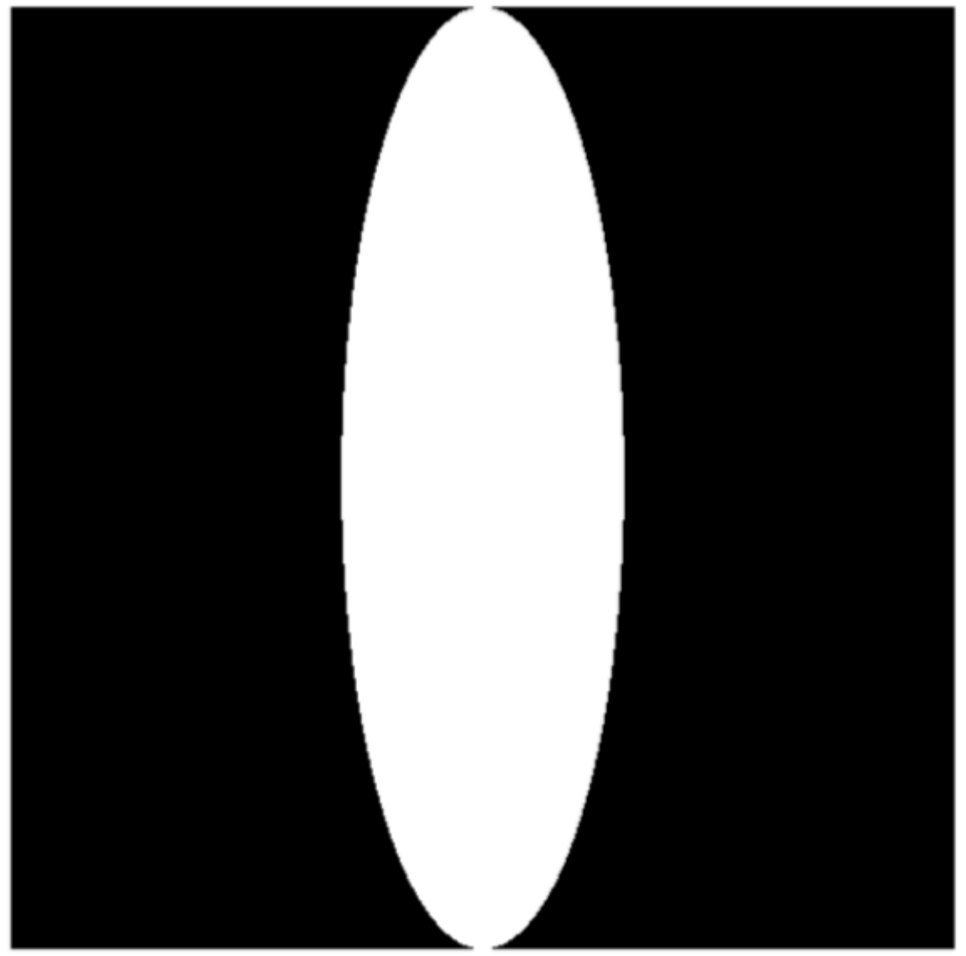
Variance for Stationary Point Processes

$$Var[\hat{I}] = \frac{1}{\lambda}\int_V f(x)^2 dx \ + \ \frac{1}{\lambda^2}\int_V a_f(h)g(h)dh \ - \ I^2$$

Oztireli [2016]

Variance for Stationary Point Processes

$$Var[\hat{I}] = \frac{1}{\lambda}\int_V f(x)^2 dx \;+\; \frac{1}{\lambda^2}\int_V a_f(h) g(h) dh \;-\; I^2$$



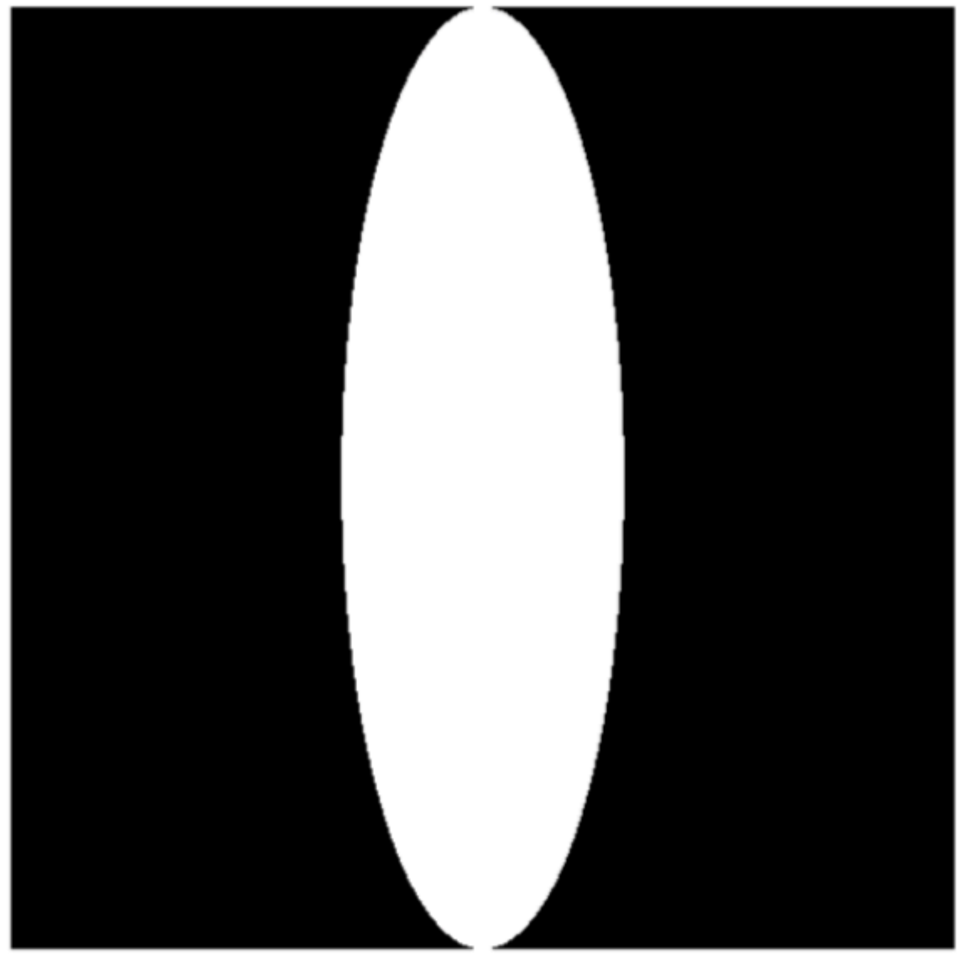
$f(x, y)$

Oztireli [2016]

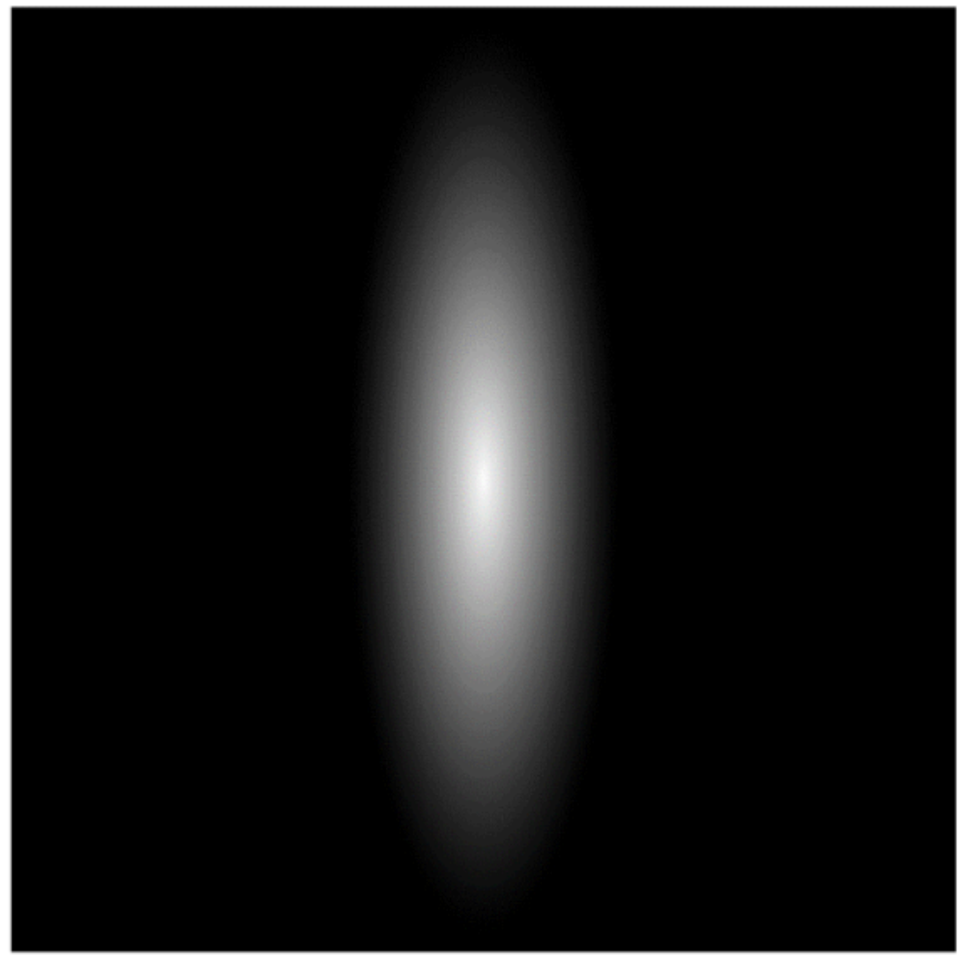
Variance for Stationary Point Processes

$$Var[\hat{I}] = \frac{1}{\lambda}\int_V f(x)^2 dx \;+\; \frac{1}{\lambda^2}\int_V a_f(h) g(h) dh \;-\; I^2$$

Autocorrelation



$f(x, y)$

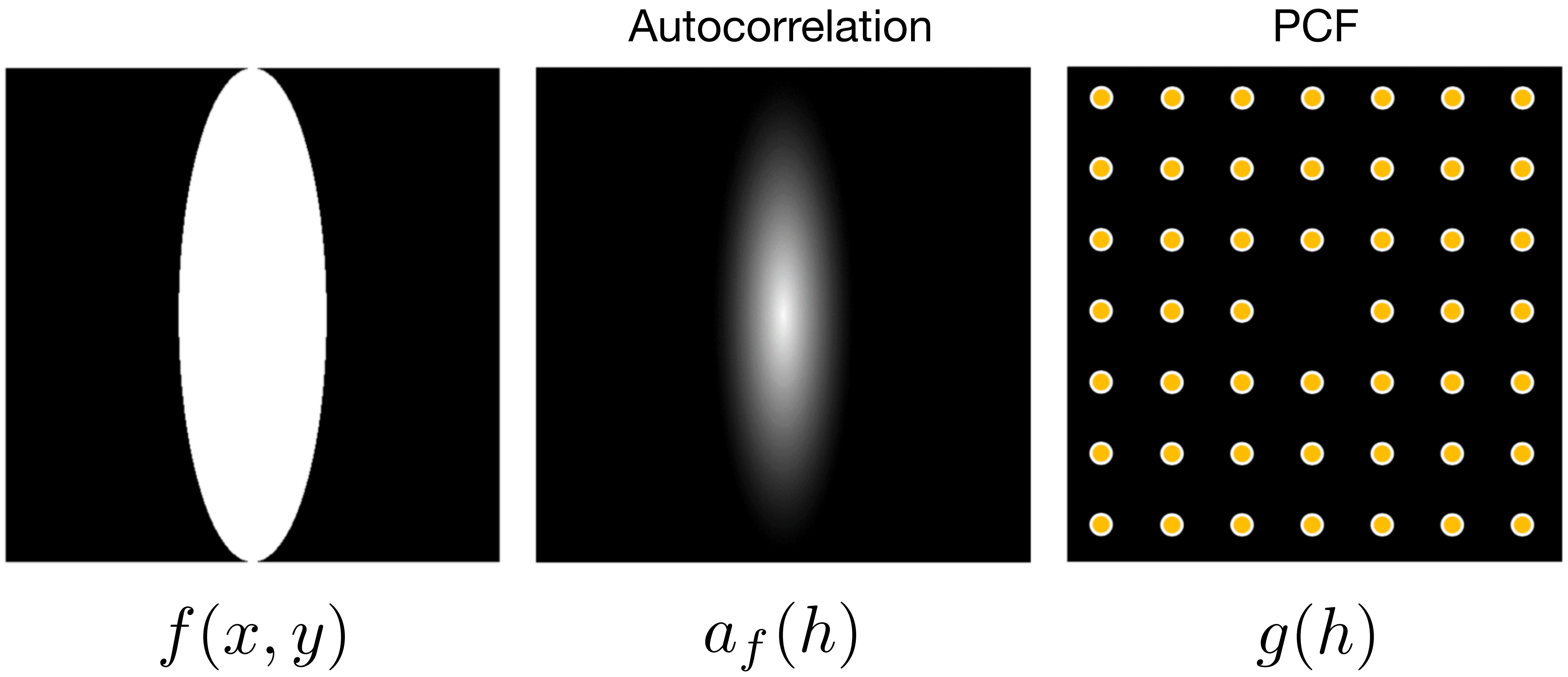


$a_f(h)$

Oztireli [2016]

Variance for Stationary Point Processes

$$Var[\hat{I}] = \frac{1}{\lambda}\int_V f(x)^2 dx \;+\; \frac{1}{\lambda^2}\int_V a_f(h)g(h)dh \;-\; I^2$$



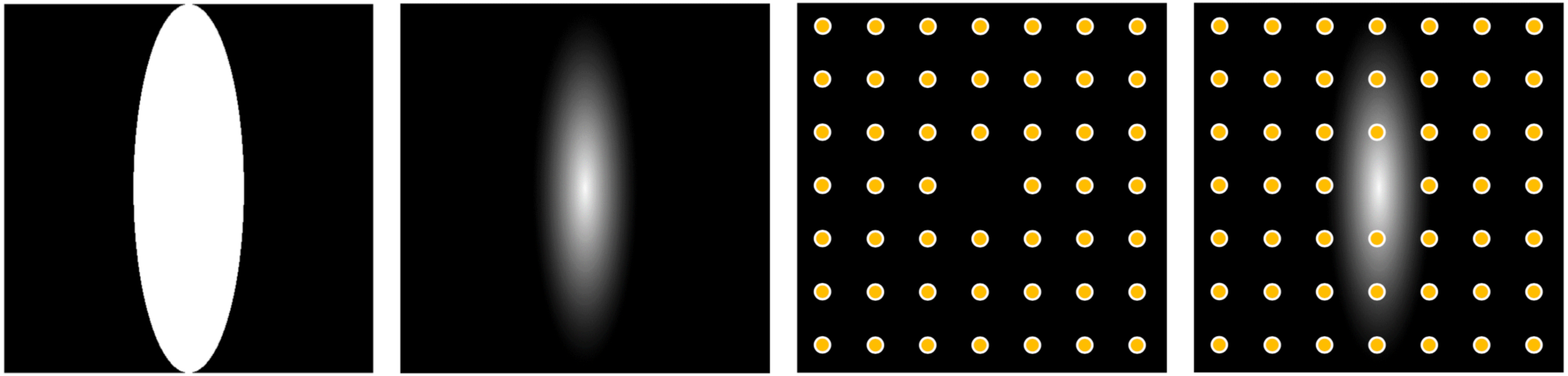
Oztireli [2016]

Variance for Stationary Point Processes

$$Var[\hat{I}] = \frac{1}{\lambda}\int_V f(x)^2 dx \;+\; \frac{1}{\lambda^2}\int_V a_f(h) g(h) dh \;-\; I^2$$

Autocorrelation

PCF

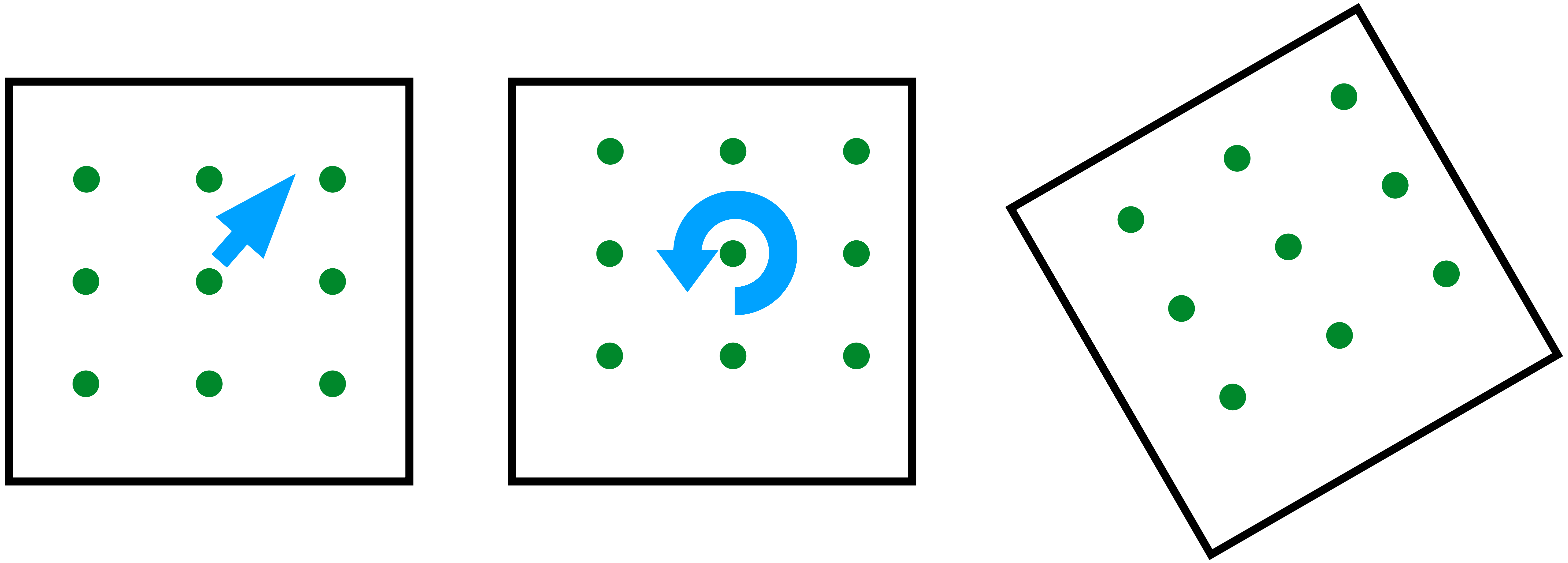


$f(x,y)$ $a_f(h)$ $g(h)$ $a_f(h)g(h)$

Oztireli [2016]

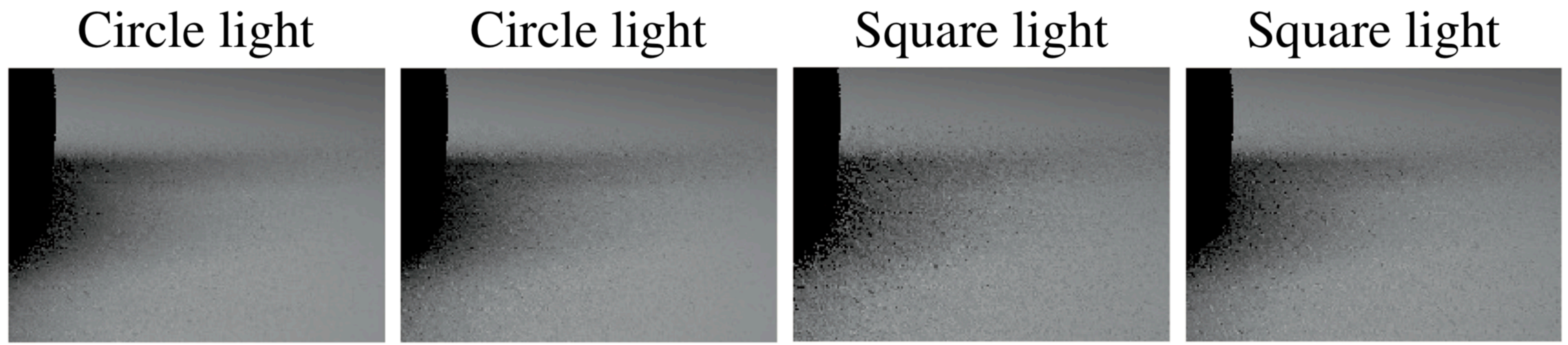
Uniform and Isotropic Jittered Samples

Regular grid   Uniform Jitter   Isotropic Jitter



Rammamoorthi et al.[2012]   **Oztireli 2016**

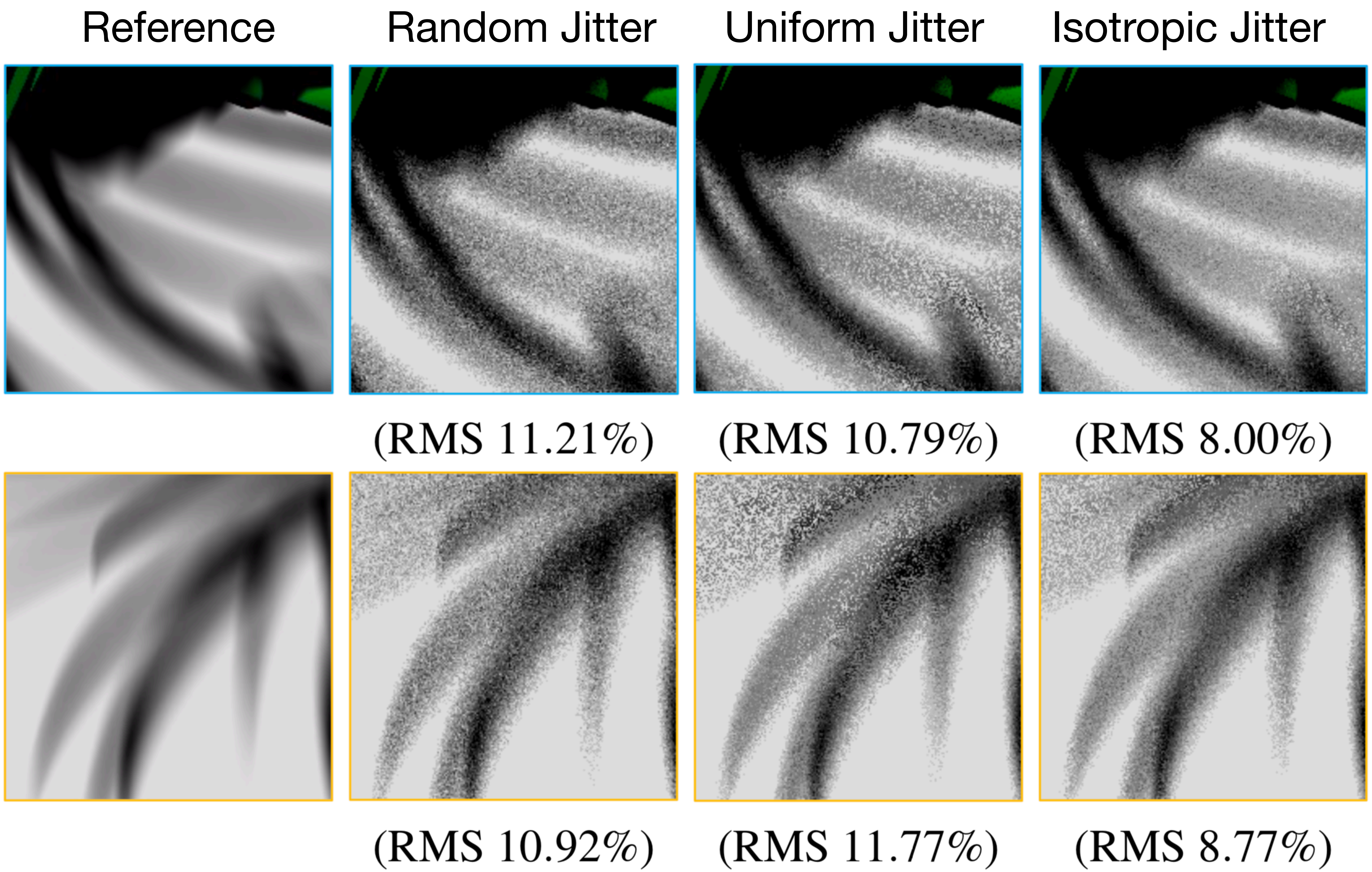
Uniform and Isotropic Jittered Samples



(a) Uniform jitter (RMS 6.59%) (b) Random jitter (RMS 8.32%) (c) Uniform jitter (RMS 13.4%) (d) Random jitter (RMS 10.4%)

Rammamoorthi et al.[2012]

Uniform and Isotropic Jittered Samples



Oztireli [2016]

- Error Formulation in the Spatial Domain
- **Error Formulation in the Fourier Domain**
- Practical Results
- Conclusion: Design Principles

Variance for Stationary Point Processes

$$Var[\hat{I}] = \frac{1}{\lambda}\int_V f(x)^2 dx \ - \ \frac{1}{\lambda^2}\int_V a_f(h)g(h)dh \ - \ I^2$$

Relation between the Spatial and Fourier Statistics

$$\mathcal{F}(a_f)(\nu) = \mathcal{P}_f(\nu)$$

Power spectrum

$$\mathbb{E}\langle \mathcal{P}_{S_N}(\nu) \rangle = \lambda G(\nu) + 1$$

Variance for Stationary Point Processes

Spatial Formulation

$$Var[\hat{I}] = \frac{1}{\lambda}\int_V f(x)^2 dx \; - \; \frac{1}{\lambda^2}\int_V a_f(h)g(h)dh \; - \; I^2$$

Fourier Formulation

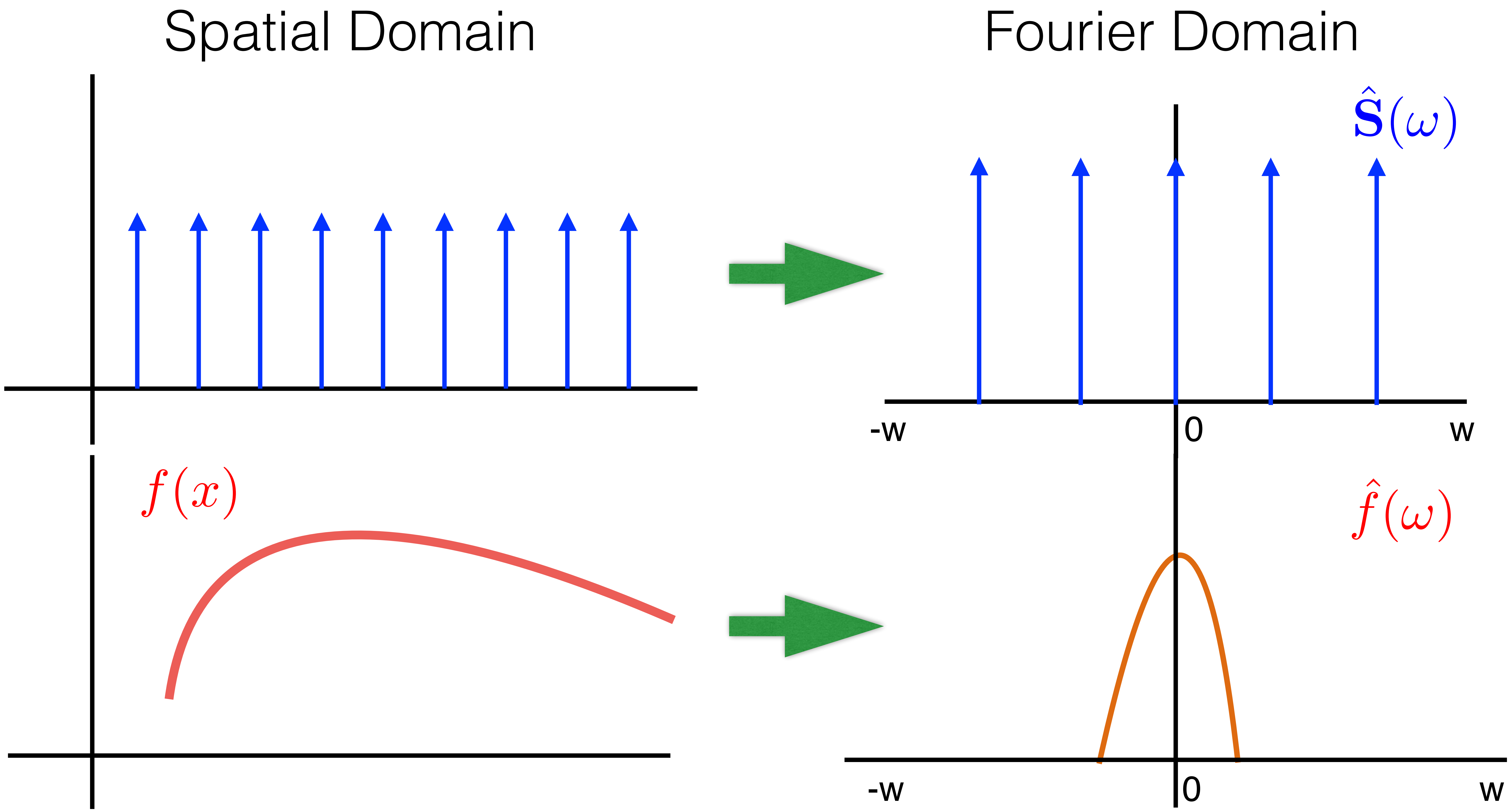
$$\mathrm{Var}[\hat{I}] = \int_\Omega \mathbb{E}\langle \mathcal{P}_{S_N}(\nu)\rangle \mathcal{P}_f(\nu)d\nu - \mathcal{P}_f(0)$$

$$\mathcal{F}(a_f)(\nu) = \mathcal{P}_f(\nu)$$

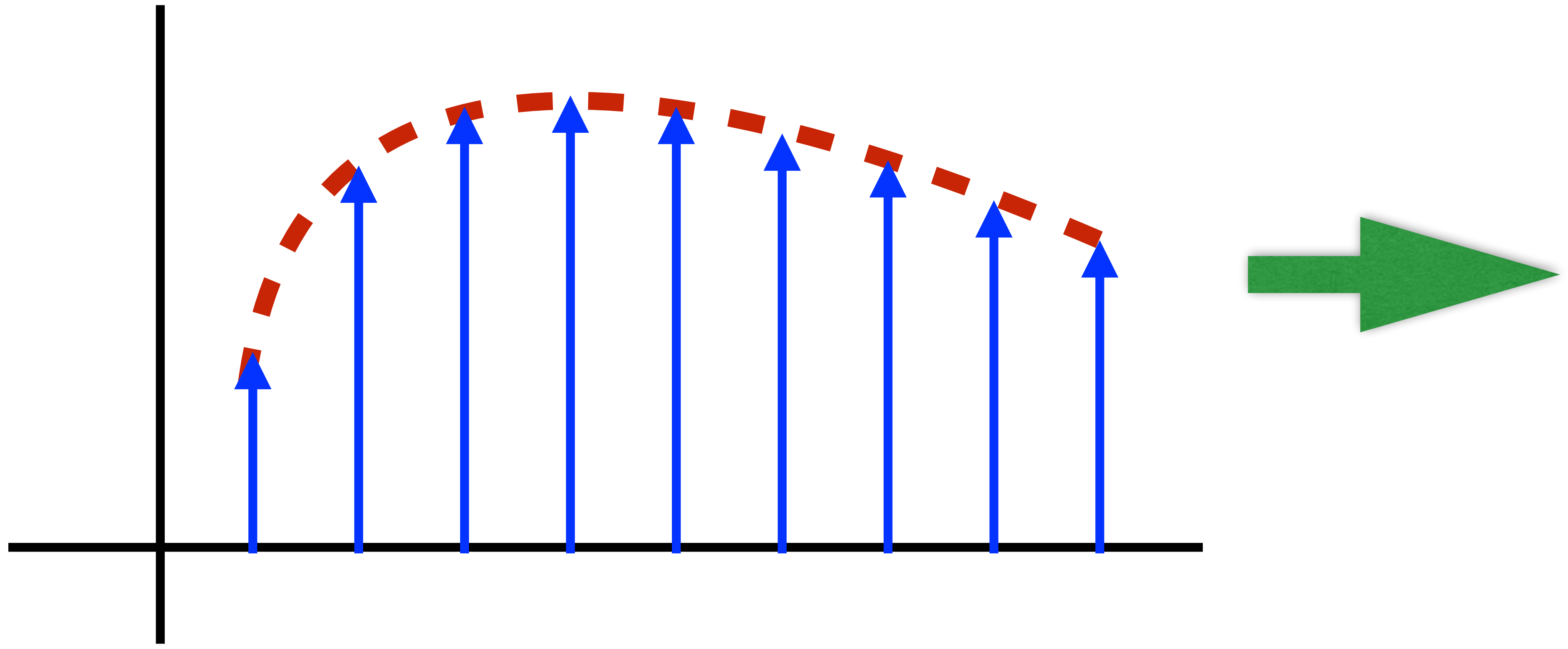
Power spectrum

$$\mathbb{E}\langle \mathcal{P}_{S_N}(\nu)\rangle = \lambda G(\nu) + 1$$

Samples and function in Fourier Domain



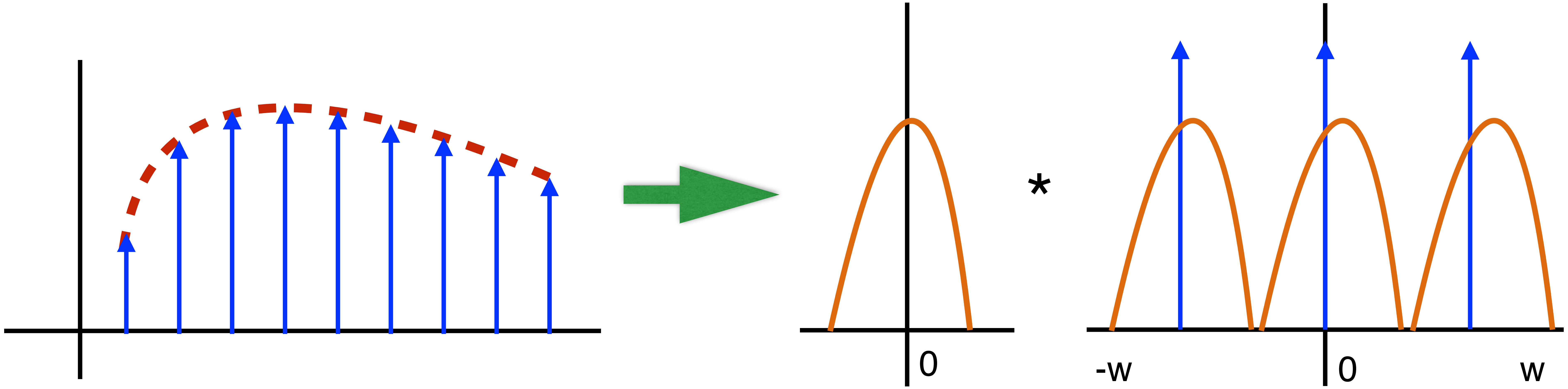
Sampling in Primal Domain is Convolution in Fourier Domain



$f(x)\,\mathbf{S}(x)$

Fredo Durand [2011]

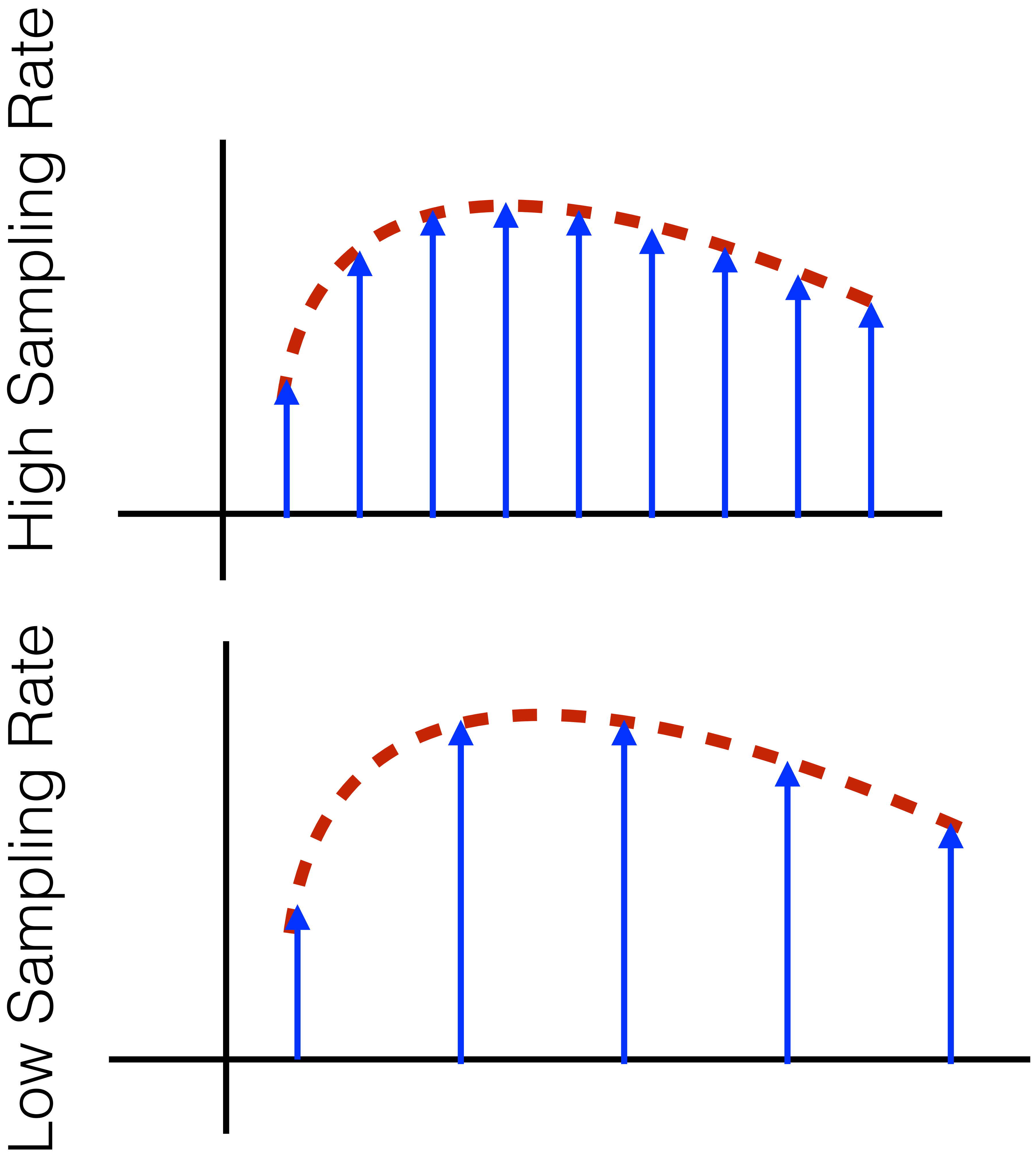
Sampling in Primal Domain is Convolution in Fourier Domain

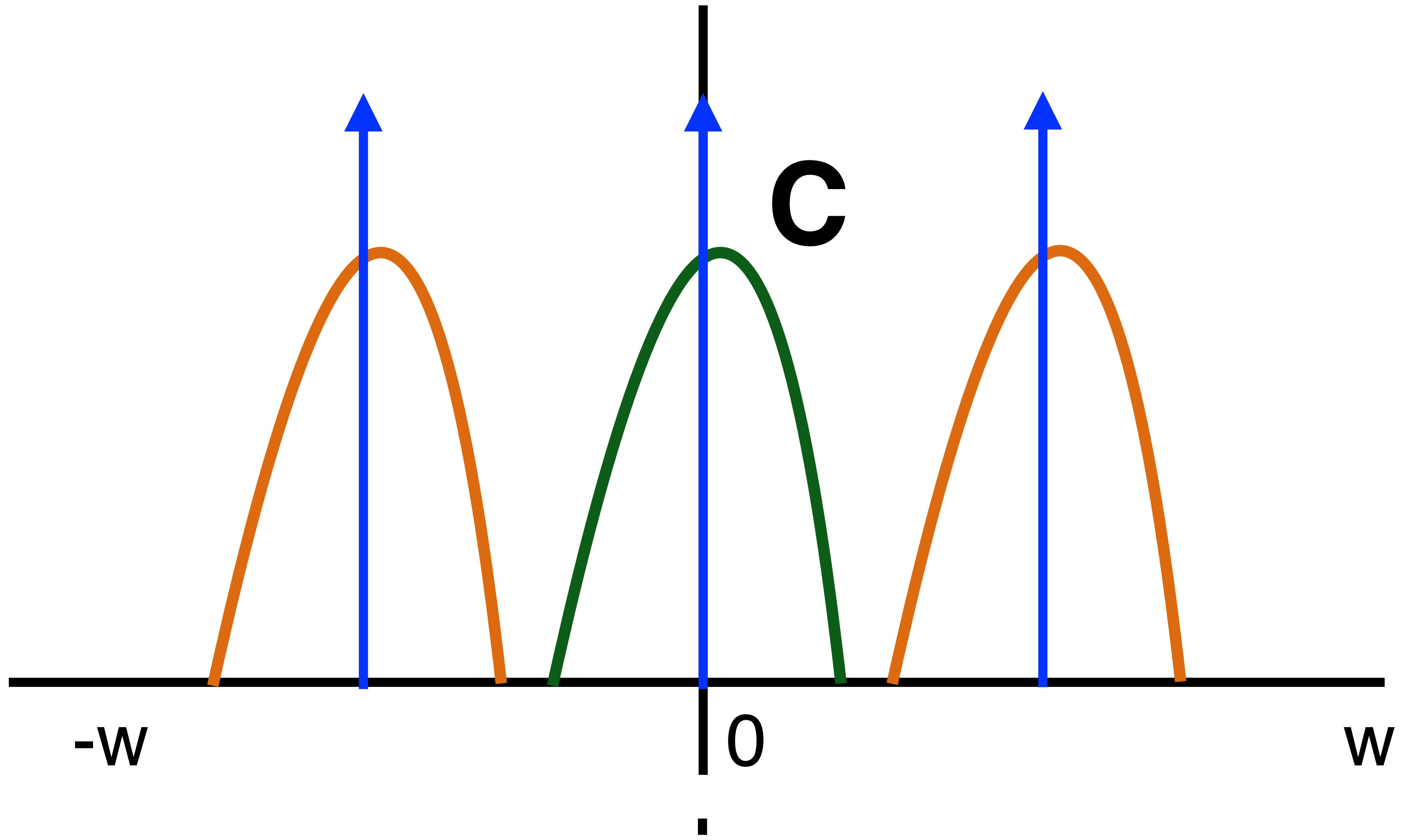


$f(x)\,\mathbf{S}(x)$ $\hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$

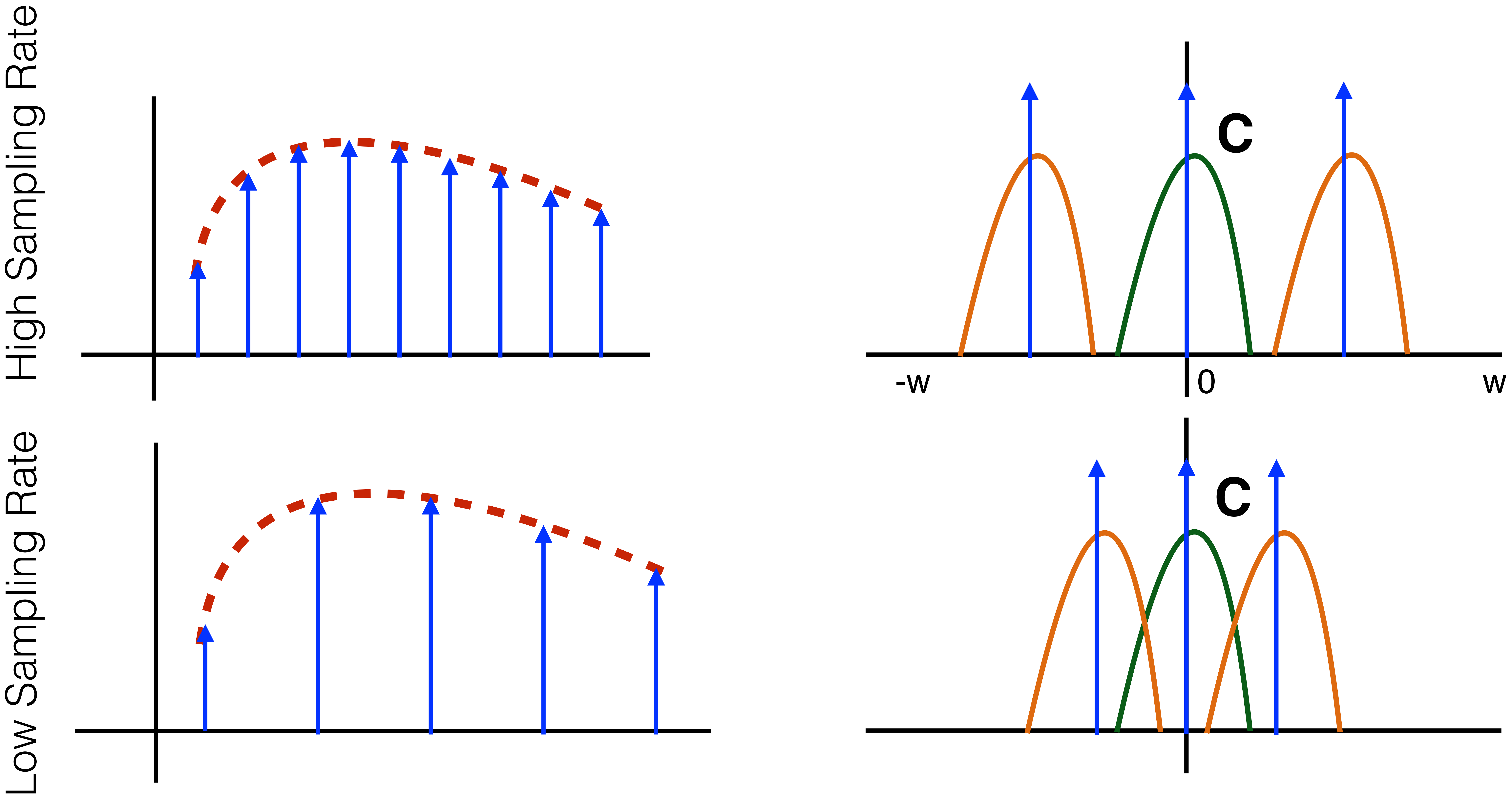
Fredo Durand [2011]

Aliasing in Reconstruction

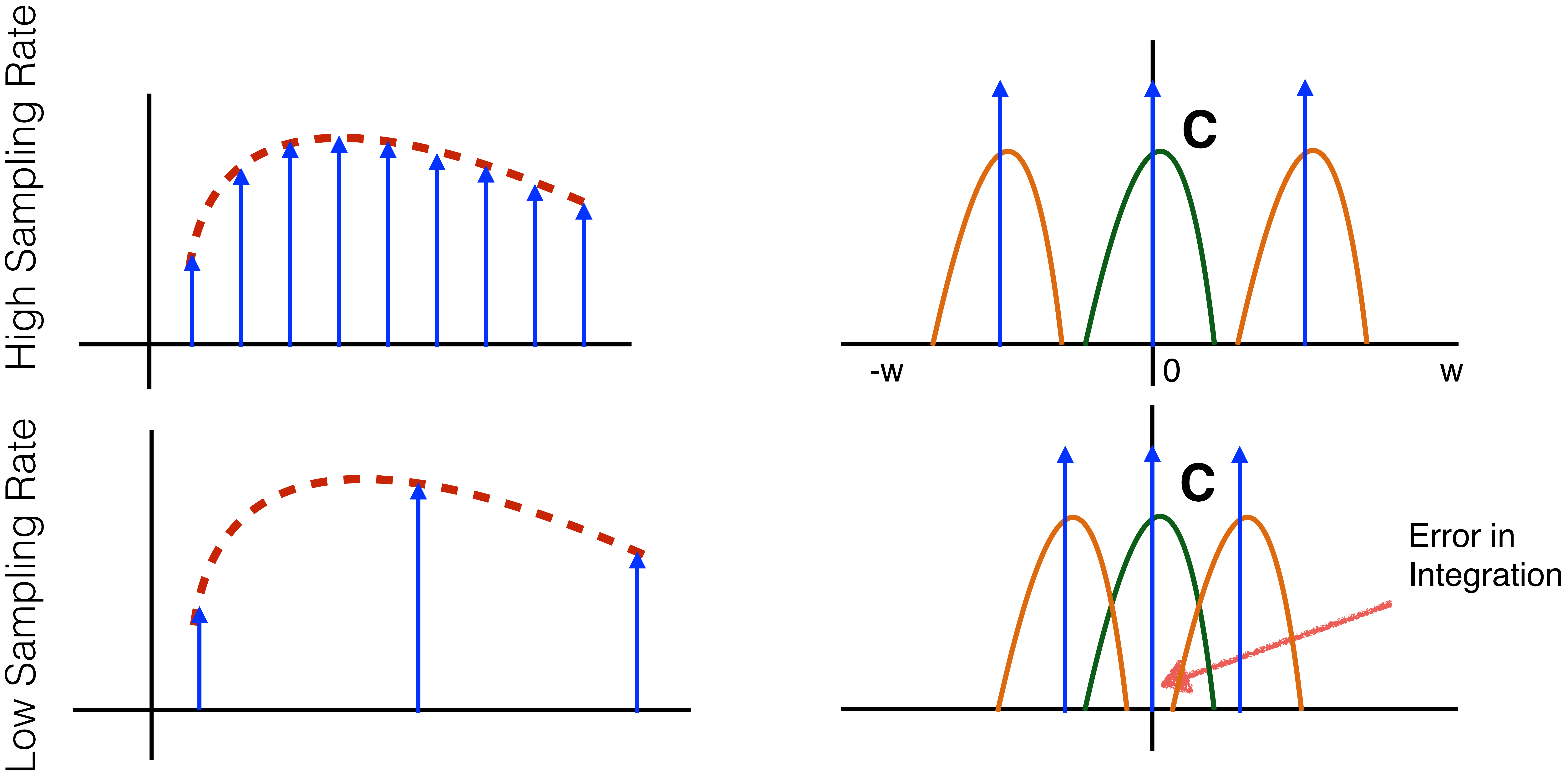




Aliasing in Reconstruction

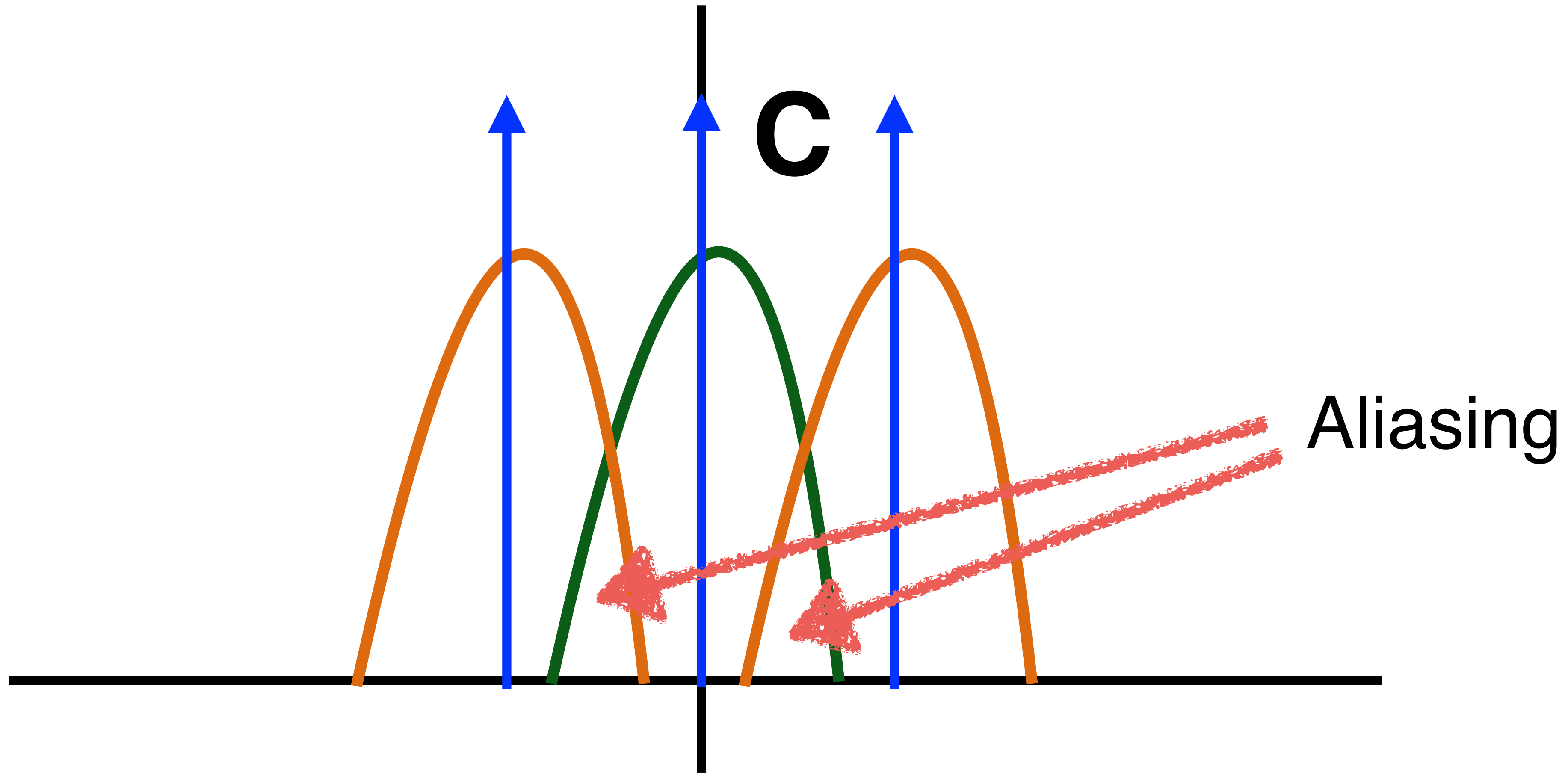


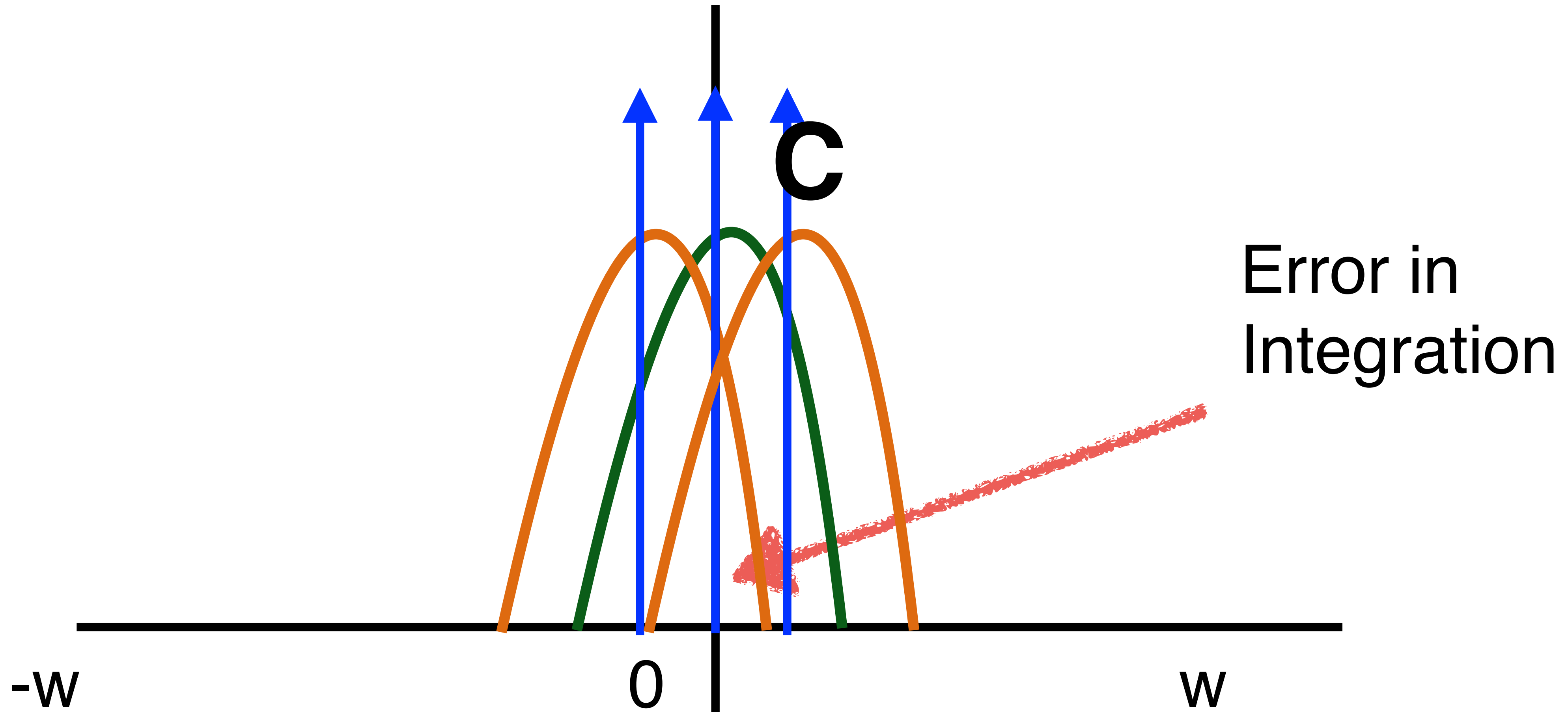
Error in Monte Carlo Integration



Aliasing (Reconstruction) vs. Error (Integration)

Fredo Durand [2011]
Belcour et al. [2013]

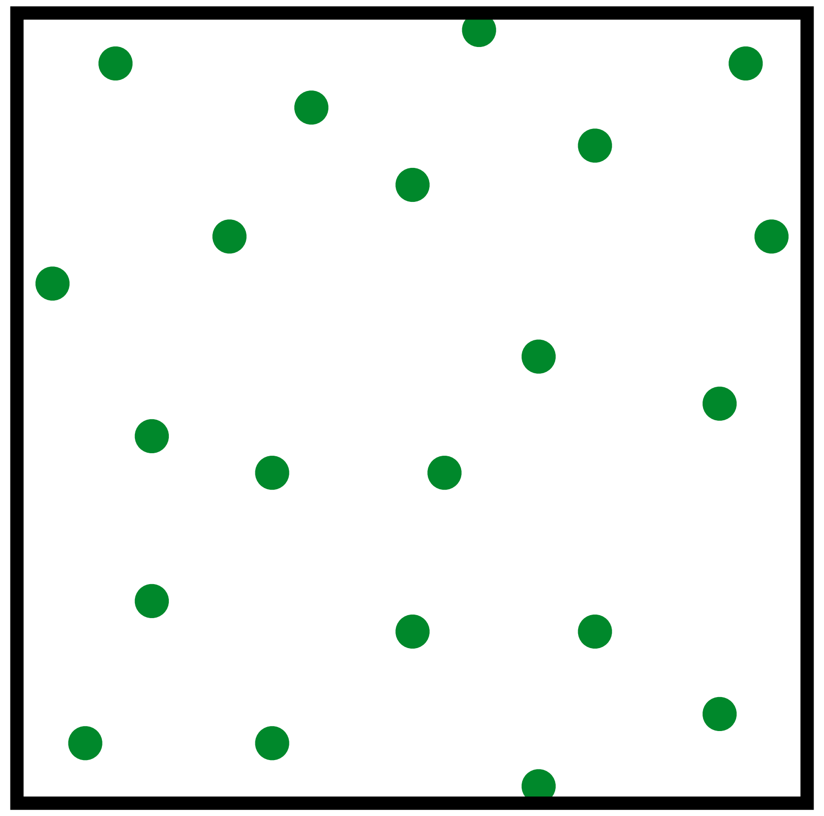




Monte Carlo Estimator

$$\hat{I} = \frac{1}{N}\sum_{k=1}^{N} f(\vec{x}_k) = \int_0^1 \frac{1}{N}\sum_{k=1}^{N} \delta(\vec{x} - \vec{x}_k)\, f(\vec{x})\, d\vec{x} = \int_0^1 S_N(\vec{x})\, f(\vec{x})\, d\vec{x}$$

$$S_N(\vec{x}) = \frac{1}{N}\sum_{k=1}^{N} \delta(\vec{x} - \vec{x}_k)$$



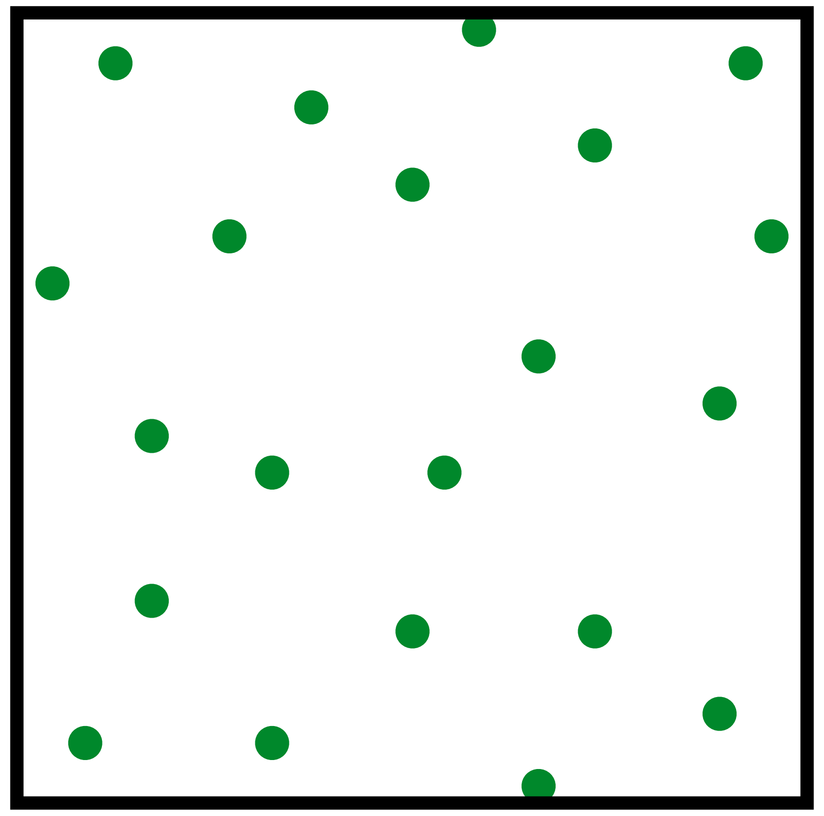
Fredo Durand [2011]

Samples Power Spectrum

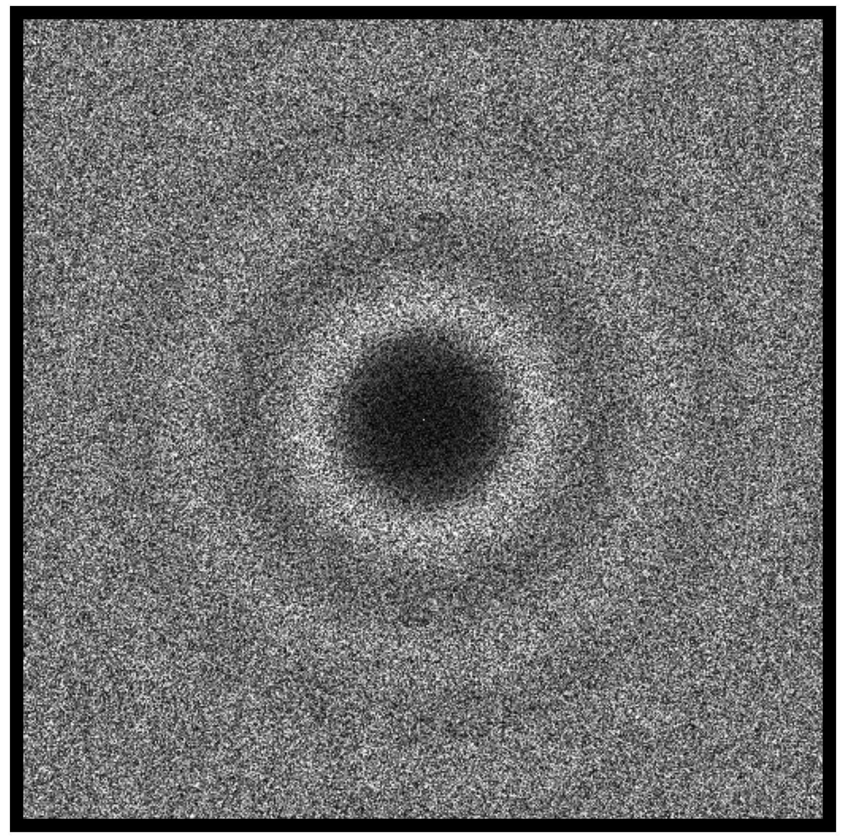
$$\hat{I} = \int_0^1 S_N(\vec{x}) f(\vec{x})\, d\vec{x}$$

Samples

Poisson Disk



Spectrum



$$S_N(\vec{x}) = \frac{1}{N} \sum_{k=1}^{N} \delta(\vec{x} - \vec{x}_k)$$

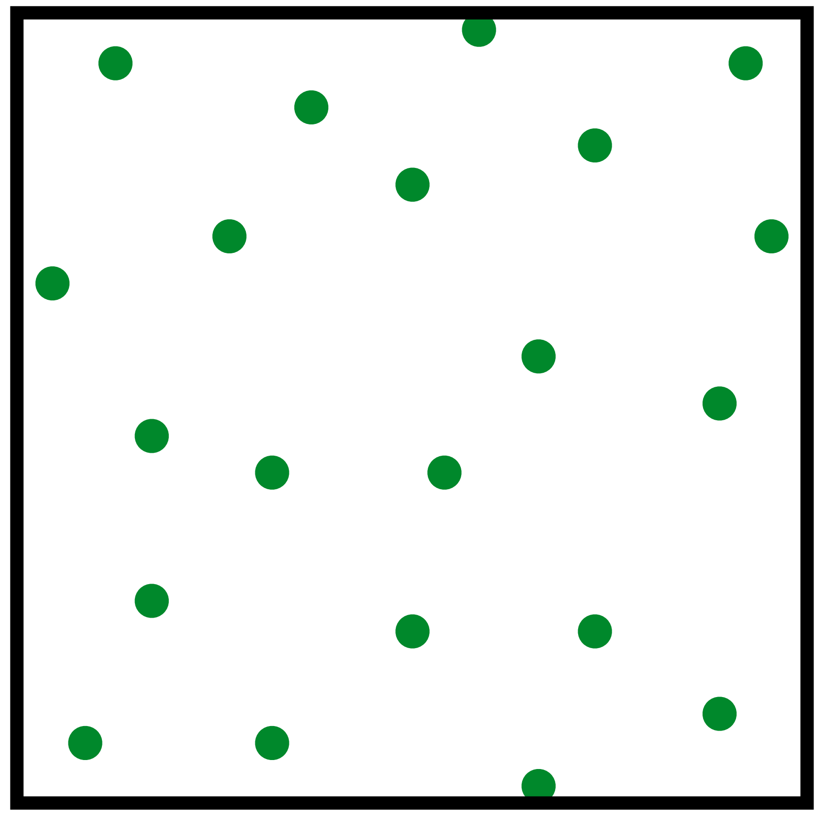
$$\mathcal{P}_{S_N}(\nu) = \left| \frac{1}{N} \sum_{k=1}^{N} e^{-i2\pi\nu\cdot\vec{x}_k} \right|^2$$

Samples Power Spectrum

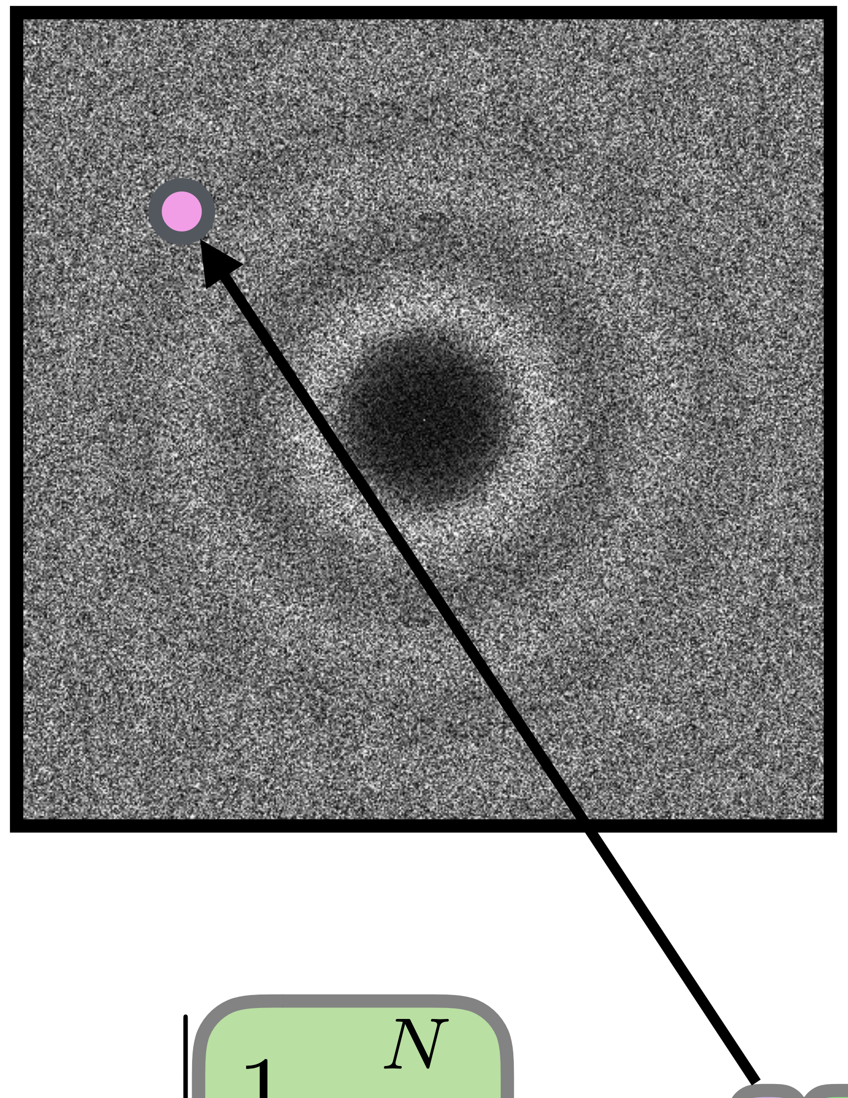
$$\hat{I} = \int_0^1 S_N(\vec{x}) f(\vec{x})\, d\vec{x}$$

Samples

Poisson Disk



Spectrum

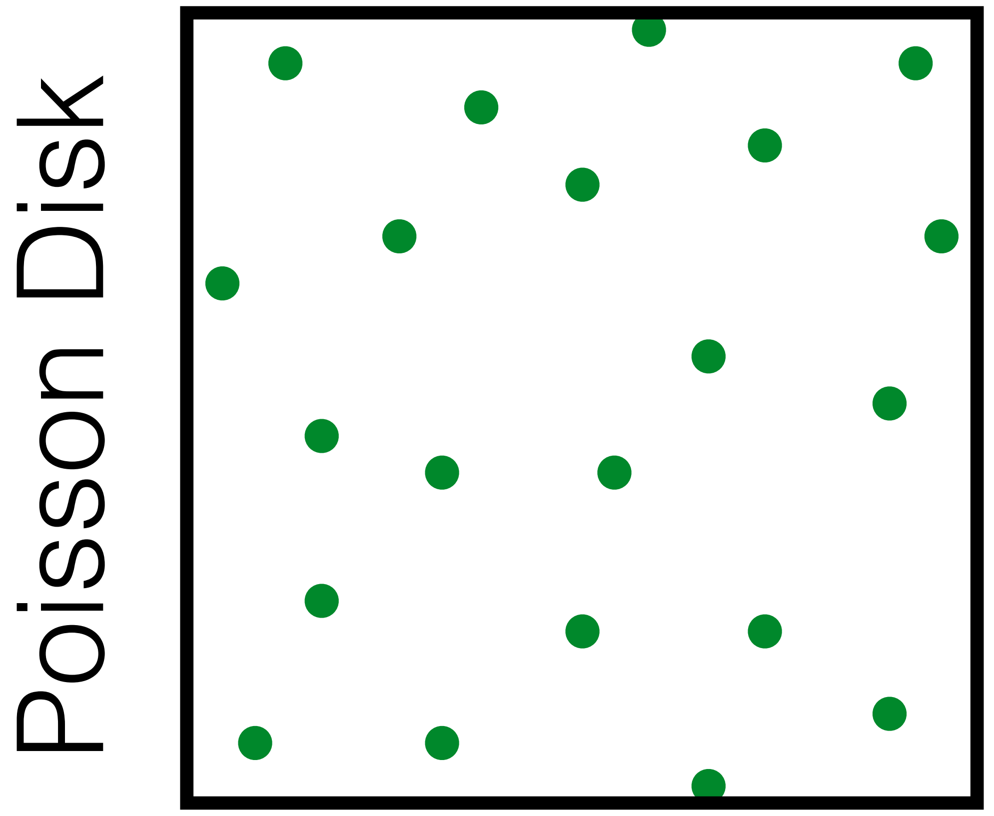


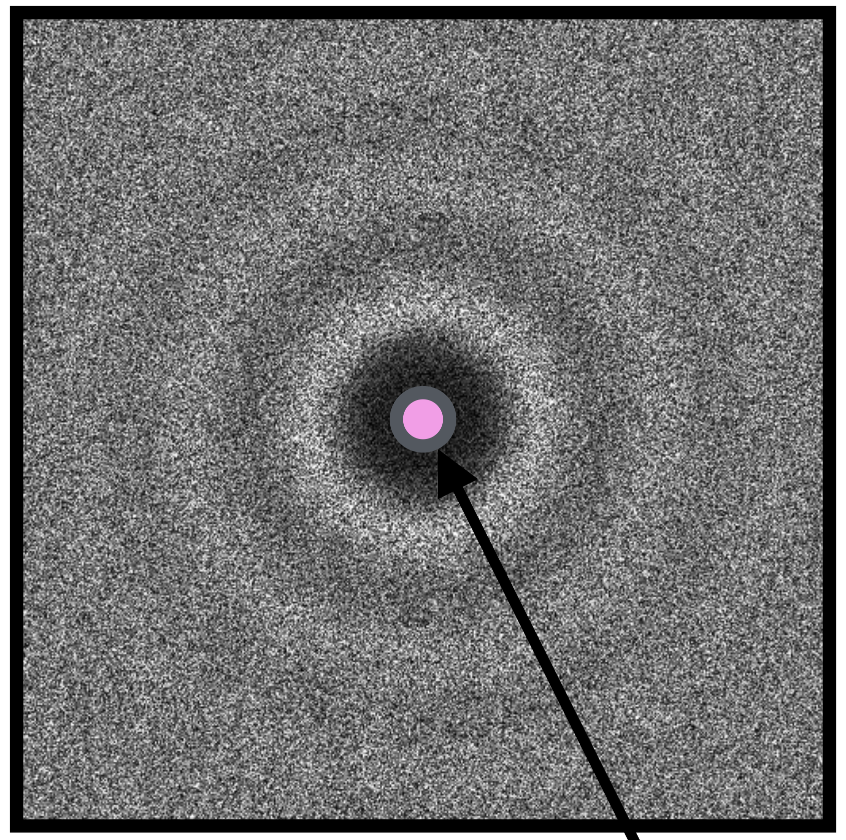
$$S_N(\vec{x}) = \frac{1}{N}\sum_{k=1}^{N} \delta(\vec{x} - \vec{x}_k)$$

$$\mathcal{P}_{S_N}(\nu) = \left| \frac{1}{N}\sum_{k=1}^{N} e^{-i2\pi\nu\cdot\vec{x}_k} \right|^2$$

Samples Power Spectrum

$$\hat{I} = \int_0^1 S_N(\vec{x}) f(\vec{x})\, d\vec{x}$$





$$S_N(\vec{x}) = \frac{1}{N}\sum_{k=1}^{N} \delta(\vec{x} - \vec{x}_k)$$

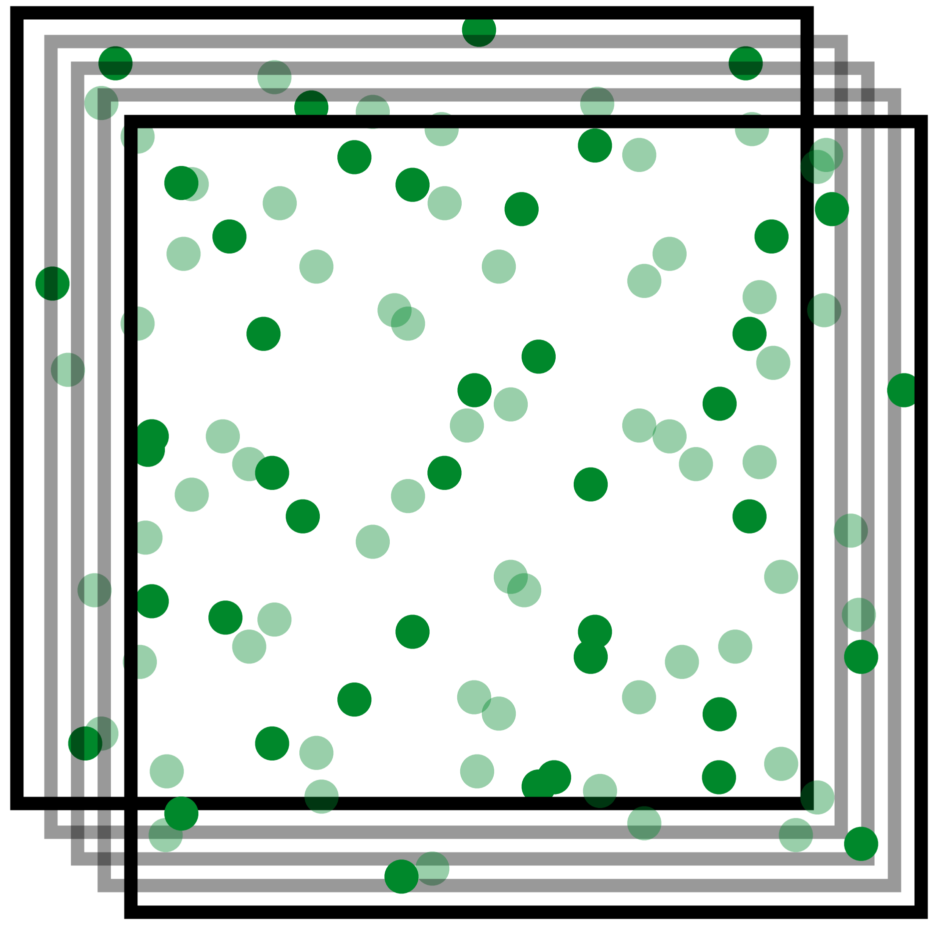
$$\mathcal{P}_{S_N}(\nu) = \left| \frac{1}{N} \sum_{k=1}^{N} e^{-i2\pi\nu\cdot\vec{x}_k} \right|^2$$

Expected Sampling Power Spectra

$$\hat{I} = \int_0^1 S_N(\vec{x}) f(\vec{x})\, d\vec{x}$$

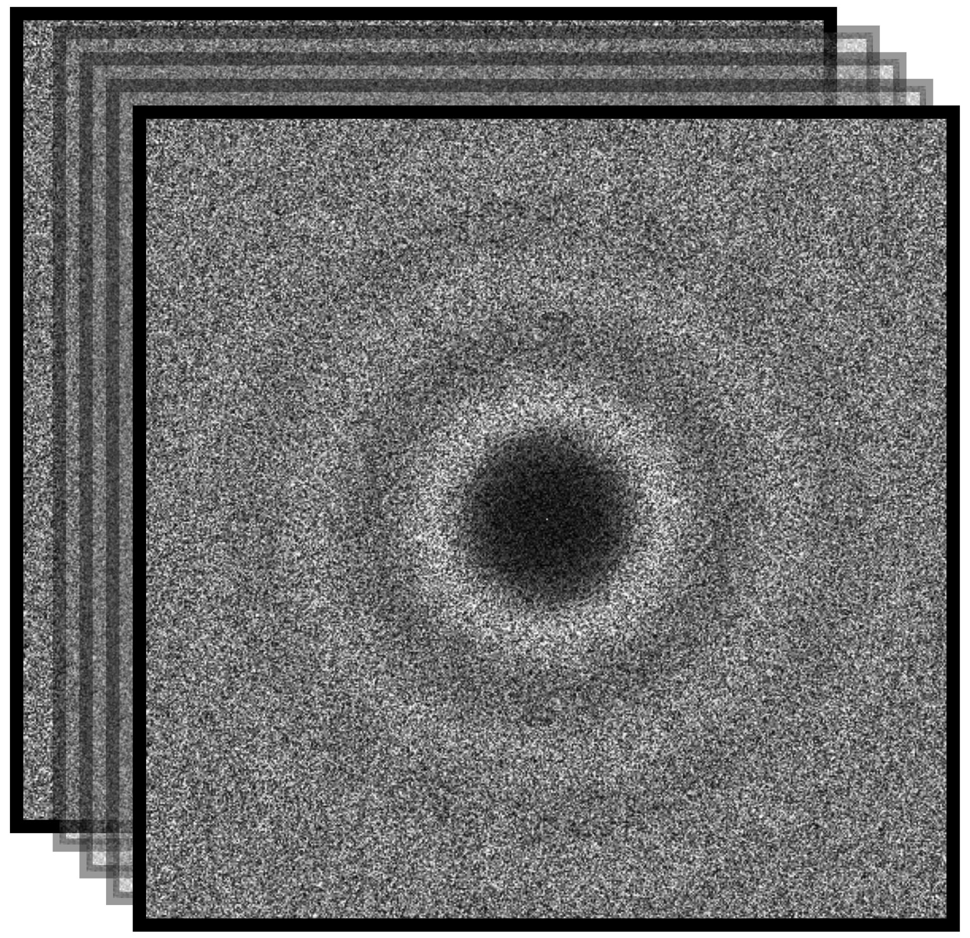
Samples

Poisson Disk



$$S_N(\vec{x}) = \frac{1}{N}\sum_{k=1}^{N} \delta(\vec{x}-\vec{x}_k)$$

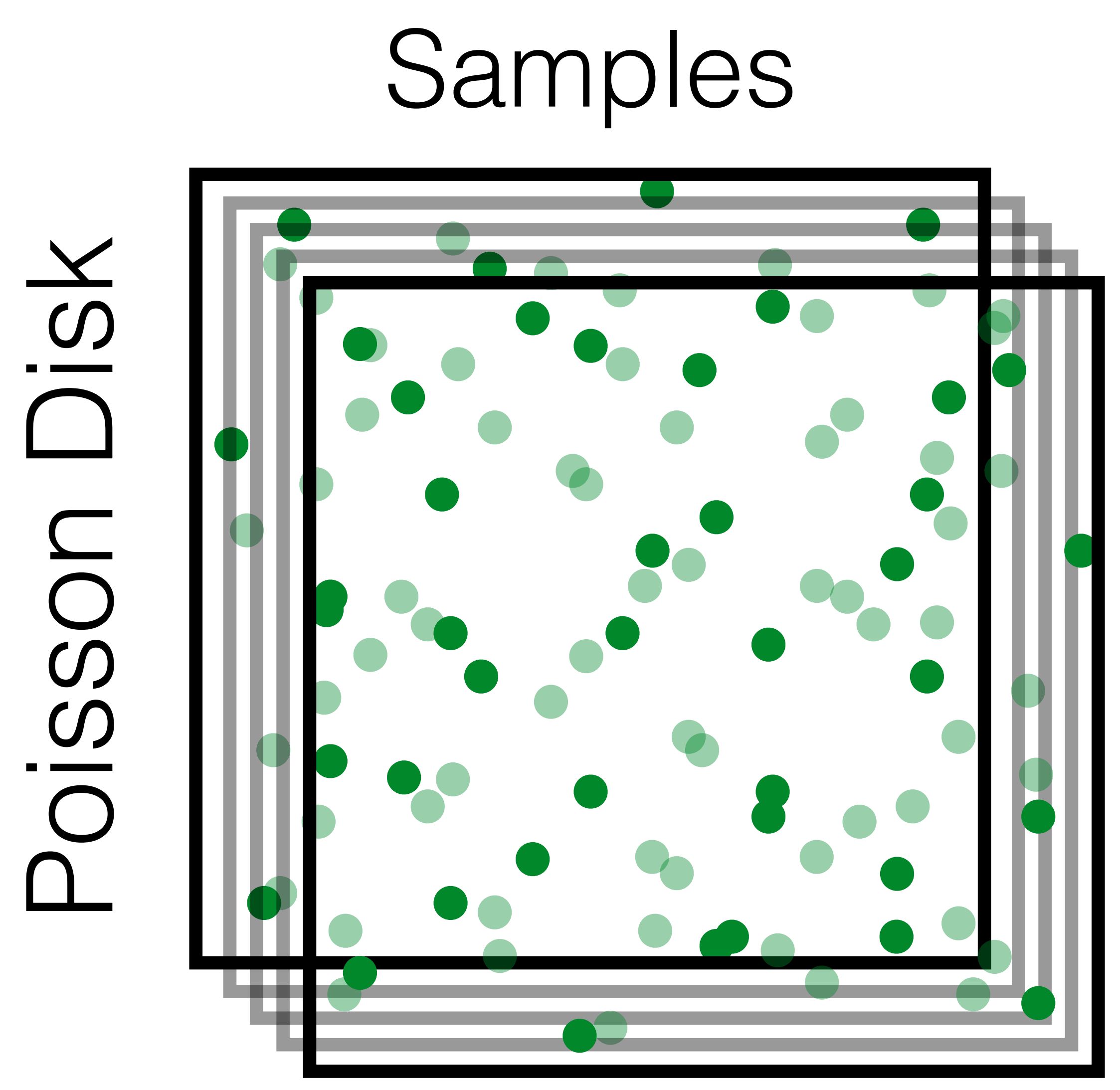
Spectrum

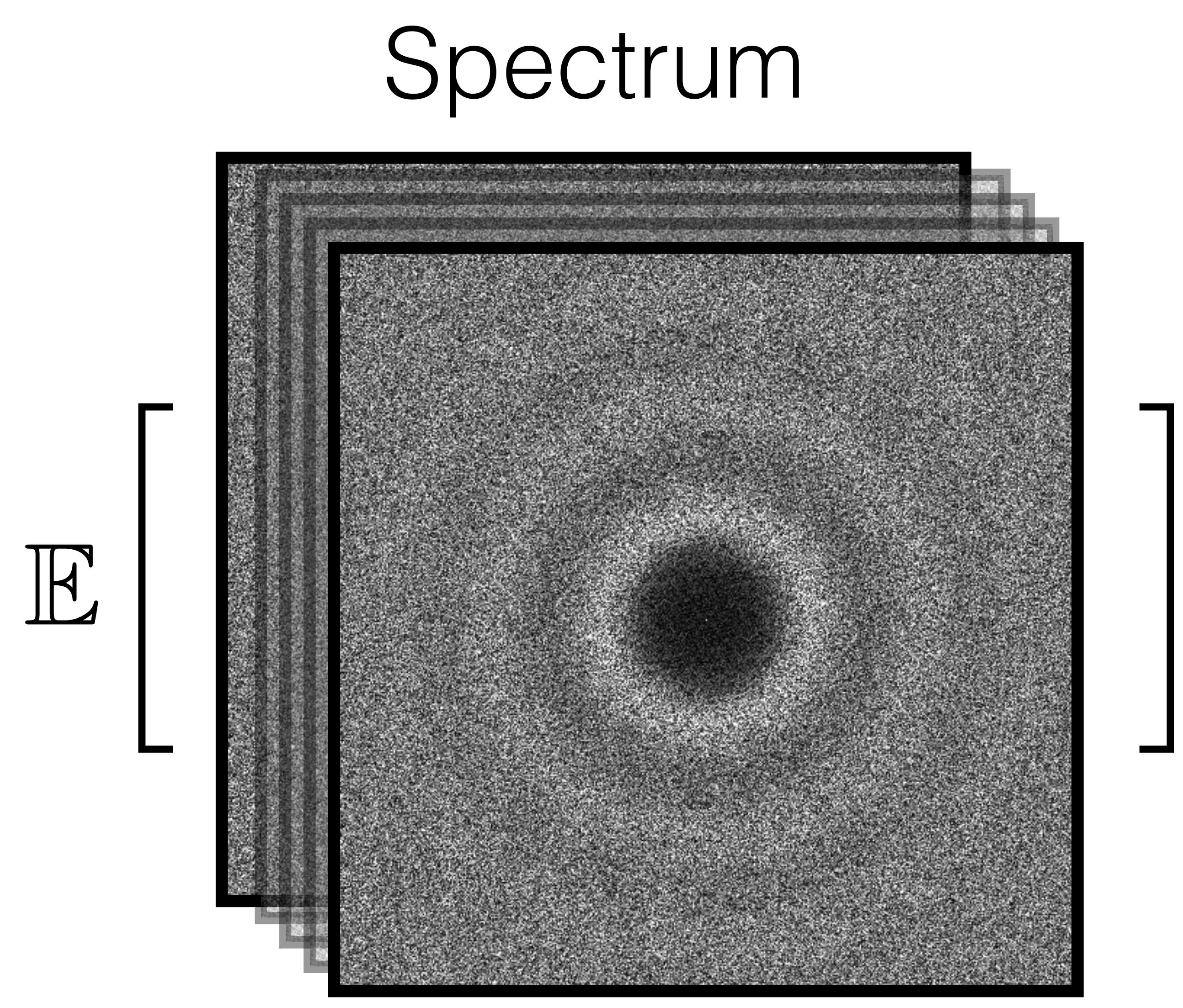


$$\mathcal{P}_{S_N}(\nu) = \left|\frac{1}{N}\sum_{k=1}^{N} e^{-i2\pi\nu\cdot\vec{x}_k}\right|^2$$

Expected Sampling Power Spectra

$$\hat{I} = \int_0^1 S_N(\vec{x}) f(\vec{x})\, d\vec{x}$$





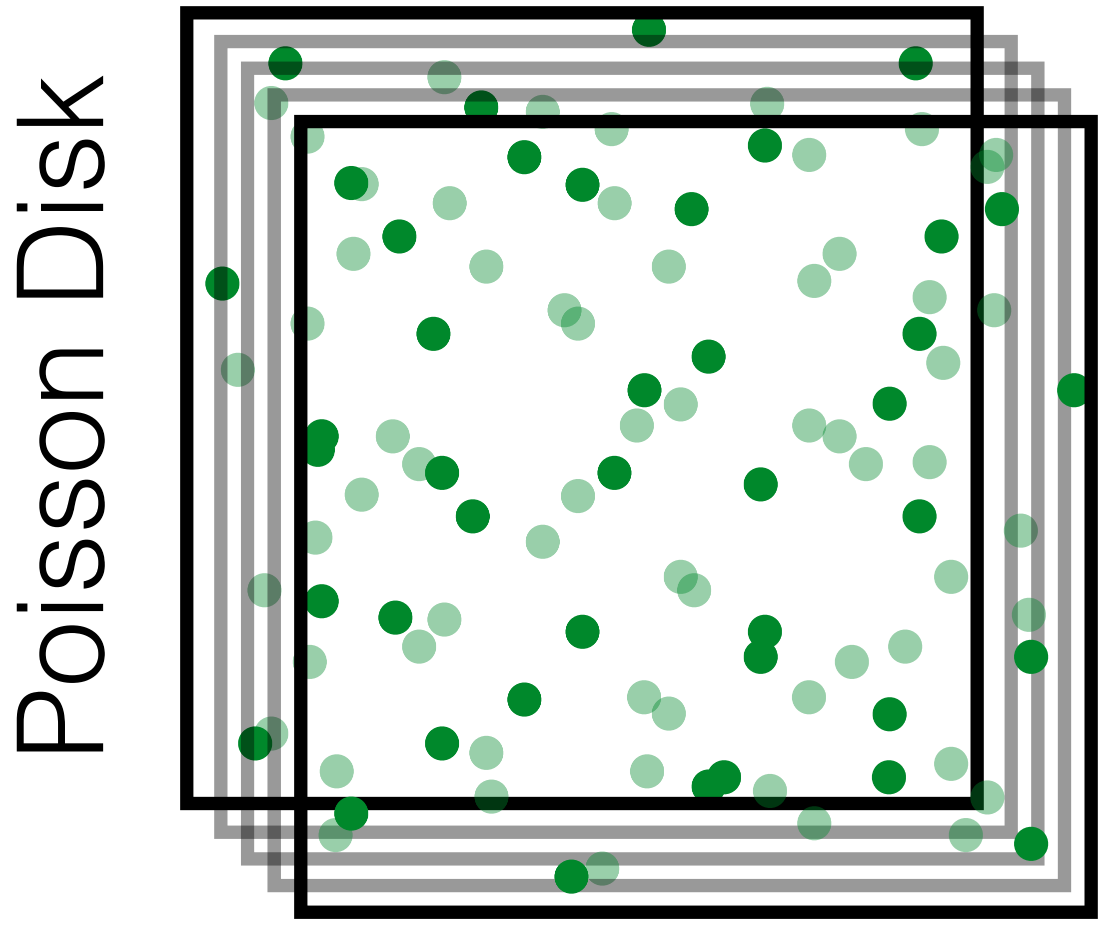
$$S_N(\vec{x}) = \frac{1}{N} \sum_{k=1}^{N} \delta(\vec{x} - \vec{x}_k)$$

$$\mathbb{E}\left[\mathcal{P}_{S_N}(\nu)\right] = \left[\left| \frac{1}{N} \sum_{k=1}^{N} e^{-i2\pi\nu\cdot\vec{x}_k} \right|^2 \right]$$

Expected Sampling Power Spectra

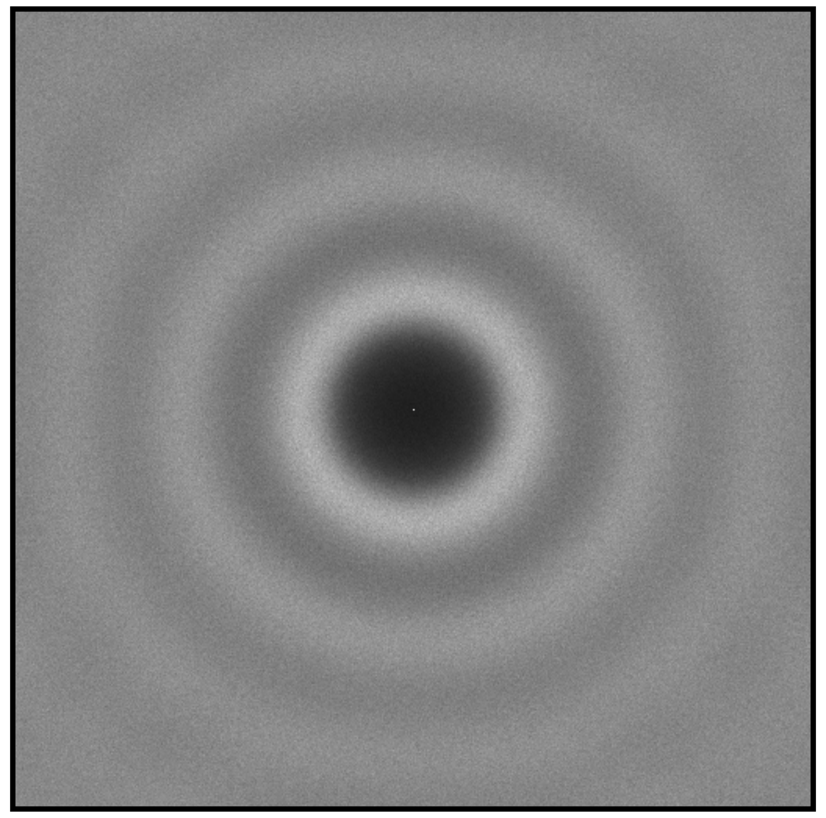
$$\hat{I} = \int_0^1 S_N(\vec{x}) f(\vec{x})\, d\vec{x}$$

Samples



$$S_N(\vec{x}) = \frac{1}{N}\sum_{k=1}^{N} \delta(\vec{x} - \vec{x}_k)$$

Expected Spectrum



$$\mathbb{E}\left[\mathcal{P}_{S_N}(\nu)\right] = \left[\left|\frac{1}{N}\sum_{k=1}^{N} e^{-i2\pi\nu\cdot\vec{x}_k}\right|^2\right]$$

Monte Carlo Integration in Fourier Domain

$$\hat{I} = \int_0^1 S_N(\vec{x}) f(\vec{x})\, d\vec{x} \;=\; \int_\Omega \mathcal{F}_{S_N}(\nu) \mathcal{F}_f^*(\nu) d\nu$$

Using Convolution theorem

Monte Carlo Integration in Fourier Domain

$$\hat{I} = \int_{\Omega} \mathcal{F}_{S_N}(\nu) \mathcal{F}_f^*(\nu) d\nu$$

Monte Carlo Integration in Fourier Domain

$$\hat{I} = \int_{\Omega} \mathcal{F}_{S_N}(\nu)\mathcal{F}_f^*(\nu)d\nu$$

$$I = \int_V f(x)dx = \mathcal{F}_f(0)$$

Monte Carlo Integration in Fourier Domain

$$\hat{I} = \int_{\Omega} \mathcal{F}_{S_N}(\nu)\mathcal{F}_f^*(\nu)d\nu$$

$$I = \int_V f(x)dx = \mathcal{F}_f(0)$$

$$\text{Error: } \Delta = \hat{I} - I$$

Error: Bias Term

$$\text{Error: } \Delta = \int_\Omega \mathcal{F}_{S_N}(\nu)\mathcal{F}_f^*(\nu)d\nu - \mathcal{F}_f(0)$$

$$\text{Bias: } \mathbb{E}[\Delta] = \int_\Omega \mathbb{E}\left[\mathcal{F}_{S_N}(\nu)\right]\mathcal{F}_f^*(\nu)d\nu - \mathcal{F}_f(0)$$

$$\mathbb{E}\left[Xa\right] = \mathbb{E}[X]\,a$$

$$\mathbb{E}[a] = a$$

Error: Bias Term

$$\text{Error: } \Delta = \int_\Omega \mathcal{F}_{S_N}(\nu)\mathcal{F}_f^*(\nu)d\nu - \mathcal{F}_f(0)$$

$$\text{Bias: } \mathbb{E}[\boldsymbol{\Delta}] = \int_\Omega \mathbb{E}\left[\mathcal{F}_{S_N}(\nu)\right]\mathcal{F}_f^*(\nu)d\nu - \mathcal{F}_f(0)$$

$$\mathbb{E}\left[\mathcal{F}_{S_N}(\nu)\right] = \delta(\nu)$$

Bias goes to zero

$$w(x) = 1/p(x)$$

Subr and Kautz [2013]

Error: Variance Term

$$\text{Error: } \Delta = \int_{\Omega} \mathcal{F}_{S_N}(\nu)\mathcal{F}_f^*(\nu)d\nu - \mathcal{F}_f(0)$$

Variance: $Var[\Delta]$

Error: Variance Term

$$\text{Error: } \Delta = \int_{\Omega} \mathcal{F}_{S_N}(\nu)\mathcal{F}_f^*(\nu)d\nu - \mathcal{F}_f(0)$$

$$\text{Variance: } Var[\Delta] = Var[\hat{I} - I]$$

Error: Variance Term

$$\text{Error: } \Delta = \int_{\Omega} \mathcal{F}_{S_N}(\nu)\mathcal{F}_f^*(\nu)d\nu - \mathcal{F}_f(0)$$

$$\text{Variance: } Var[\Delta] = Var[\hat{I} - I] = Var[\hat{I}] - Var[I]$$

Error: Variance Term

$$\text{Error: } \Delta = \int_{\Omega} \mathcal{F}_{S_N}(\nu)\mathcal{F}_f^*(\nu)d\nu - \mathcal{F}_f(0)$$

$$\text{Variance: } Var[\Delta] = Var[\hat{I} - I] = Var[\hat{I}] - Var[I] = Var[\hat{I}]$$

Error: Variance Term

$$\text{Error: } \Delta = \int_{\Omega} \mathcal{F}_{S_N}(\nu)\mathcal{F}_f^*(\nu)d\nu - \mathcal{F}_f(0)$$

Variance: $$Var[\Delta] = Var[\hat{I}]$$

Error: Variance Term

$$Var[\Delta] = Var[\hat{I}]$$

$$\text{Error: } \Delta = \int_{\Omega} \mathcal{F}_{S_N}(\nu)\mathcal{F}_f^*(\nu)d\nu - \mathcal{F}_f(0)$$

$$\text{Variance: } \quad Var[\Delta] = \int_{\Omega} Var\left[\mathcal{F}_{S_N}(\nu)\right] \mathcal{F}_f^*(\nu)\mathcal{F}_f(\nu)d\nu - Var\left[\mathcal{F}_f(0)\right]$$

$$Var\left[Xa\right] = Var[X]\ a^* a$$

Error: Variance Term

$$Var[\Delta] = Var[\hat{I}]$$

$$\text{Error: } \Delta = \int_{\Omega} \mathcal{F}_{S_N}(\nu)\mathcal{F}_f^*(\nu)d\nu - \mathcal{F}_f(0)$$

$$\text{Variance: } \quad Var[\Delta] = \int_{\Omega} Var\left[\mathcal{F}_{S_N}(\nu)\right] \mathcal{F}_f^*(\nu)\mathcal{F}_f(\nu)d\nu - Var\left[\mathcal{F}_f(0)\right]$$

$$Var\left[Xa\right] = Var[X] \; a^* a$$

Error: Variance Term

$$\text{Variance:} \quad Var[\hat{I}] \quad = \int_{\Omega} Var\left[\mathcal{F}_{S_N}(\nu)\right] \mathcal{F}_f^*(\nu)\mathcal{F}_f(\nu)d\nu$$

Error: Variance Term

$$\text{Variance:} \quad Var[\hat{I}] = \int_{\Omega} Var\left[\mathcal{F}_{S_N}(\nu)\right] \mathcal{F}_f^*(\nu)\mathcal{F}_f(\nu)d\nu$$

$$= \int_{\Omega} Var\left[\mathcal{F}_{S_N}(\nu)\right] \mathcal{P}_f(\nu)d\nu$$

Error: Variance Term

$$\text{Variance:} \quad Var[\hat{I}] = \int_{\Omega} \underline{Var\left[\mathcal{F}_{S_N}(\nu)\right]}\, \mathcal{P}_f(\nu) d\nu$$

$$= \int_{\Omega/0} \mathbb{E}\left[\mathcal{P}_{S_N}(\nu)\right] \mathcal{P}_f(\nu) d\nu$$

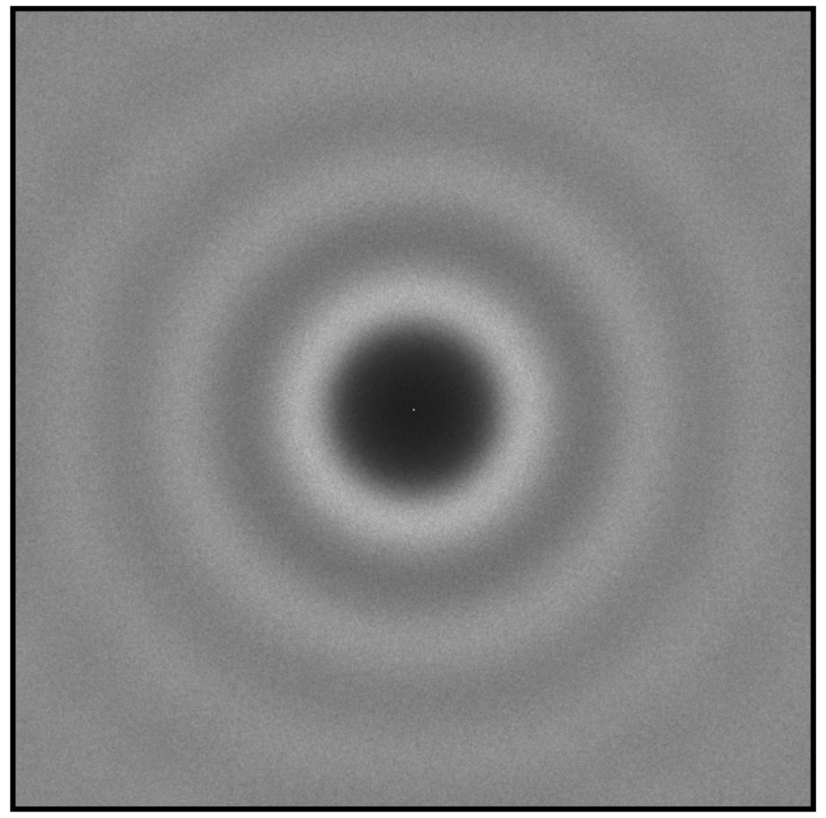
Variance of Monte Carlo Integration in Fourier Domain

$$Var[\hat{I}] \quad = \int_{\Omega/0} \mathbb{E}\left[\mathcal{P}_{S_N}(\nu)\right] \mathcal{P}_f(\nu) d\nu$$

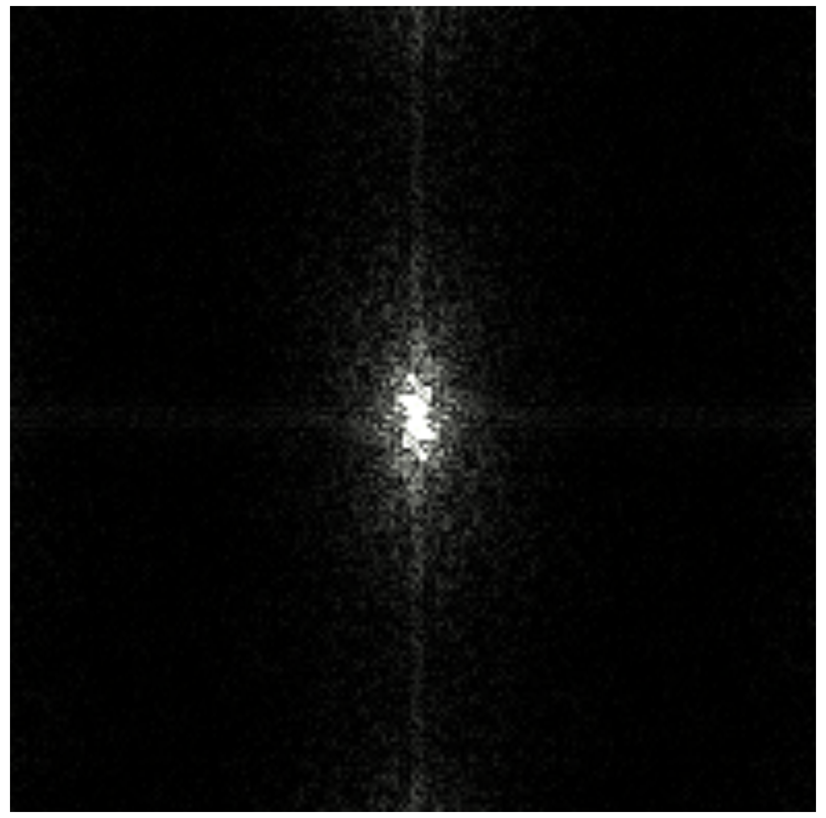
Variance of Monte Carlo Estimator

$$Var[\hat{I}] = \int_{\Omega/0} \mathbb{E}\left[\mathcal{P}_{S_N}(\nu)\right] \times \mathcal{P}_f(\nu)\, d\nu$$

$\mathbb{E}\left[\mathcal{P}_{S_N}(\nu)\right]$

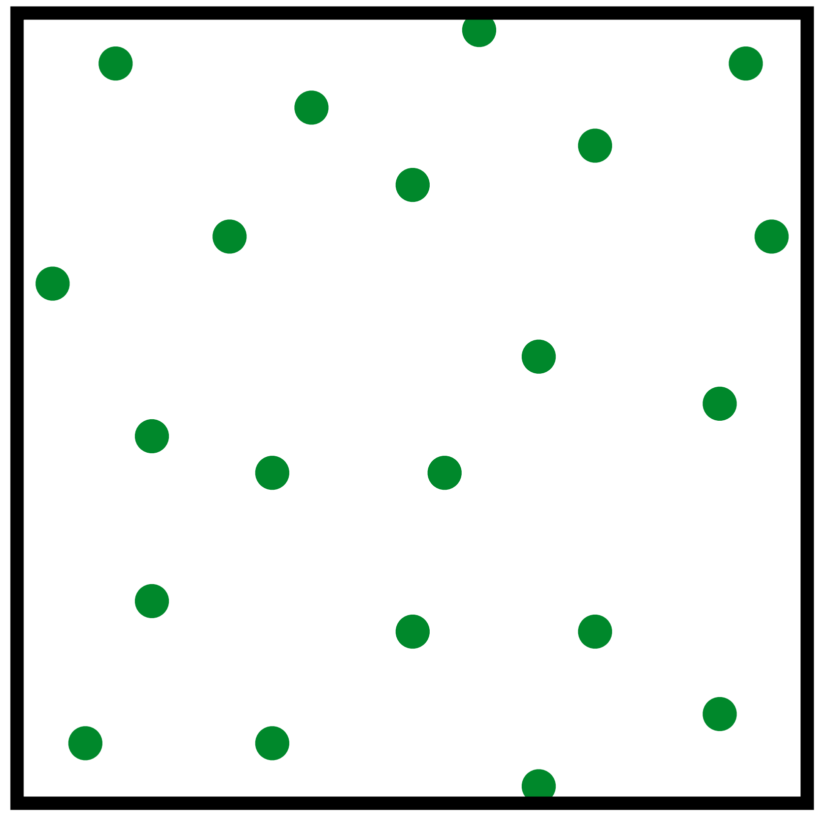


$\mathcal{P}_f(\nu)$

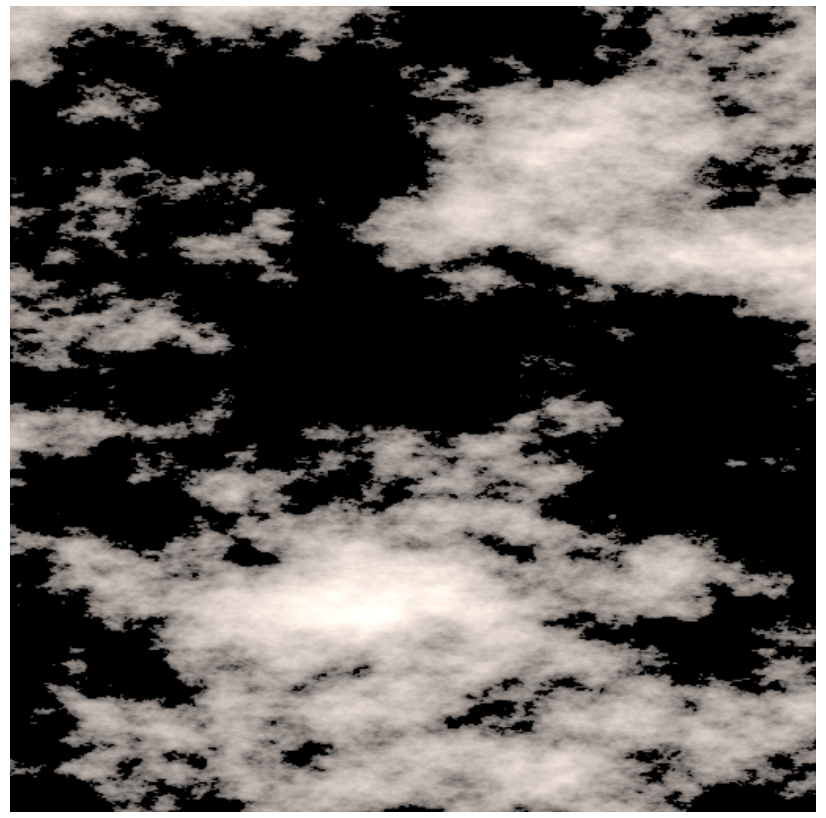


$S_N(\vec{x})$

Poisson Disk



$f(\vec{x})$



Fredo Durand [2011]

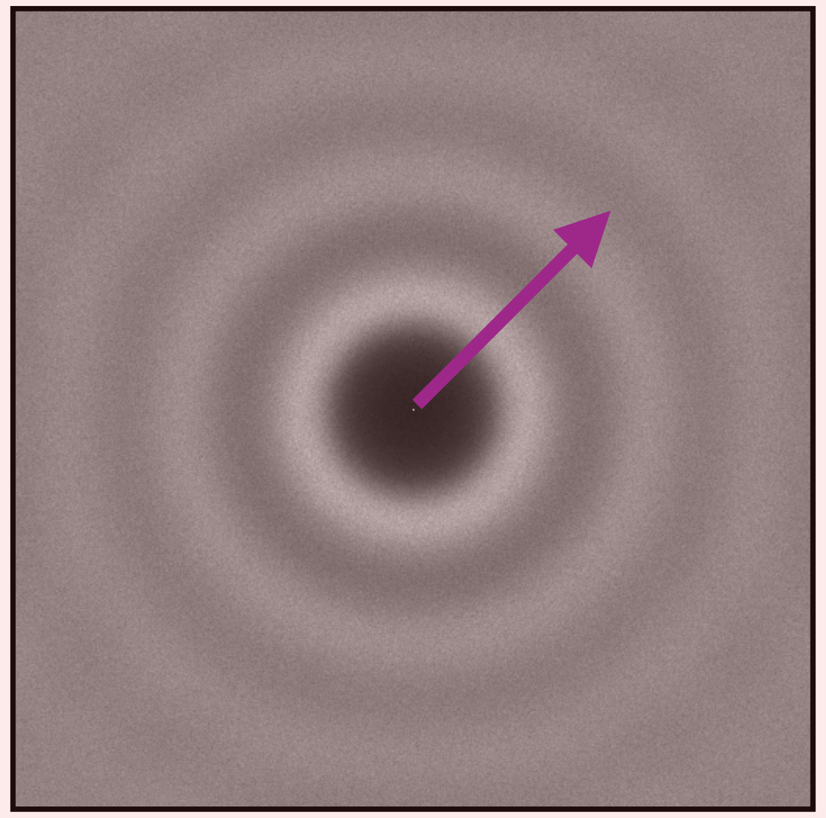
Subr & Kautz [2013]

Pilleboue et al. [2015]

Variance of Monte Carlo Estimator for Isotropic Sampling Spectra

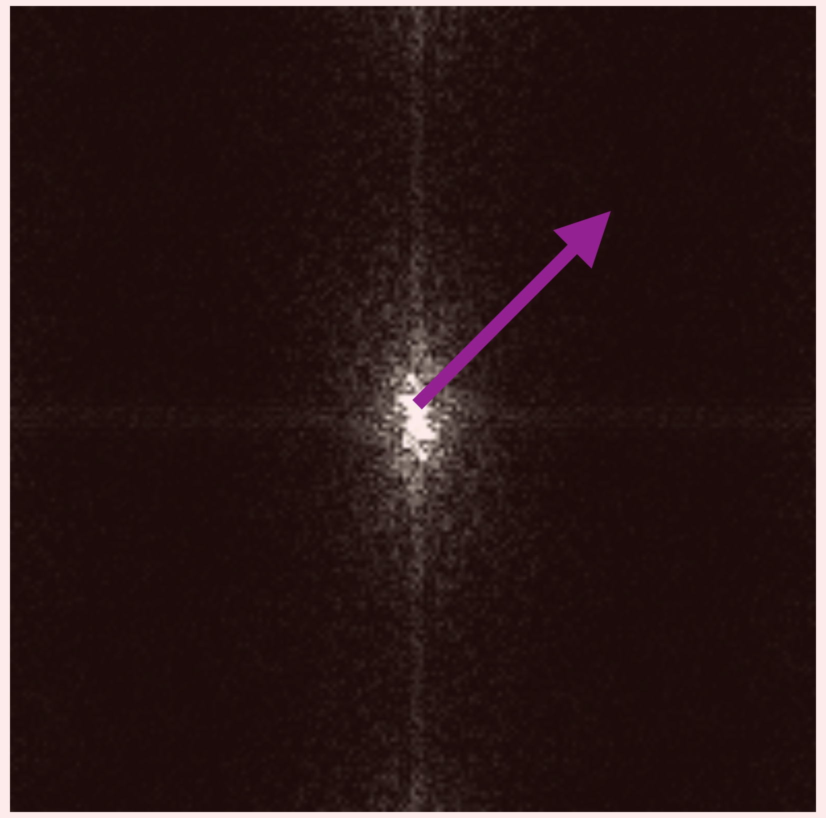
$$Var[\hat{I}] = \int_0^\infty \rho^{d-1} \int_{\mathcal{S}^{d-1}}$$

$\mathbb{E}\left[\tilde{\mathcal{P}}_{S_N}(\rho\mathbf{n})\right]$



$\times$

$\mathcal{P}_f(\rho\mathbf{n})$



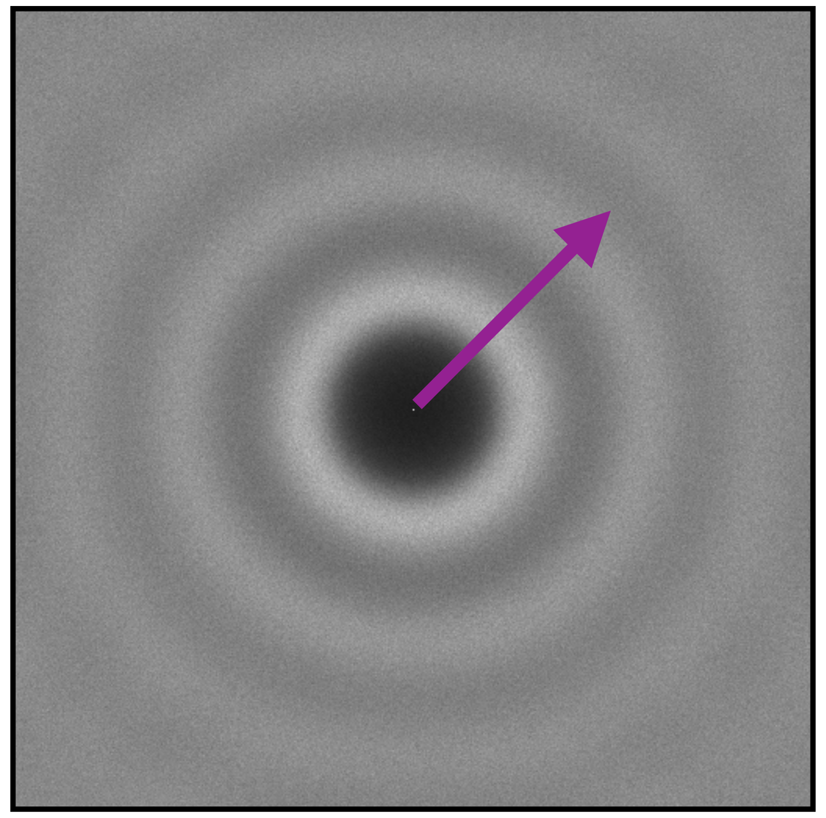
$$d\mathbf{n}\, d\rho$$

Pilleboue et al. [2015]

Variance of Monte Carlo Estimator for Isotropic Sampling Spectra

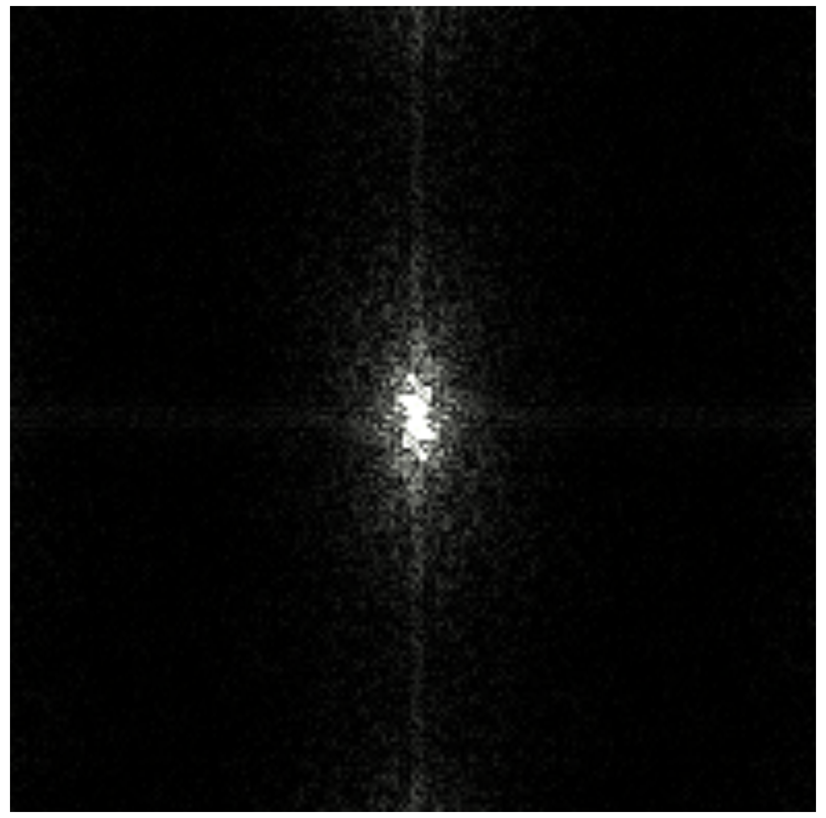
$$Var[\hat{I}] = \int_0^\infty \rho^{d-1} \int_{\mathcal{S}^{d-1}}$$

$\mathbb{E}\left[\tilde{\mathcal{P}}_{S_N}(\rho\mathbf{n})\right]$



$\times$

$\mathcal{P}_f(\rho\mathbf{n})$



$$d\mathbf{n}\, d\rho$$

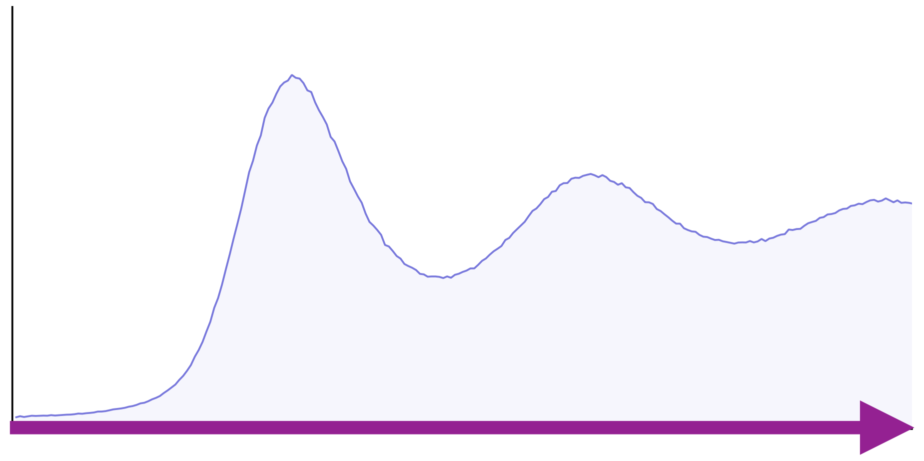
Pilleboue et al. [2015]

Variance of Monte Carlo Estimator for Isotropic Sampling Spectra

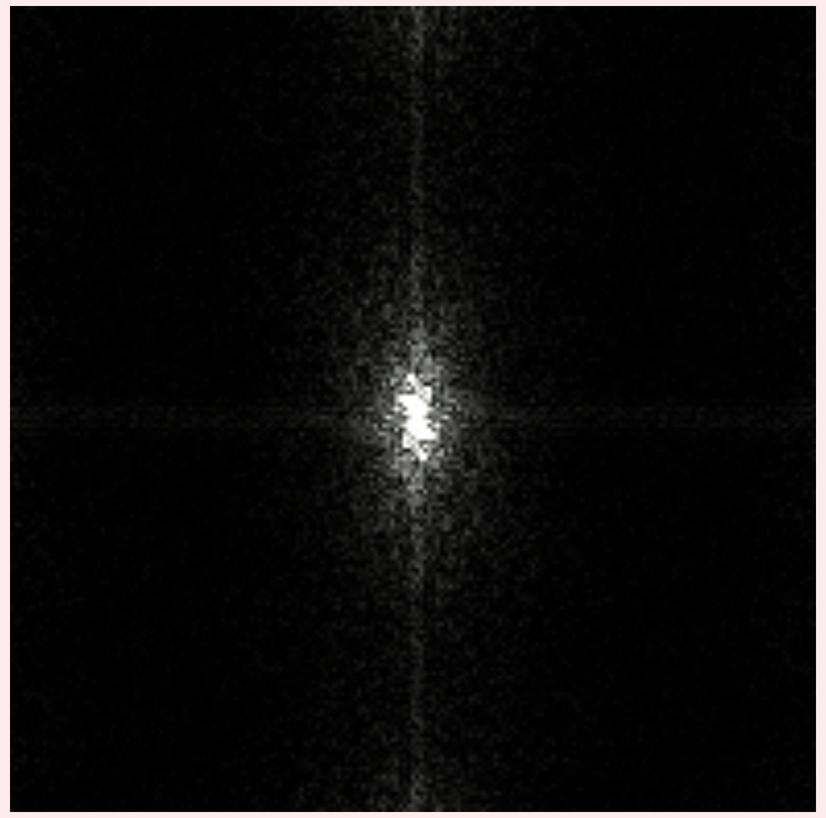
$\tilde{\mathcal{P}}_{S_N}(\rho)$

$\mathcal{P}_f(\rho\mathbf{n})$

$Var[\hat{I}] = \int_0^\infty \rho^{d-1}$



$\int_{\mathcal{S}^{d-1}}$



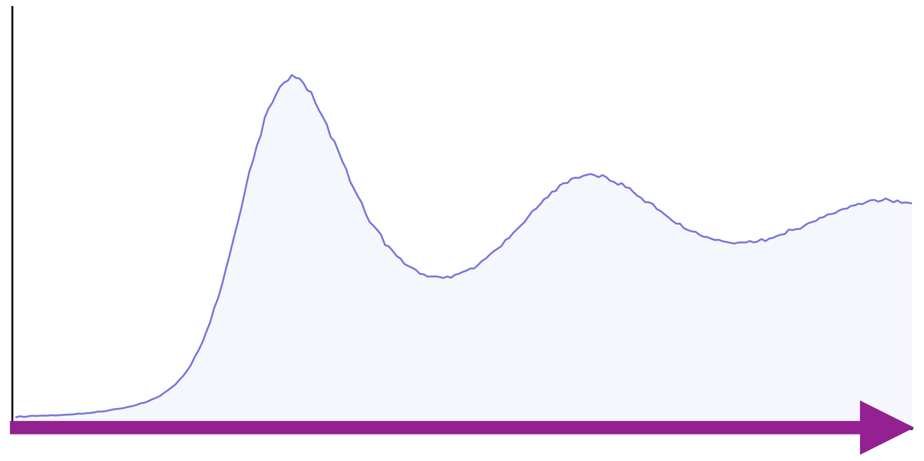
$d\mathbf{n}\, d\rho$

Pilleboue et al. [2015]

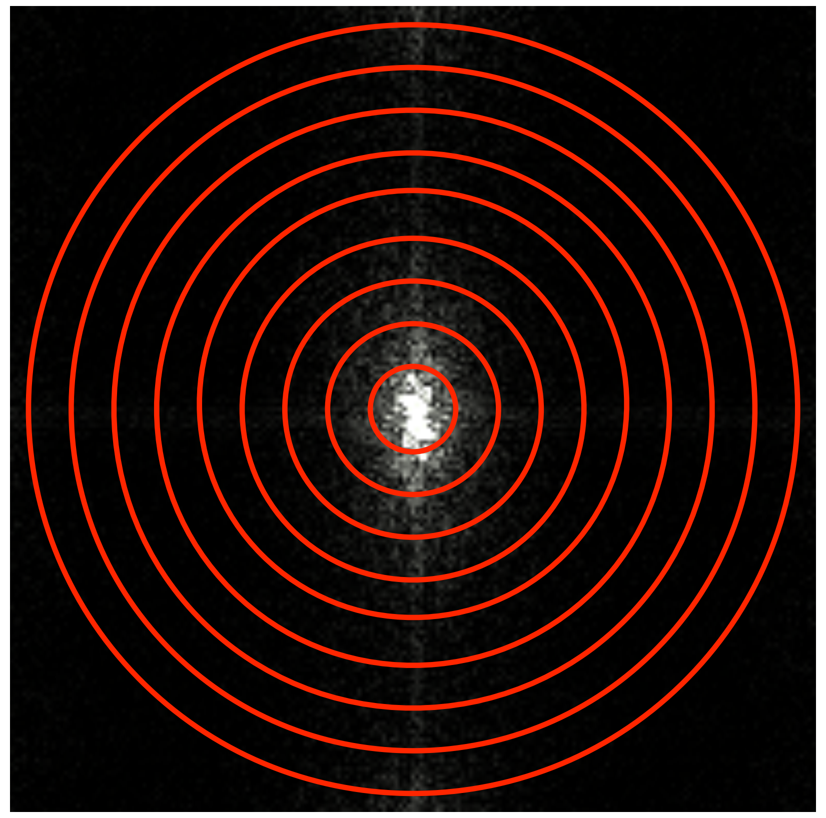
Variance of Monte Carlo Estimator for Isotropic Sampling Spectra

$\tilde{\mathcal{P}}_{S_N}(\rho)$ $\qquad$ $\mathcal{P}_f(\rho\mathbf{n})$

$$Var[\hat{I}] = \int_0^\infty \rho^{d-1}$$



$$\times$$



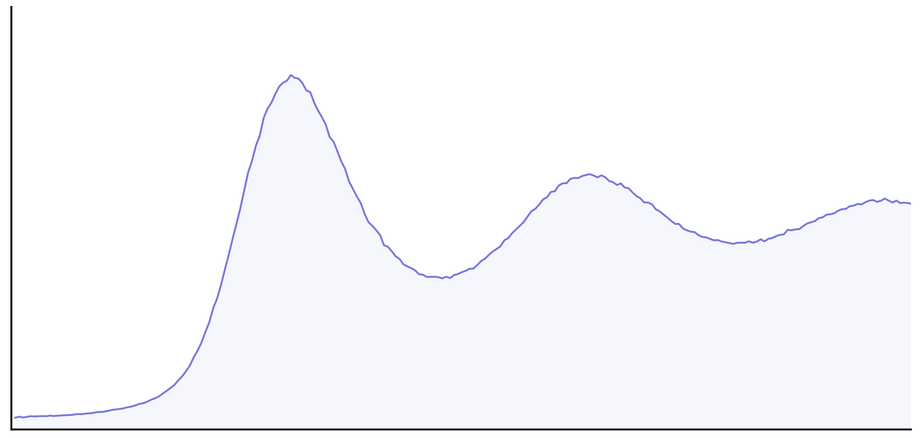
$$d\rho$$

Pilleboue et al. [2015]

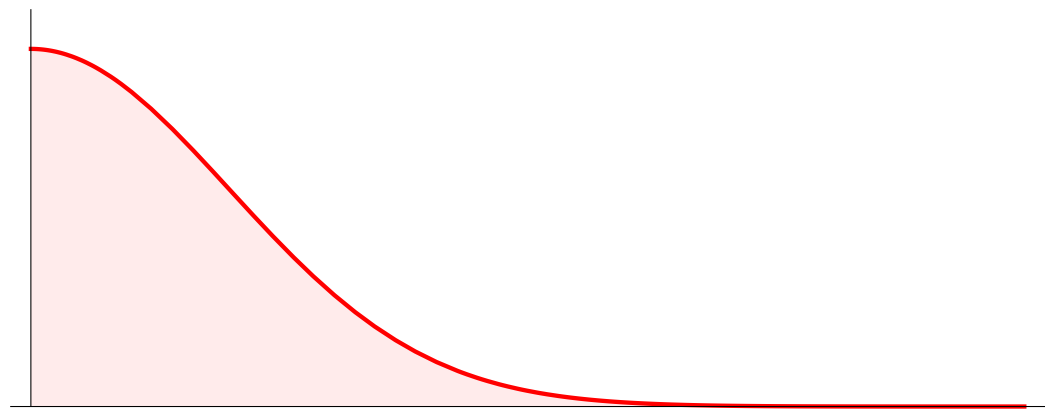
Variance of Monte Carlo Estimator for Isotropic Sampling Spectra

$\tilde{\mathcal{P}}_{S_N}(\rho)$ $\mathcal{P}_f(\rho)$

$Var[\hat{I}] = \int_0^\infty \rho^{d-1}$



$\times$

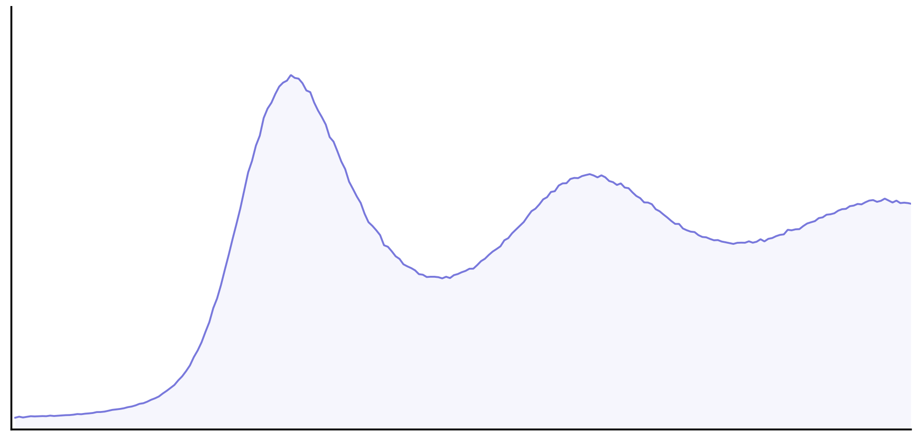


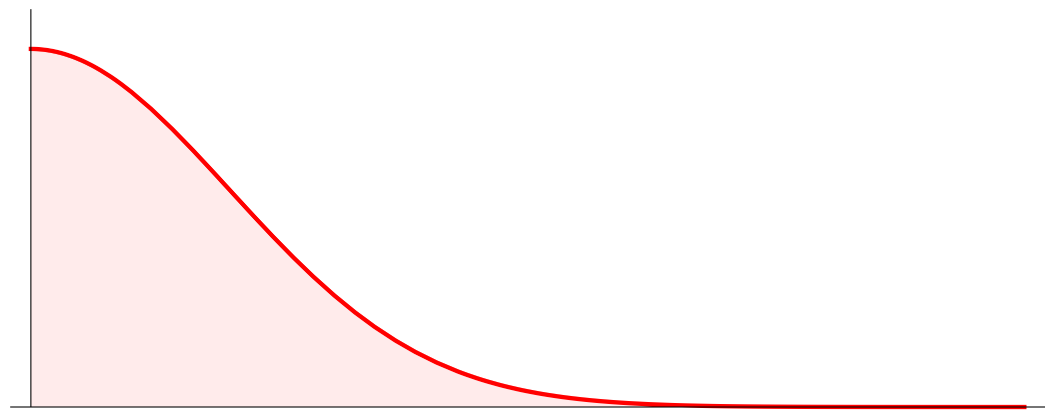
$d\rho$

Pilleboue et al. [2015]

Variance of Monte Carlo Estimator for Isotropic Sampling Spectra

$\tilde{\mathcal{P}}_{S_N}(\rho)$ $\quad$ $\mathcal{P}_f(\rho)$



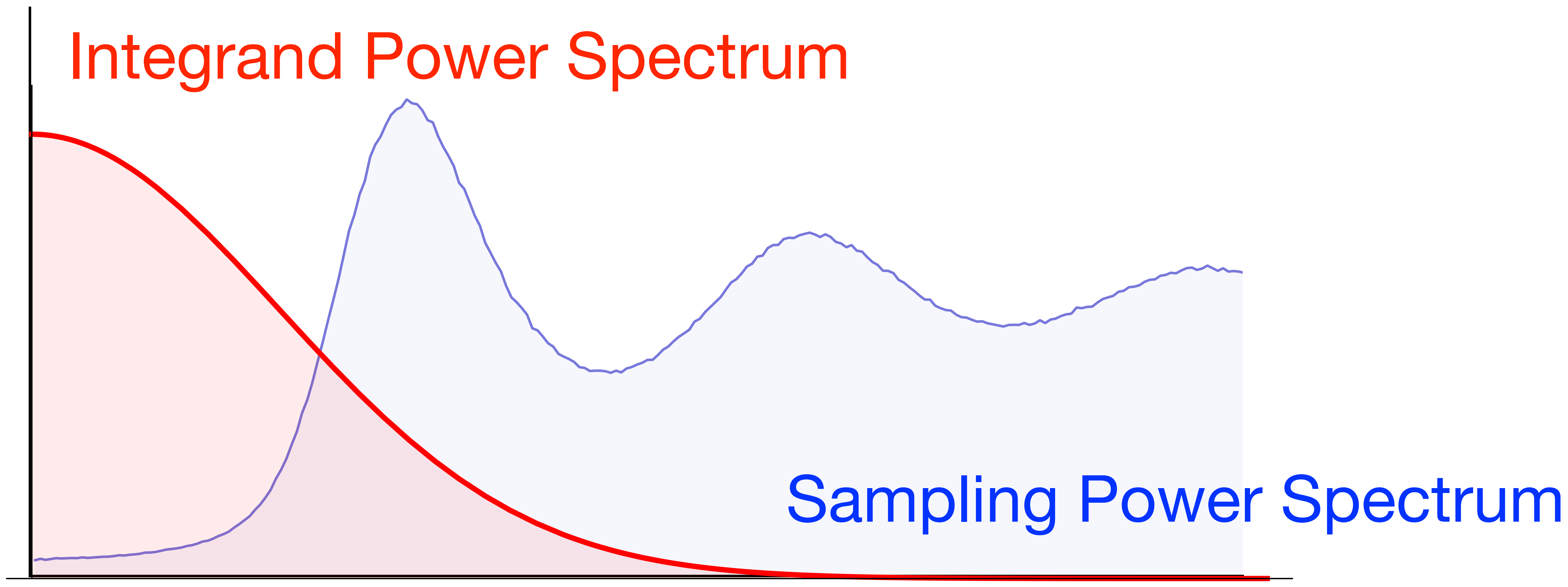


$$Var[\hat{I}] = \int_0^\infty \rho^{d-1} \; \tilde{\mathcal{P}}_{S_N}(\rho) \times \mathcal{P}_f(\rho) \; d\rho$$

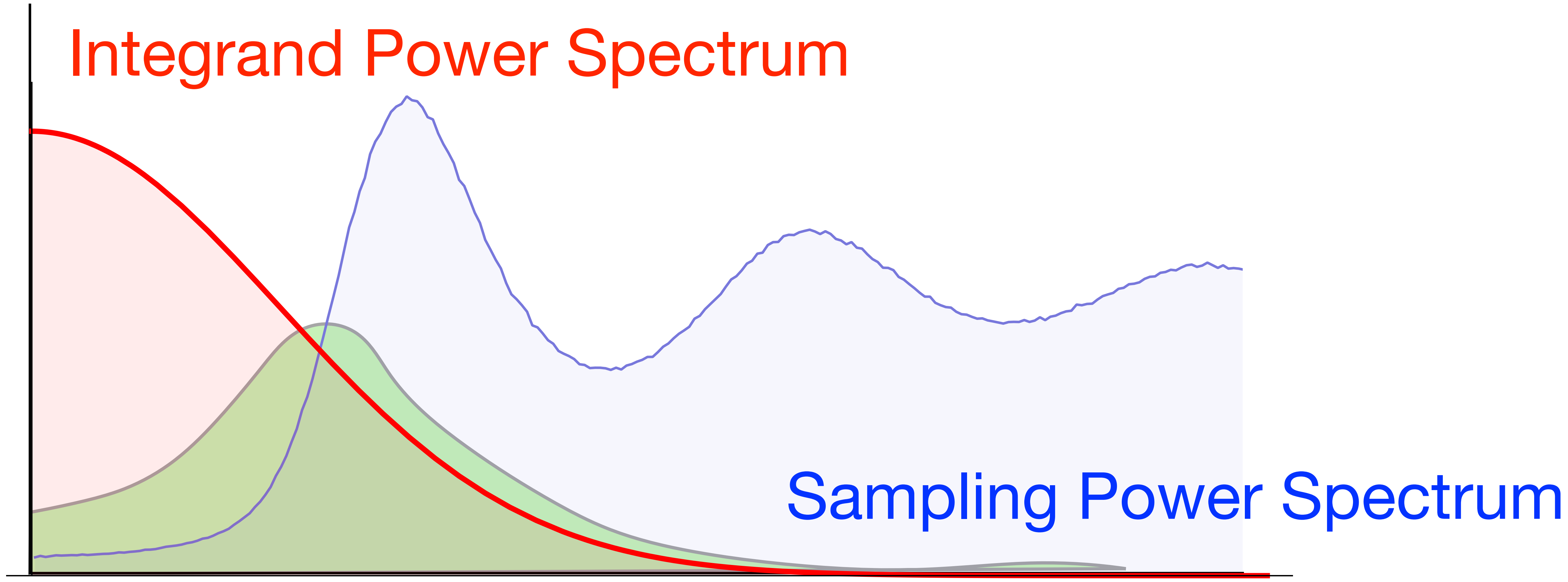
Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-3})$

Pilleboue et al. [2015]

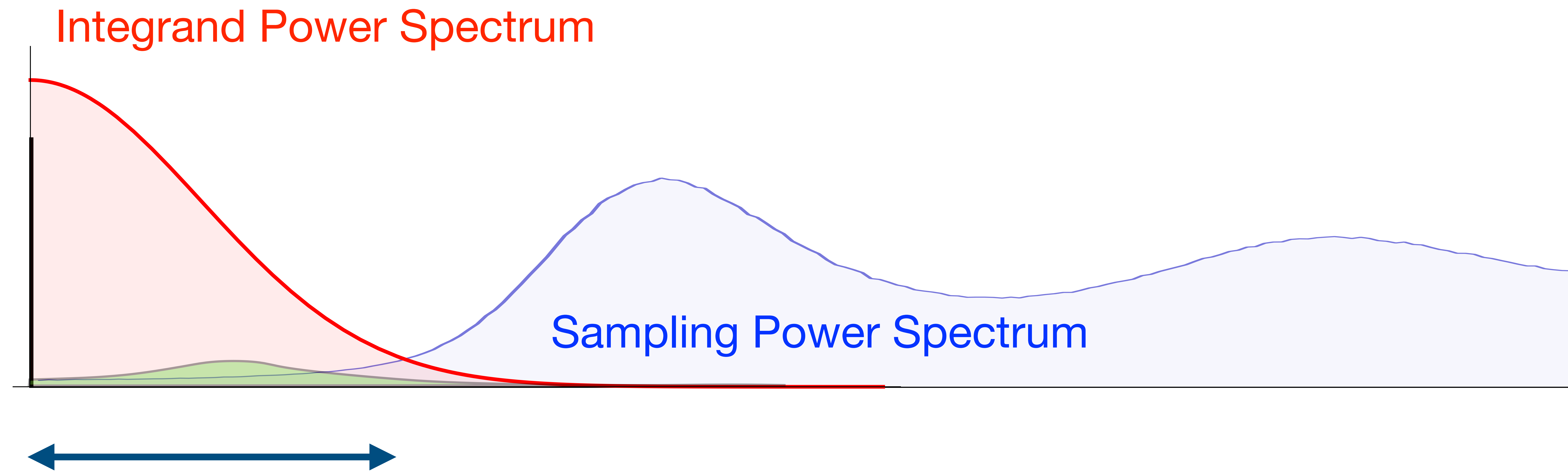
Variance: Product of Power Spectra



Variance: Product of Power Spectra

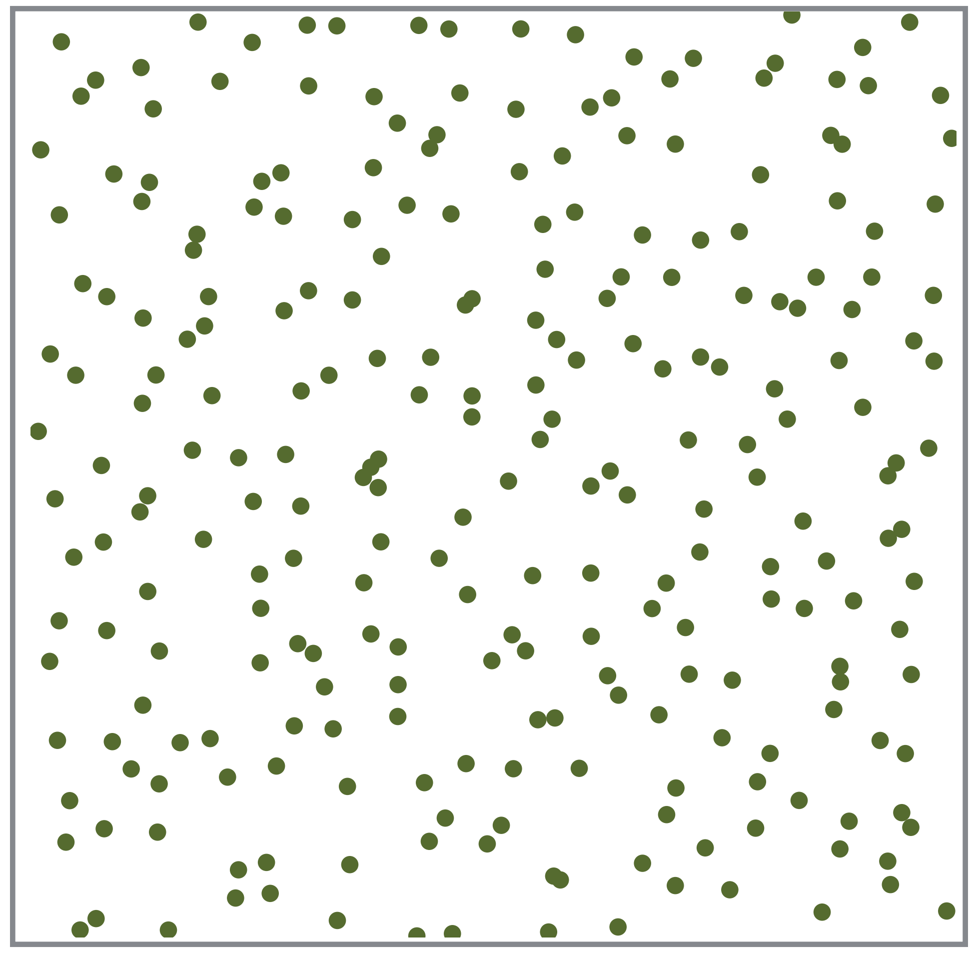


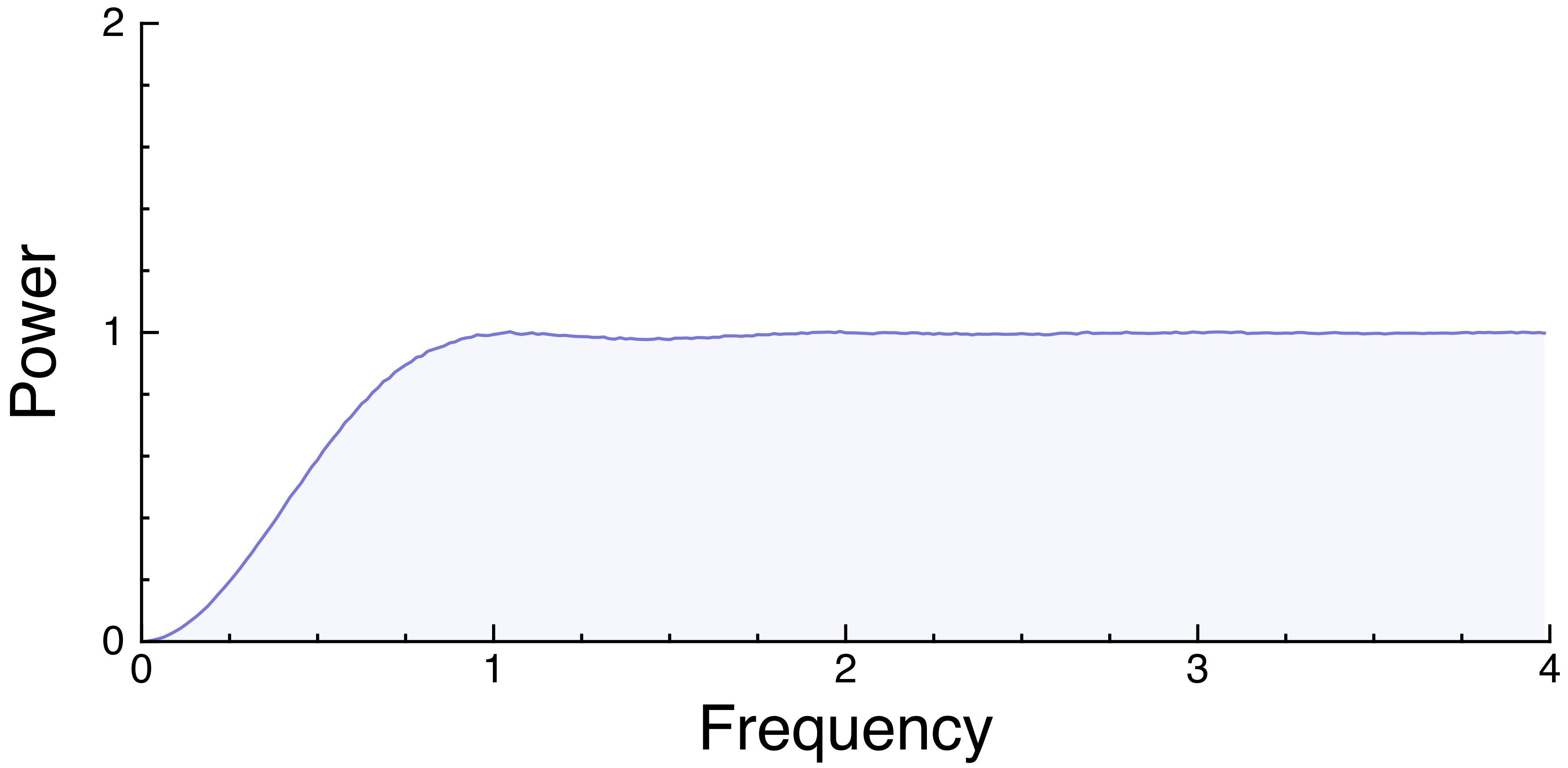
Variance: Product of Power Spectra



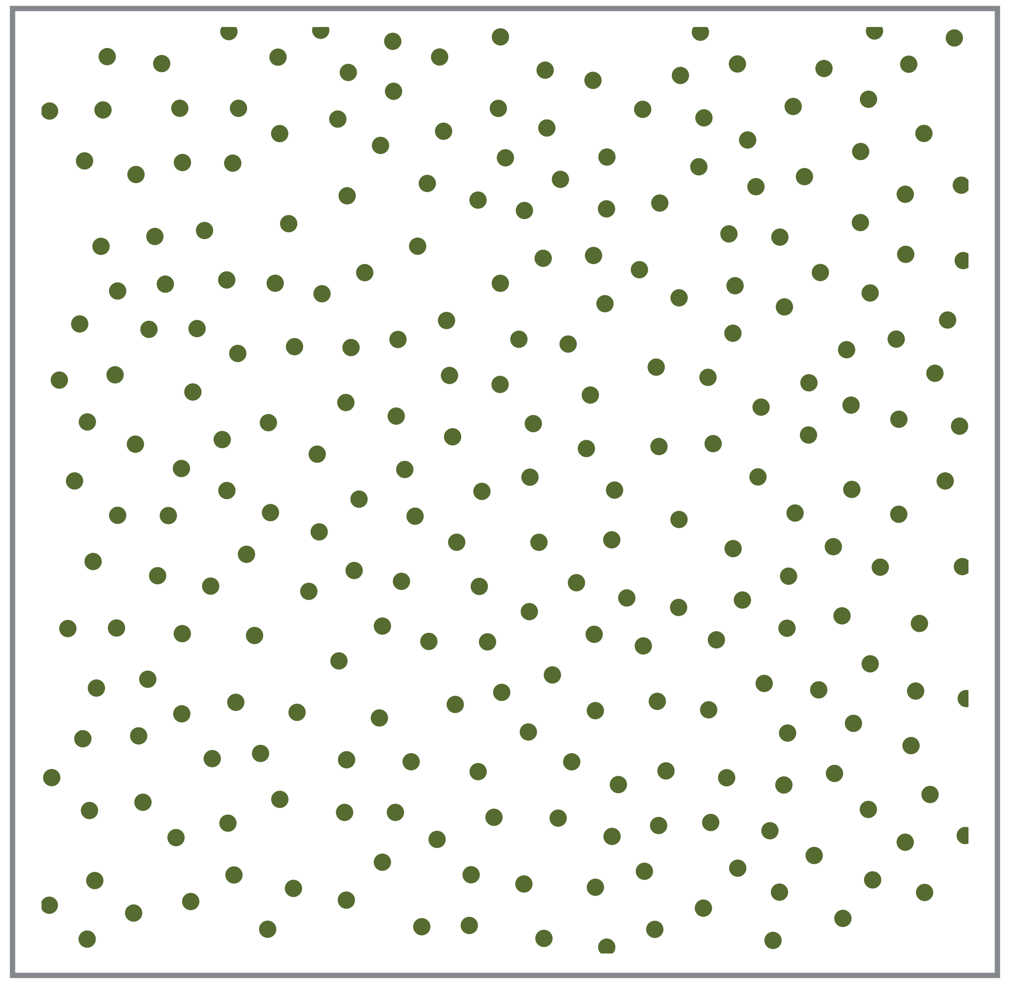
Jitter vs Poisson Disk Radial Power Spectra

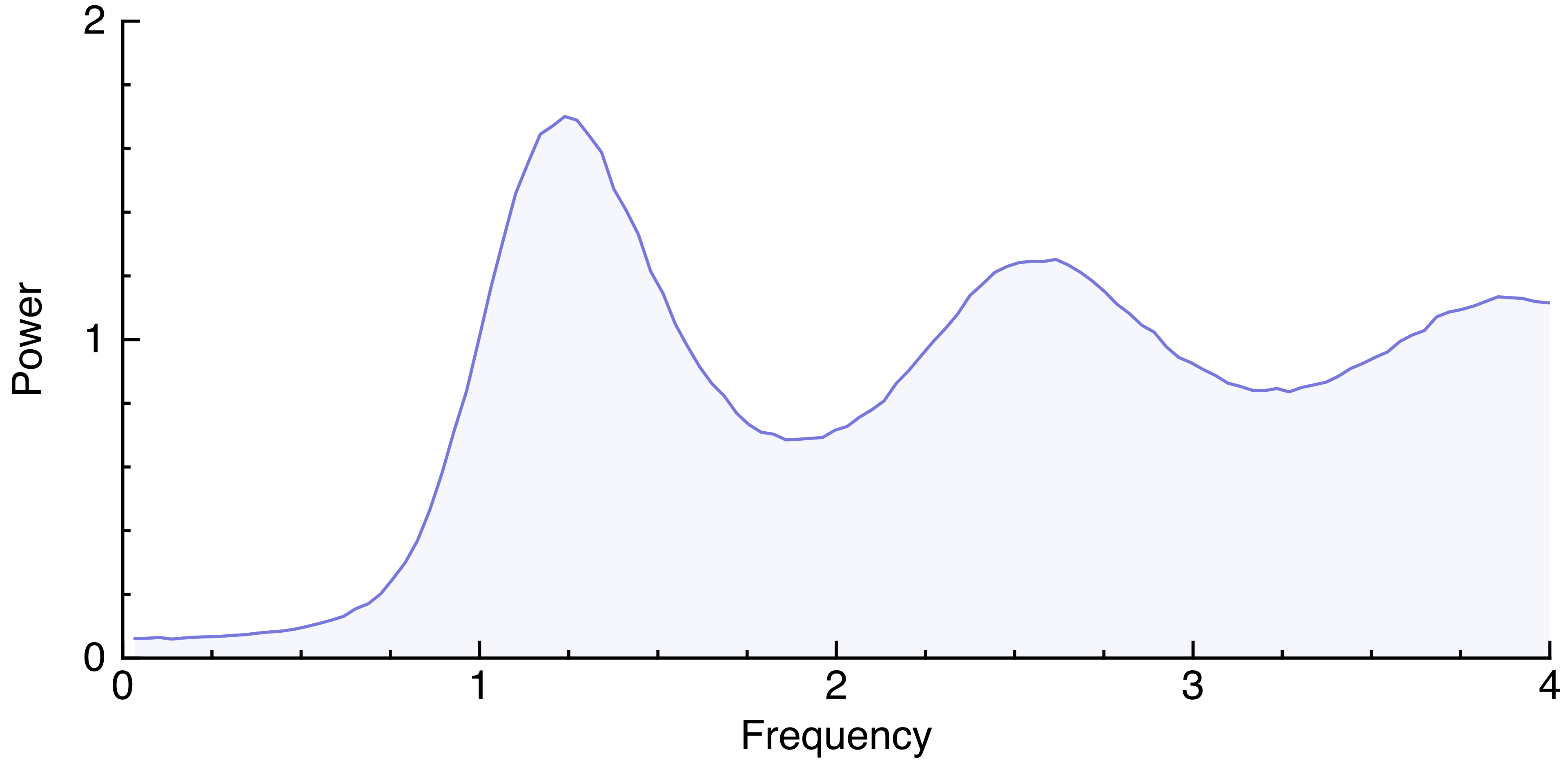
Jitter



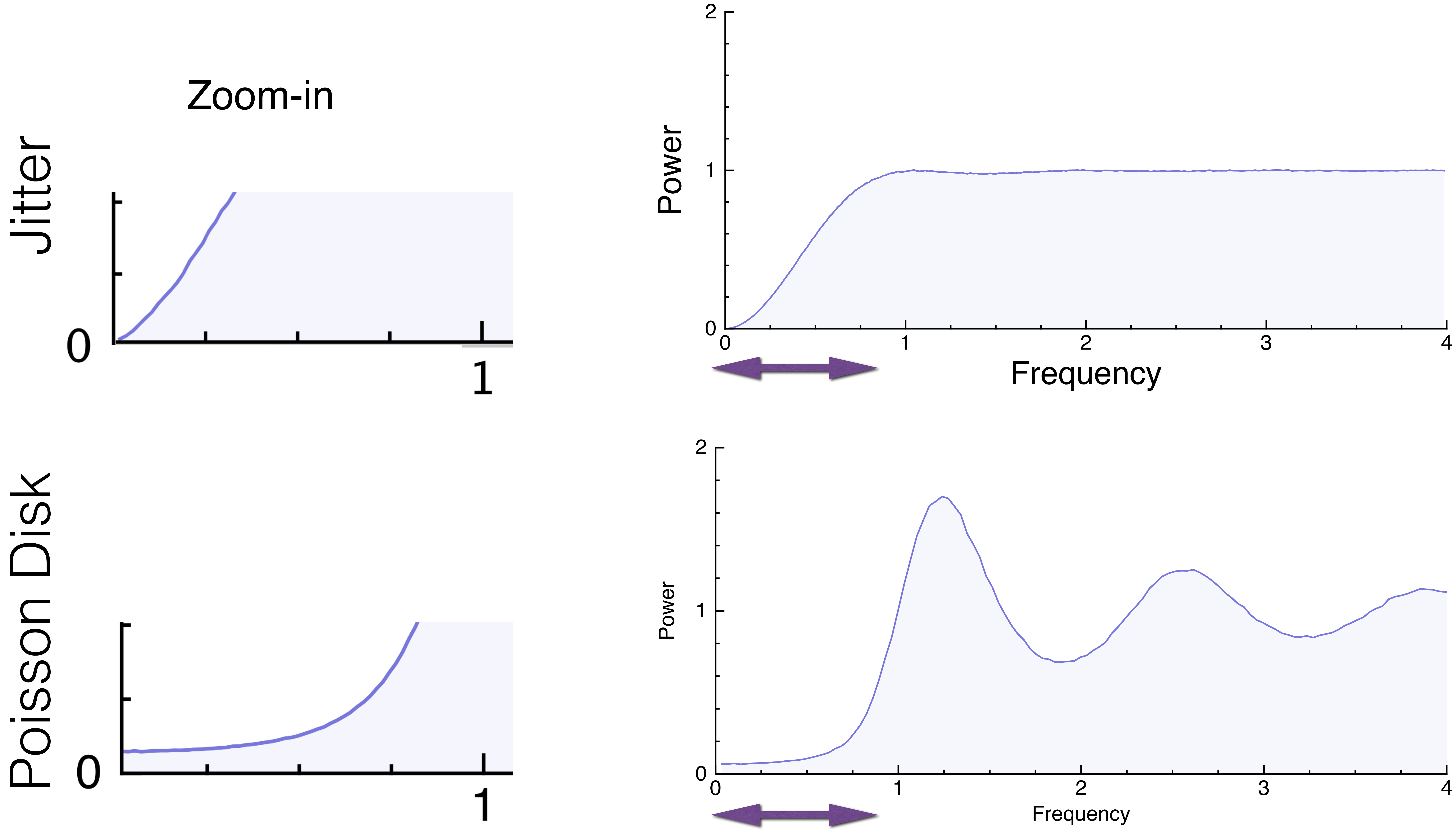


Poisson Disk





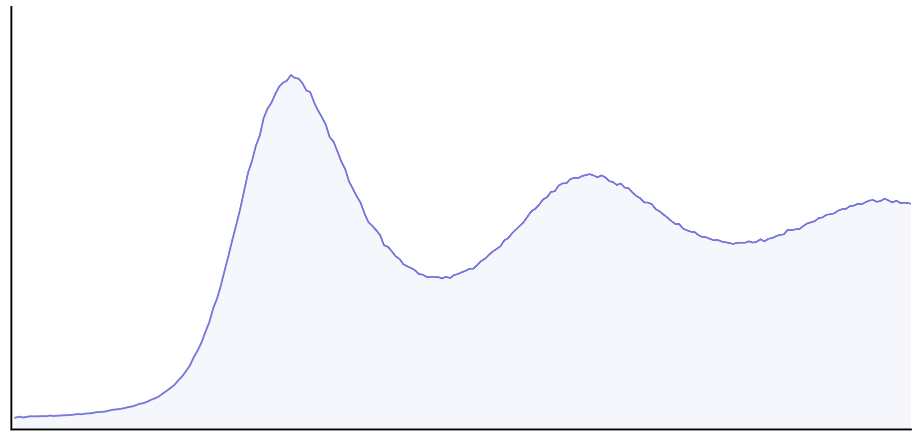
Jitter vs Poisson Disk Radial Power Spectra



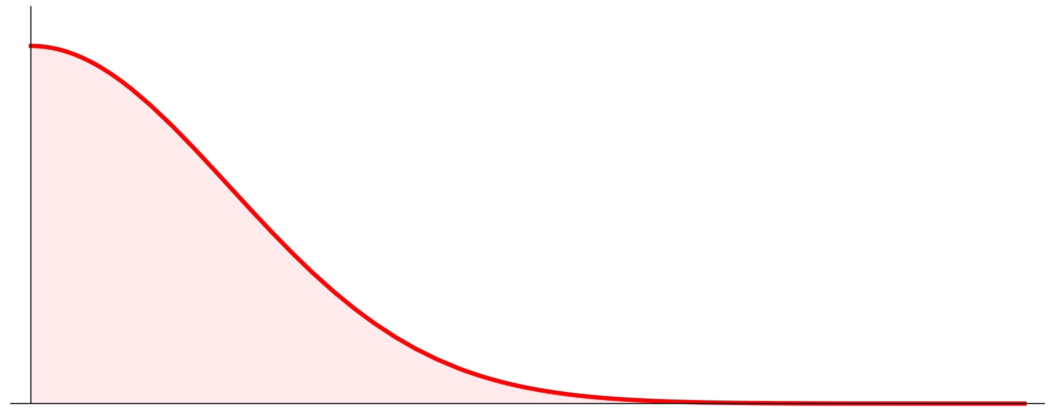
Variance of Monte Carlo Estimator for Isotropic Sampling Spectra

$\tilde{\mathcal{P}}_{S_N}(\rho)$ $\qquad$ $\mathcal{P}_f(\rho)$

$$Var[\hat{I}] = \int_0^\infty \rho^{d-1}$$



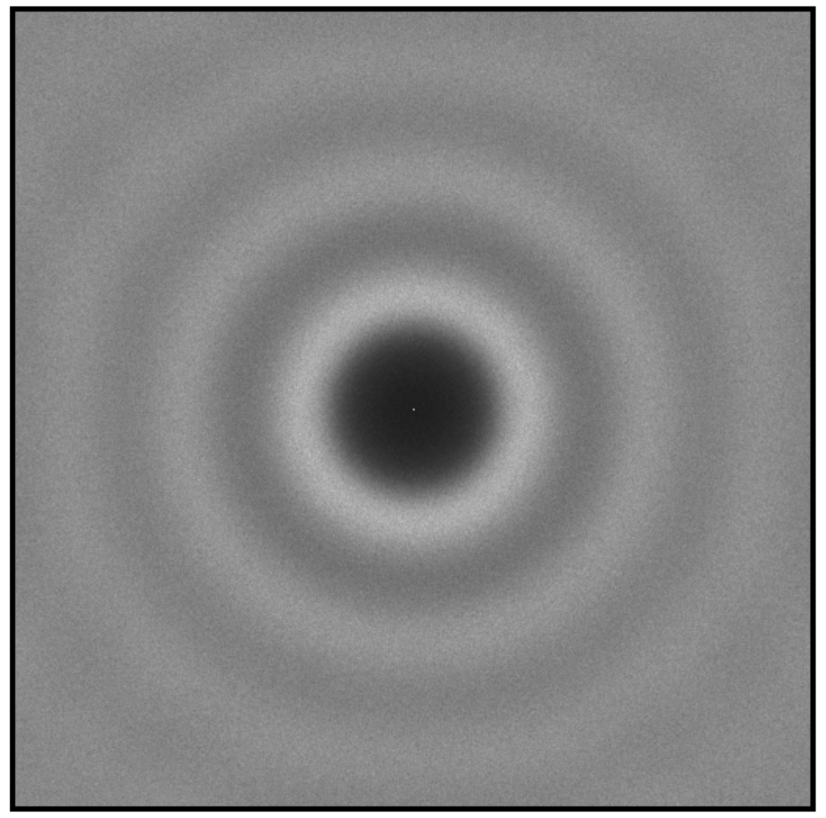
$\times$



$d\rho$

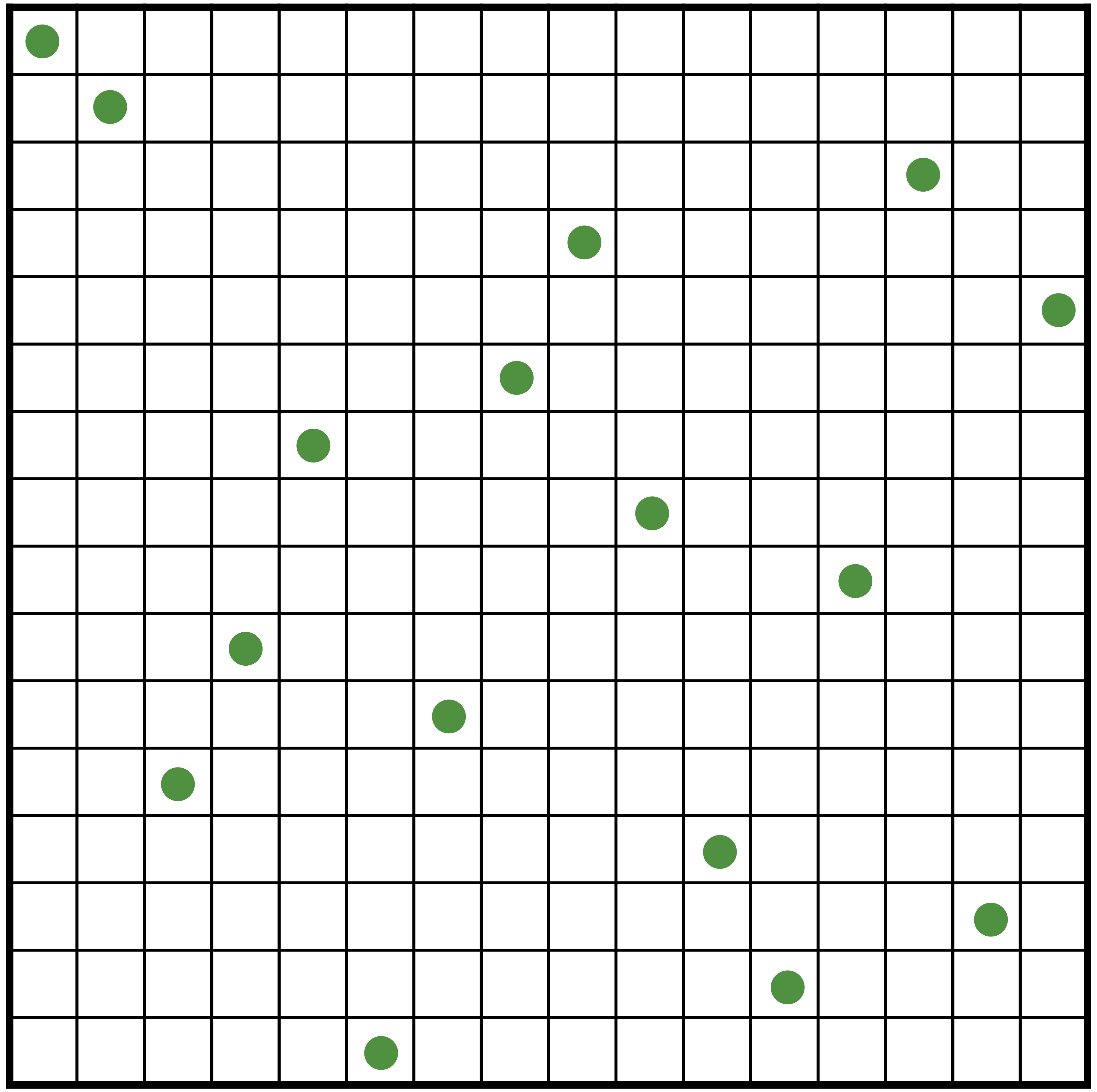
Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-3})$

Isotropic Spectrum
Poisson Disk



Pilleboue et al. [2015]

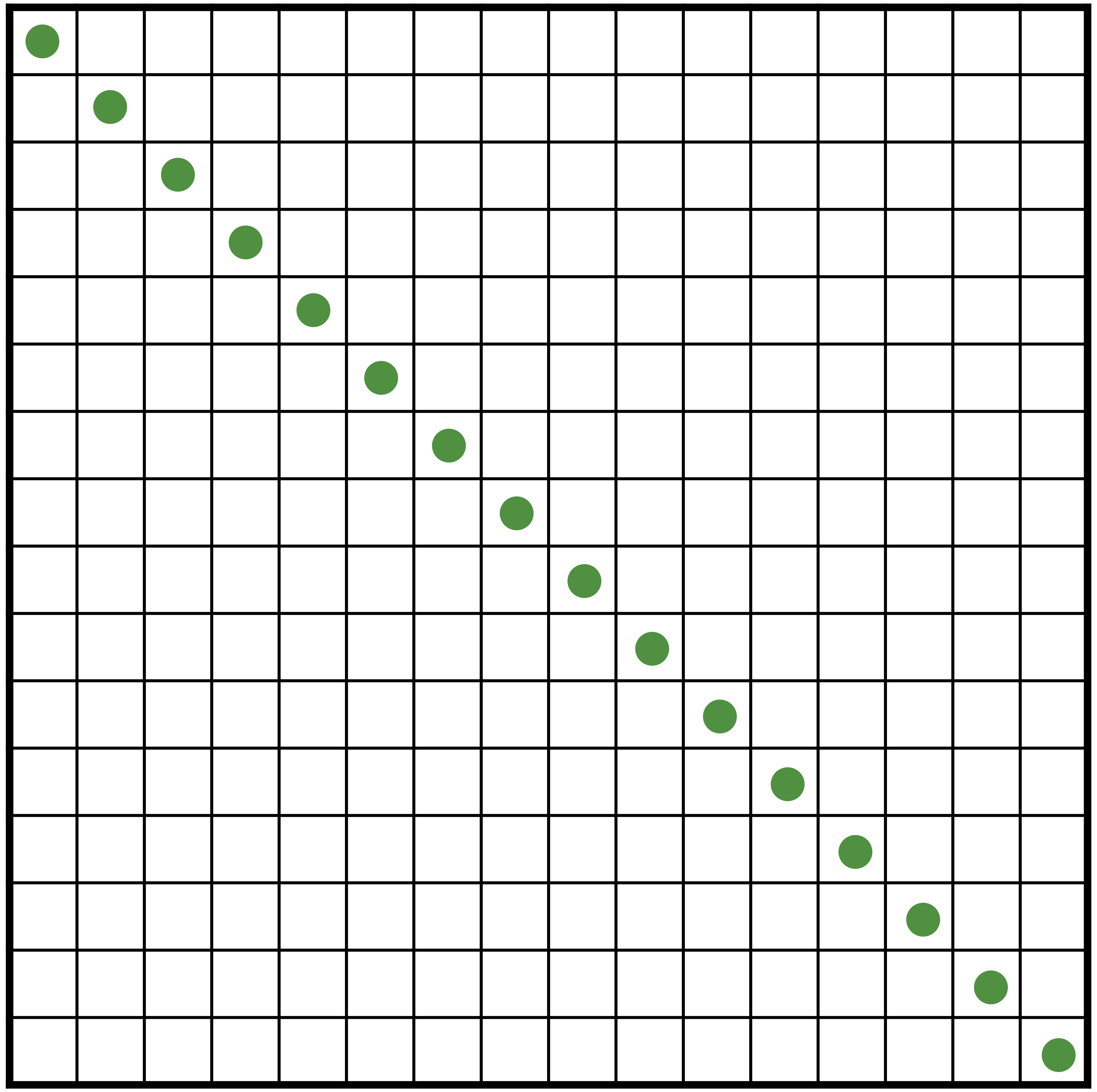
Latin Hypercube Sampler (N-rooks)



Slide after Wojciech Jarosz

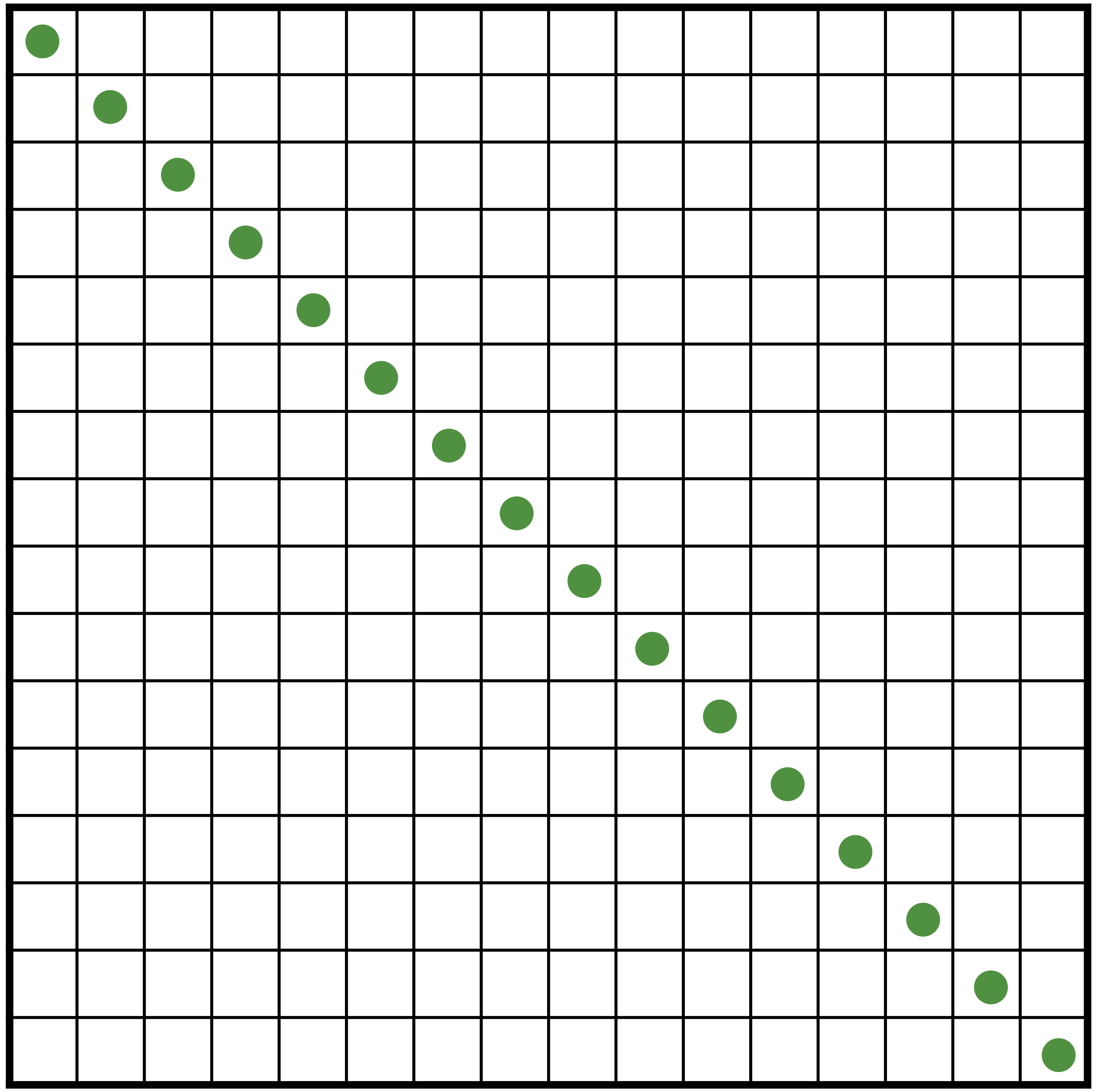
Latin Hypercube Sampler (N-rooks)

Initialize

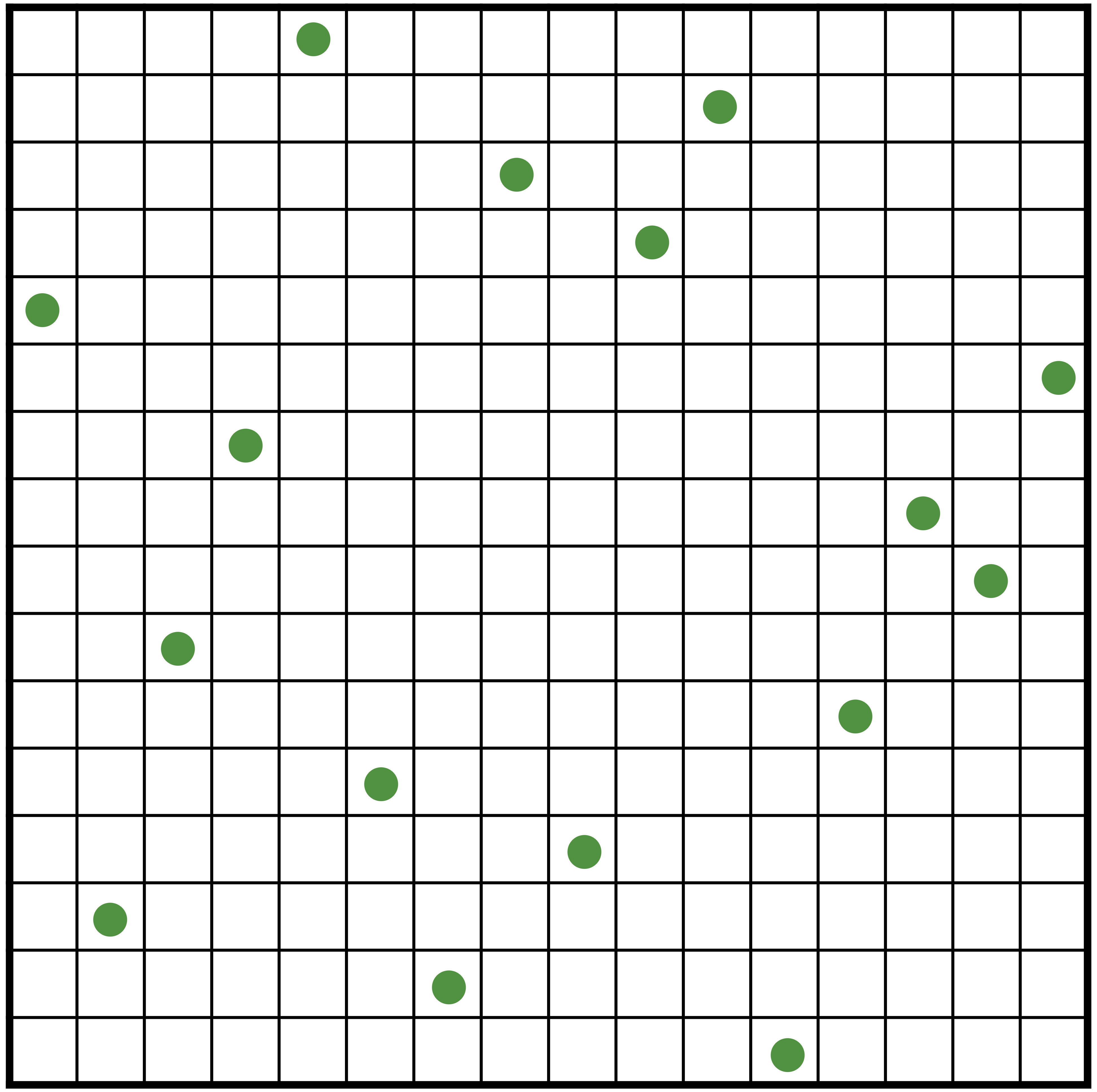


Latin Hypercube Sampler (N-rooks)

Shuffle rows

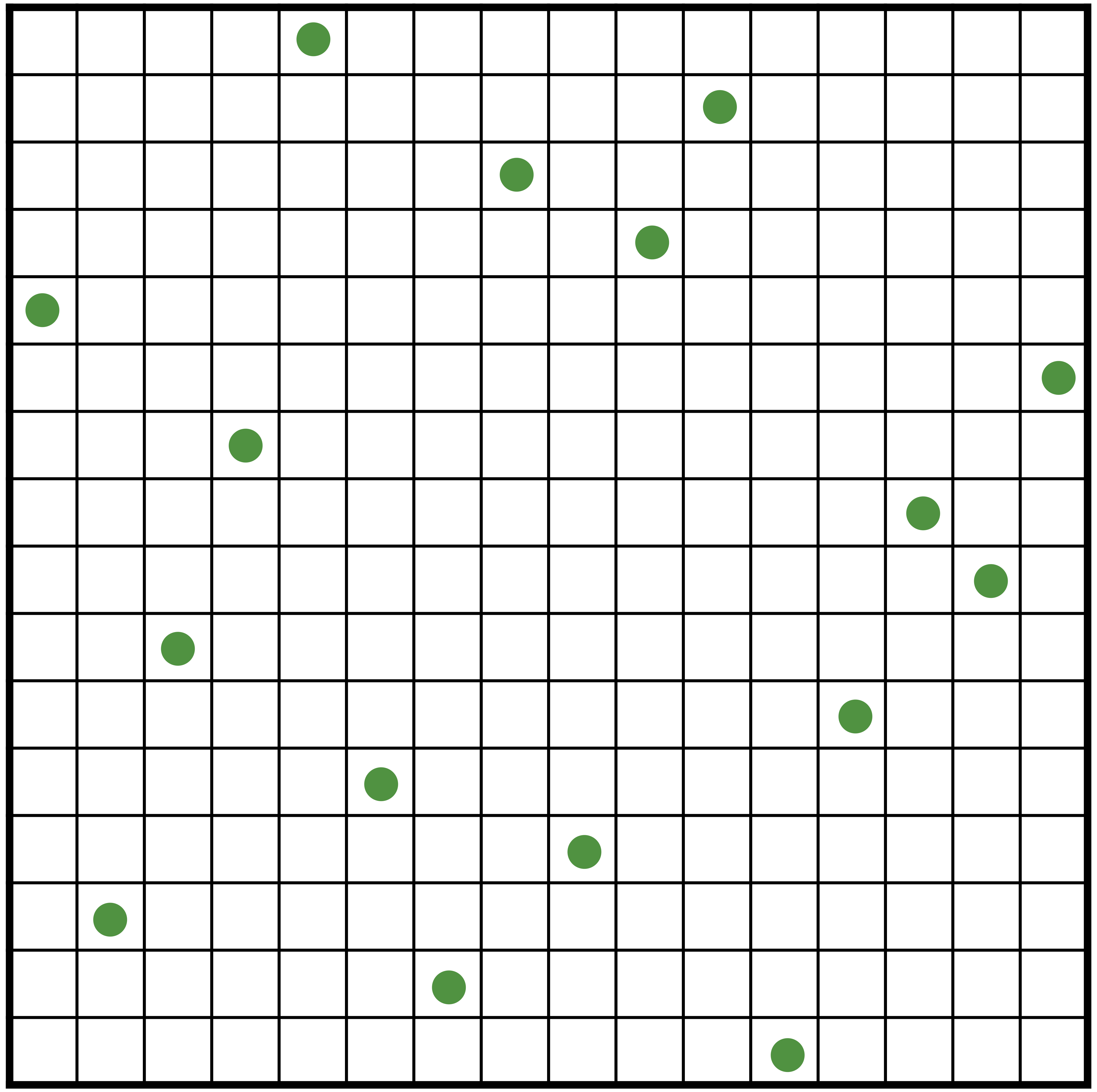


Latin Hypercube Sampler (N-rooks)

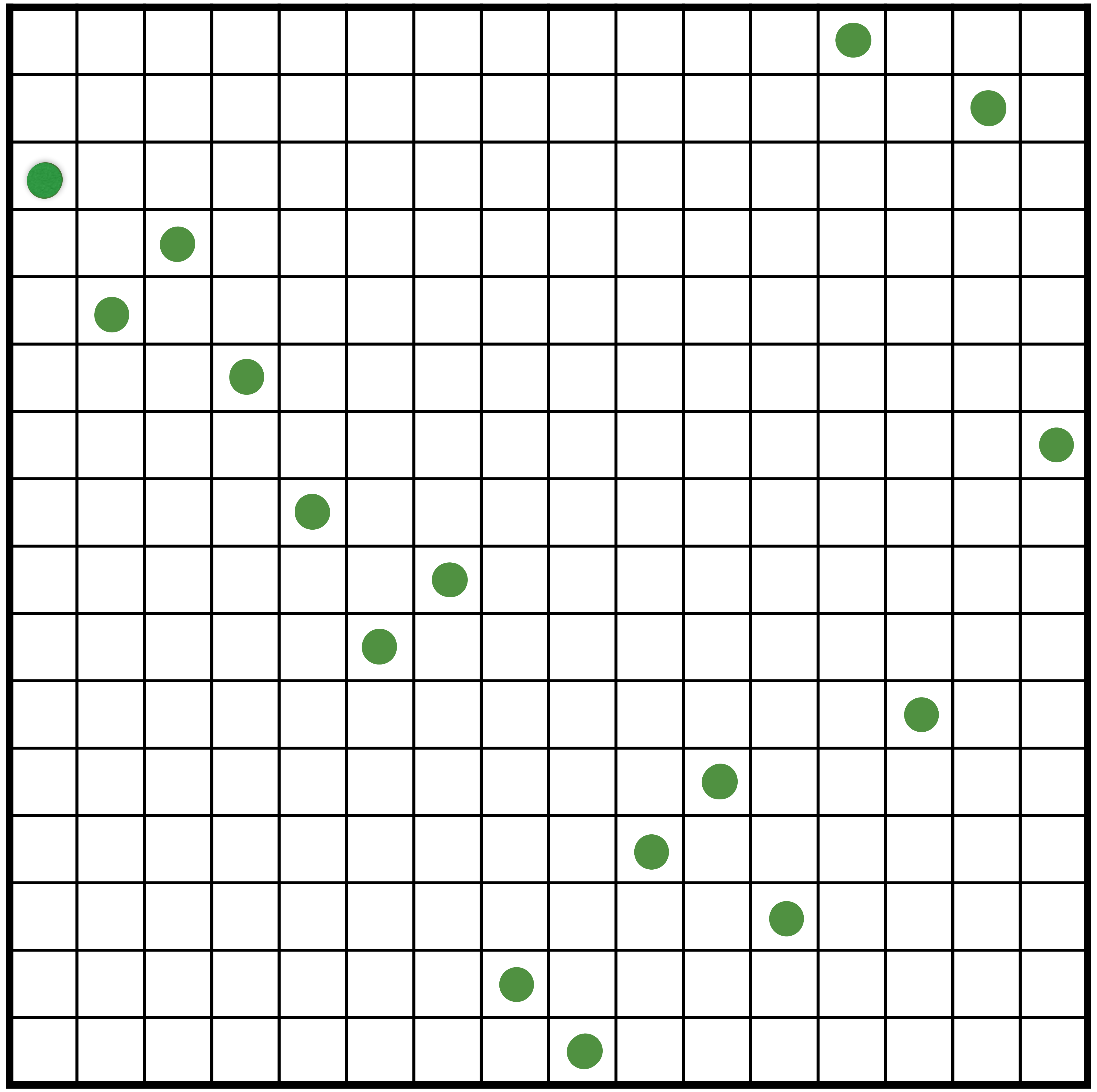


Latin Hypercube Sampler (N-rooks)

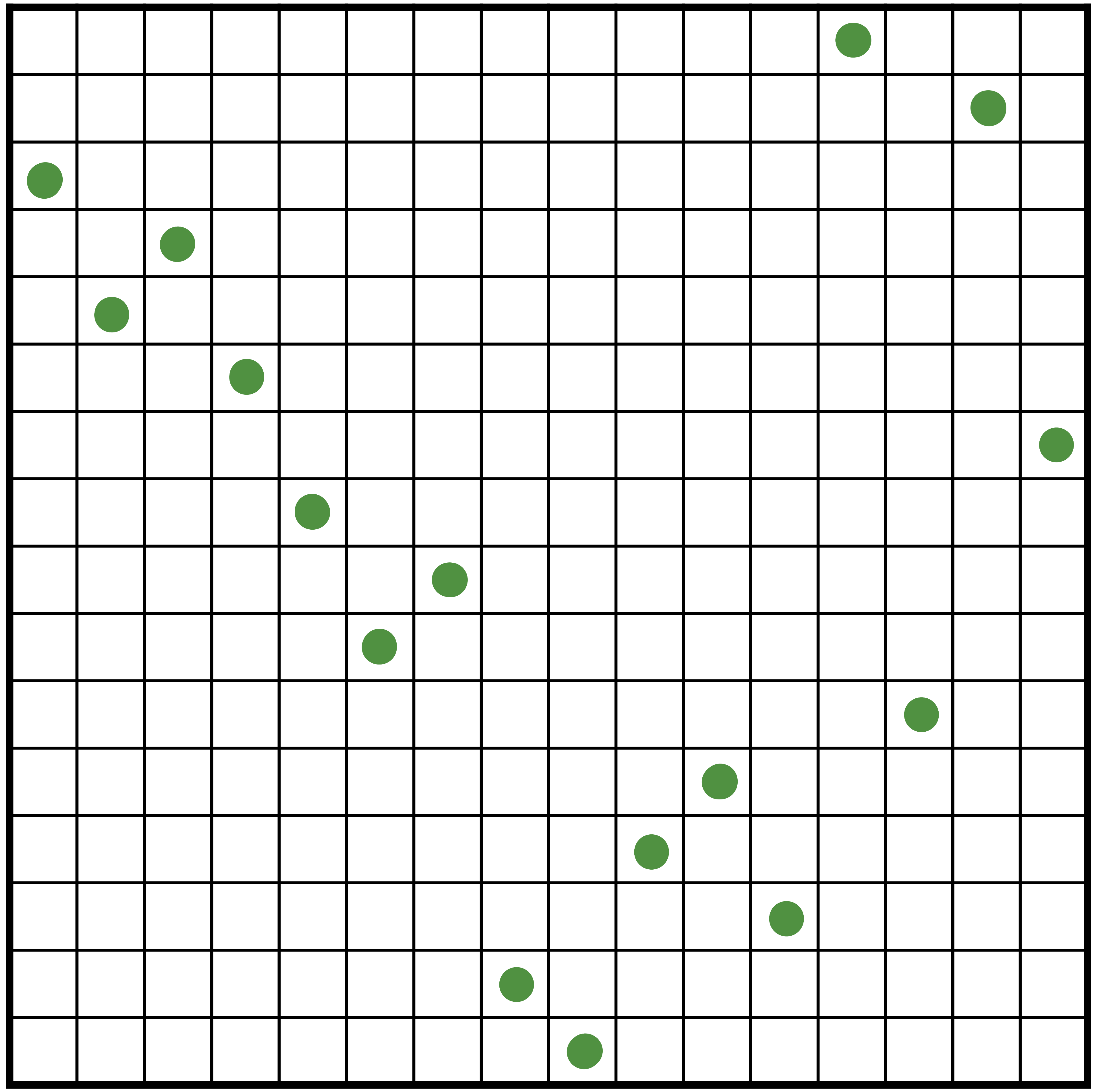
Shuffle columns



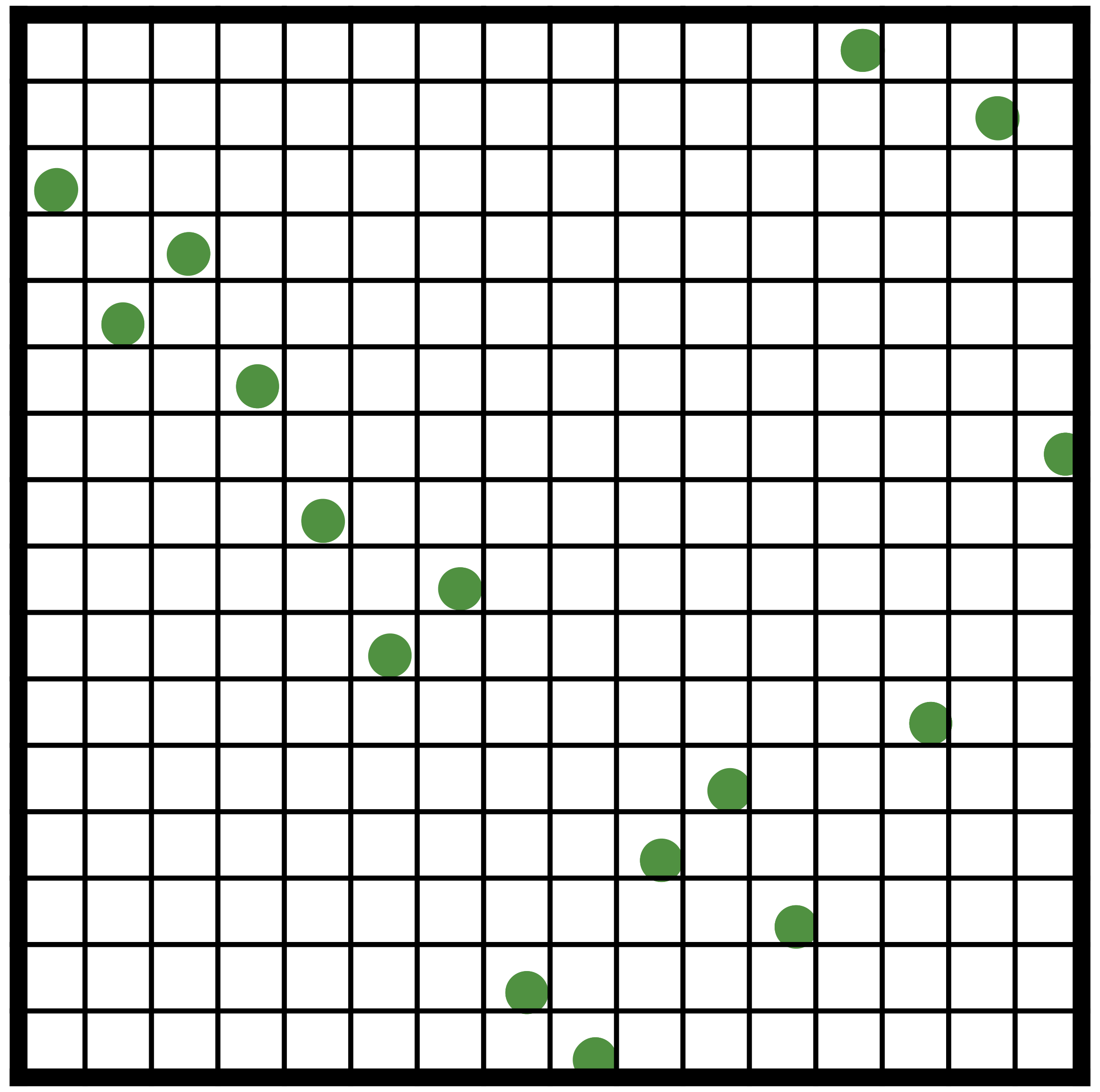
Latin Hypercube Sampler (N-rooks)



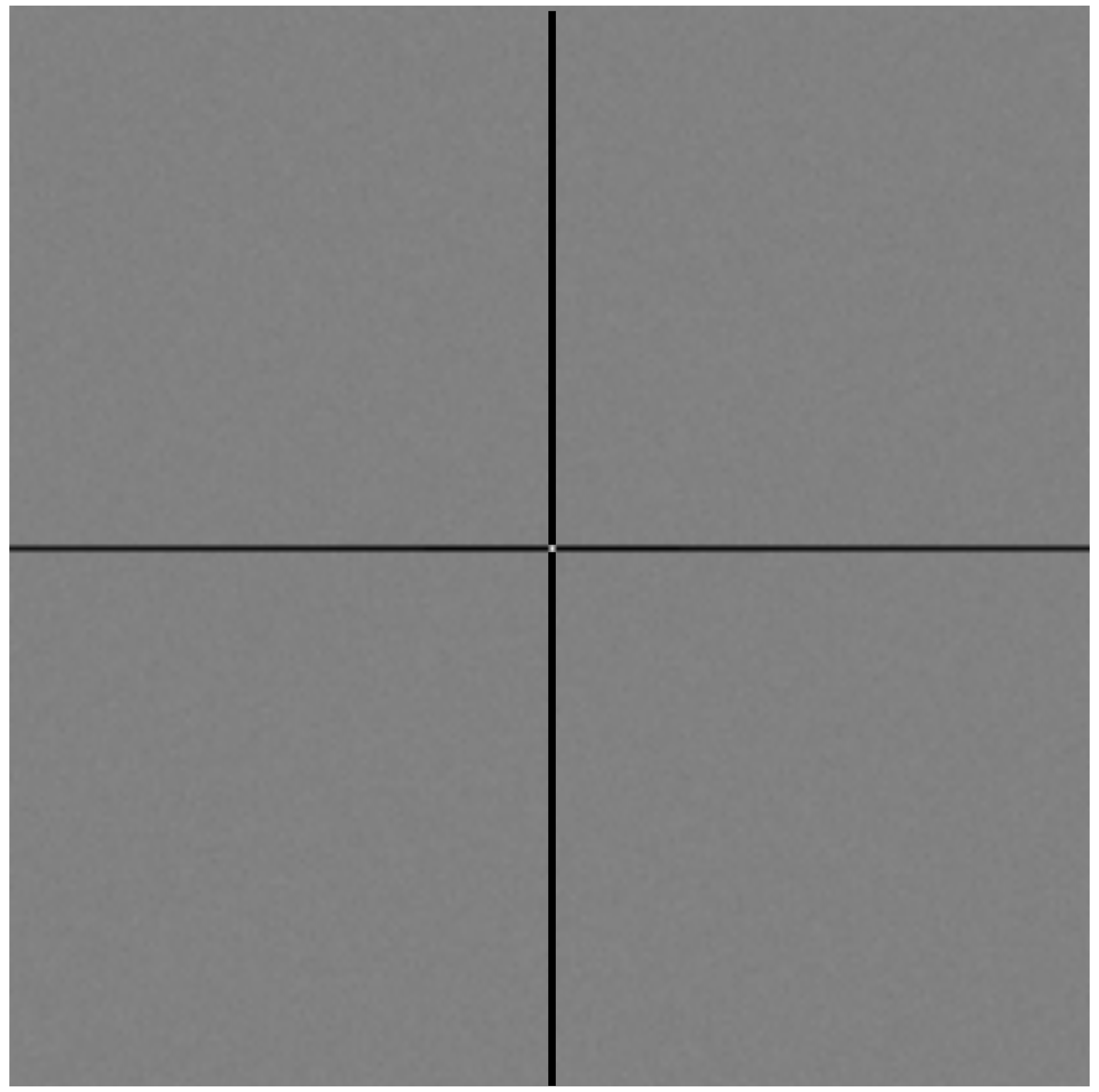
Latin Hypercube Sampler (N-rooks)



Anisotropic Sampling Power Spectra

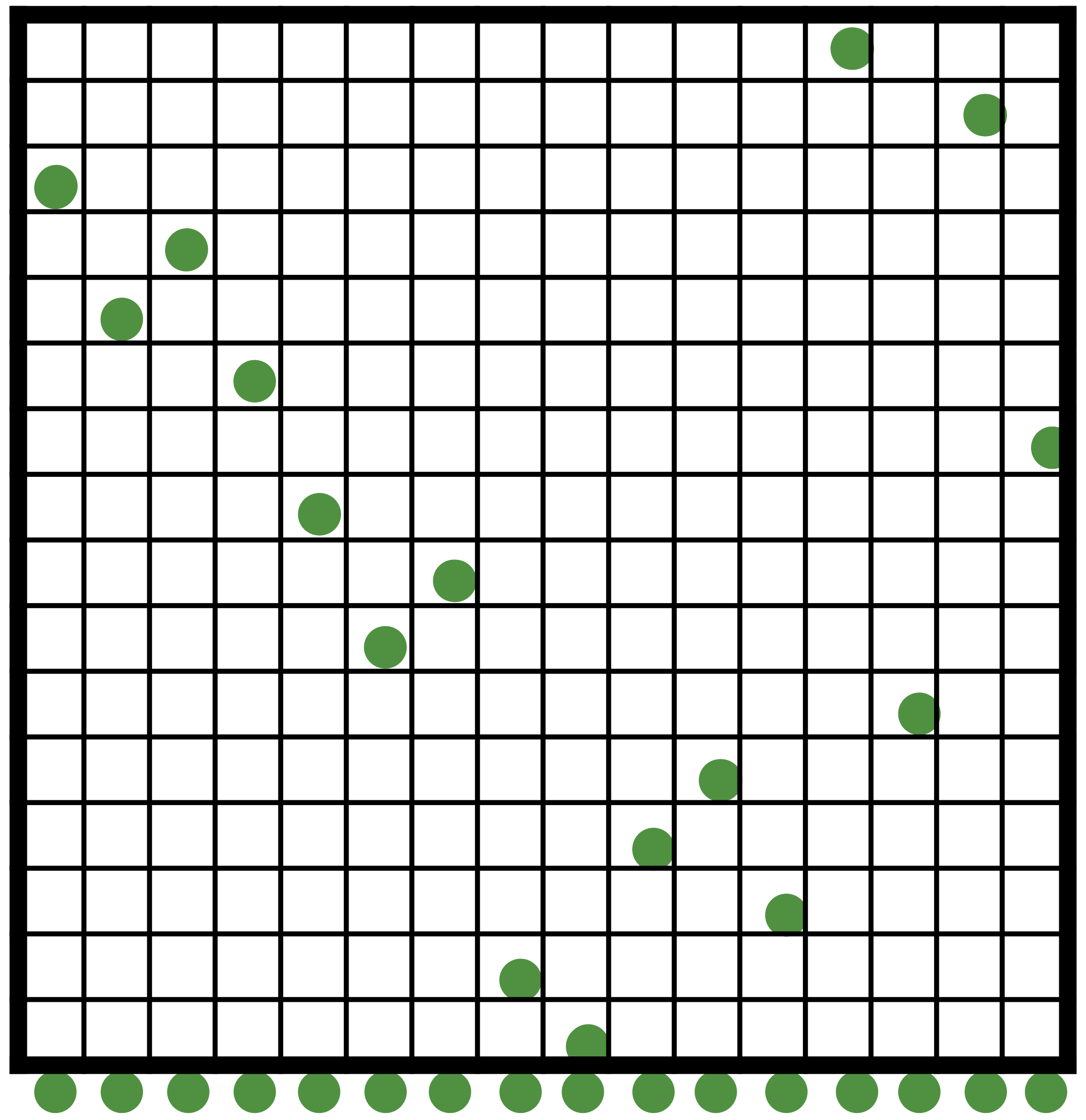


N-rooks /
Latin Hypercube

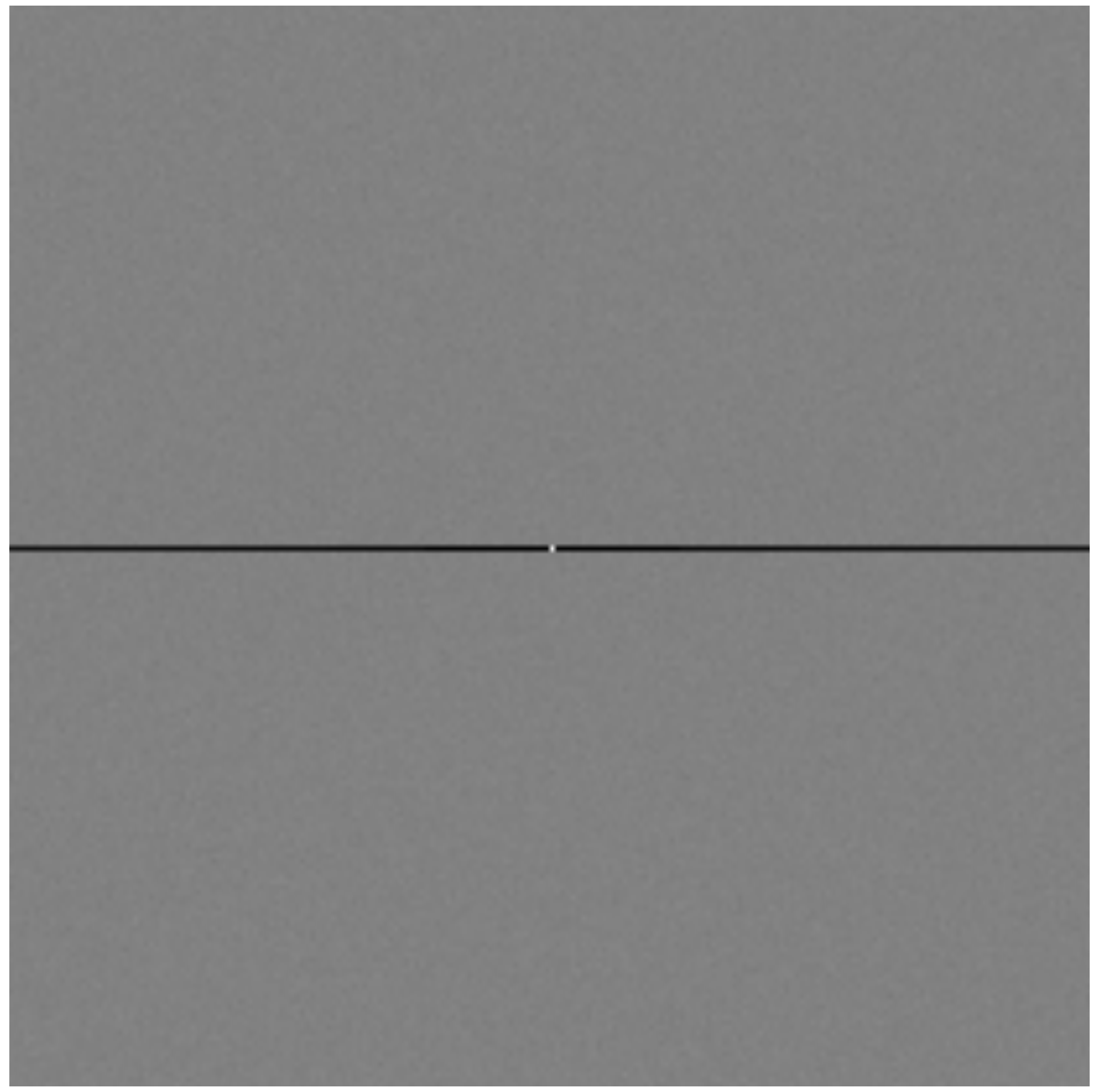


N-rooks
Spectrum

Anisotropic Sampling Power Spectra

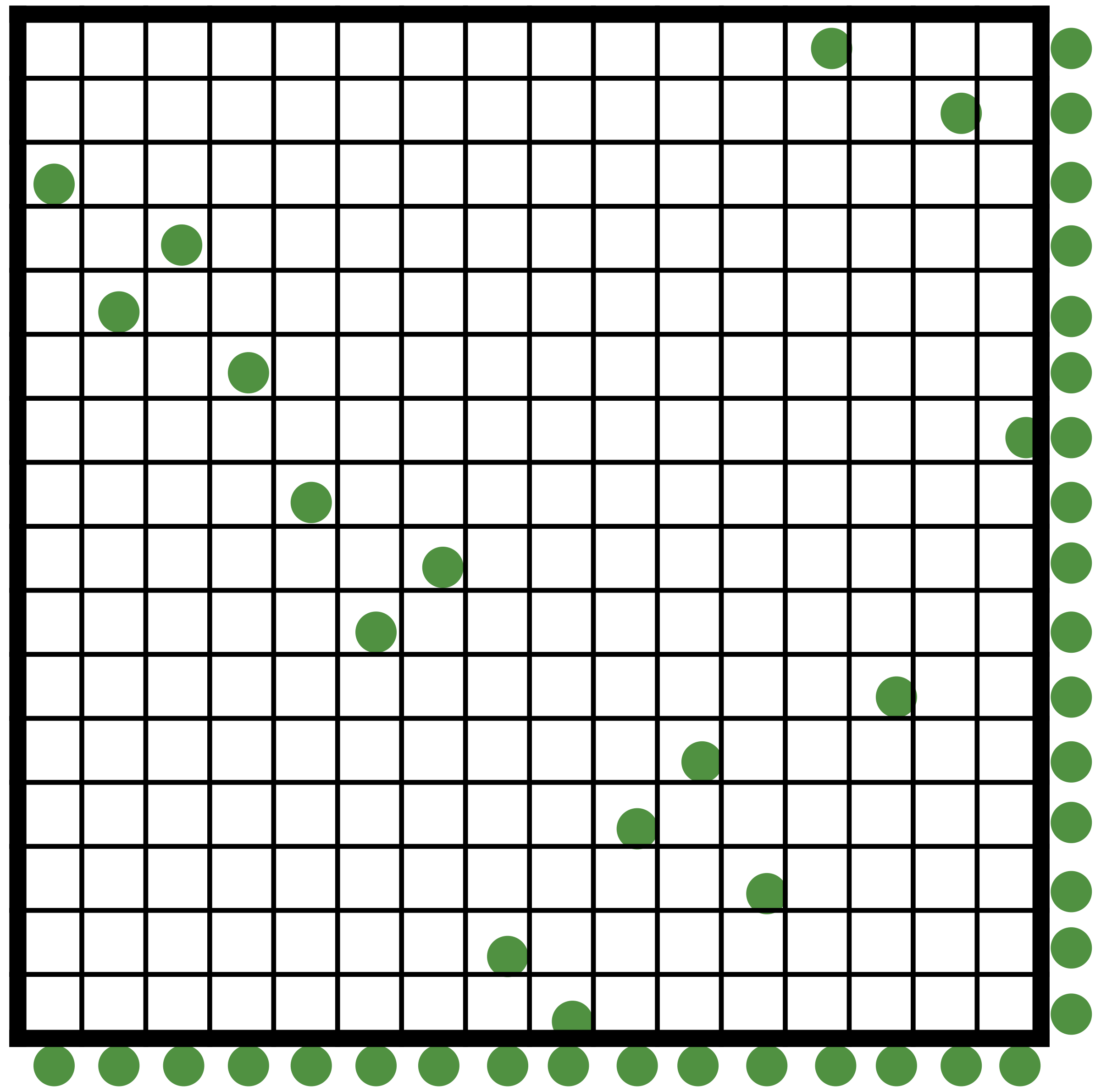


N-rooks /
Latin Hypercube

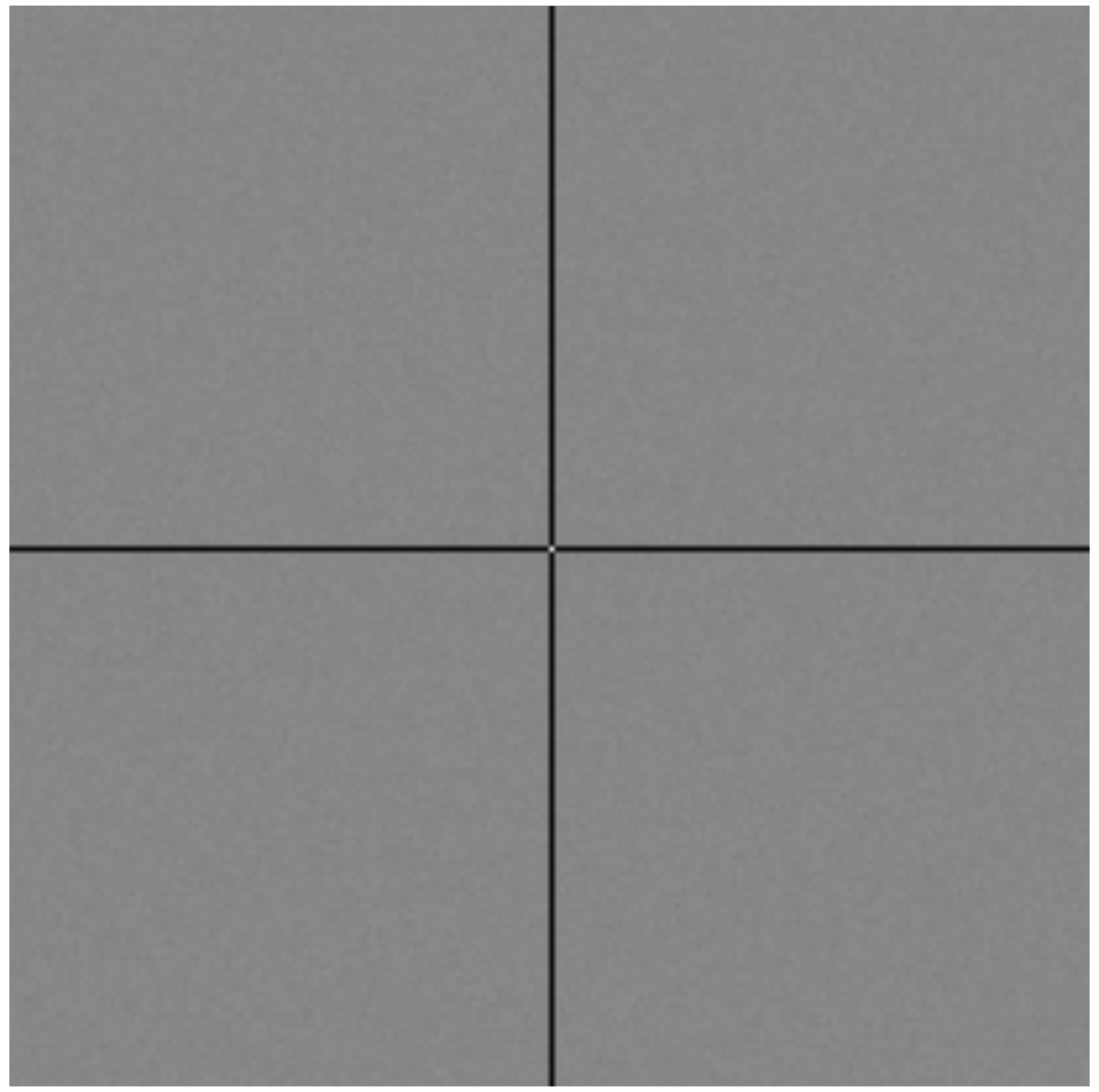


Spectrum

Anisotropic Sampling Power Spectra

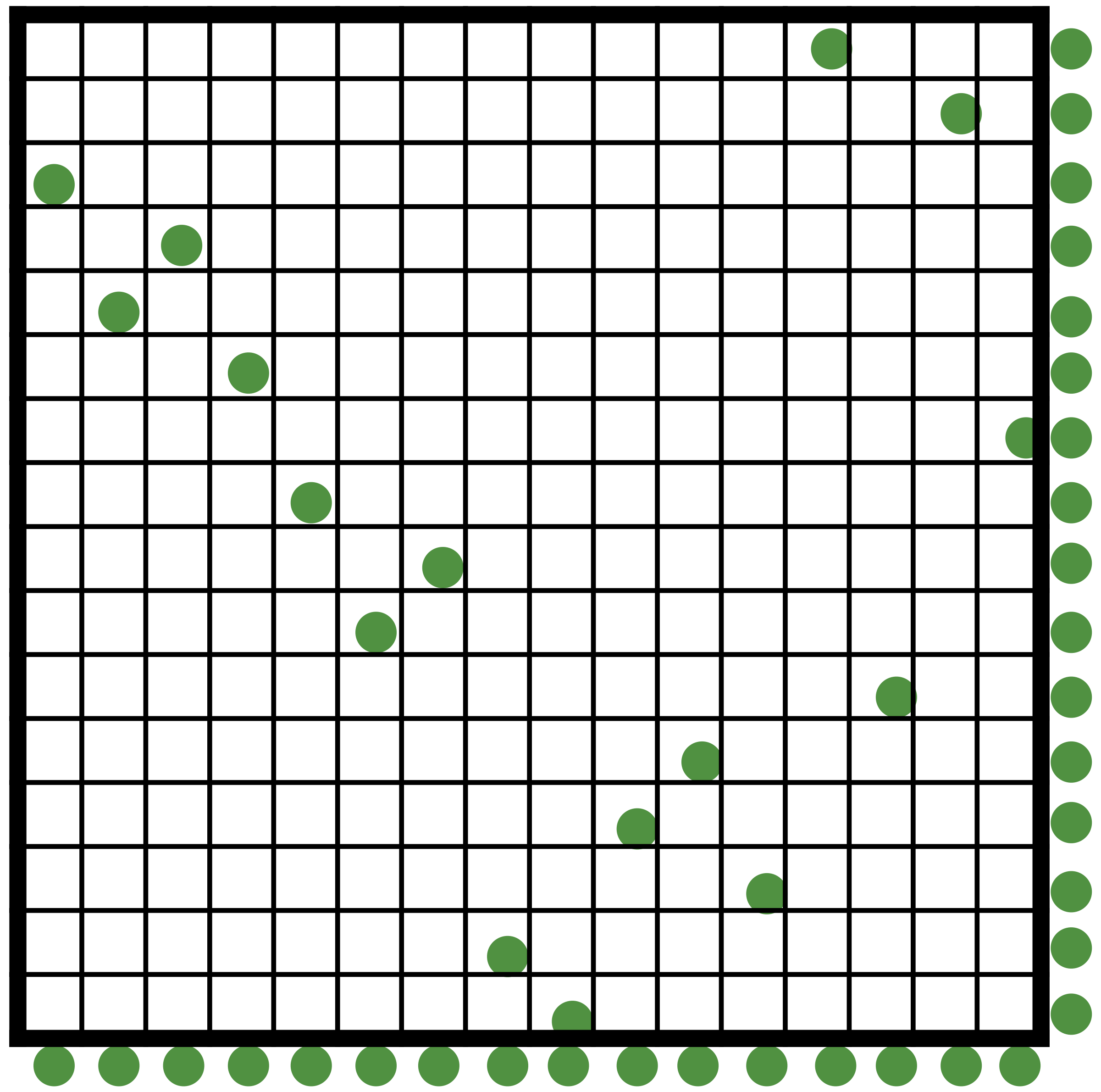


N-rooks /
Latin Hypercube

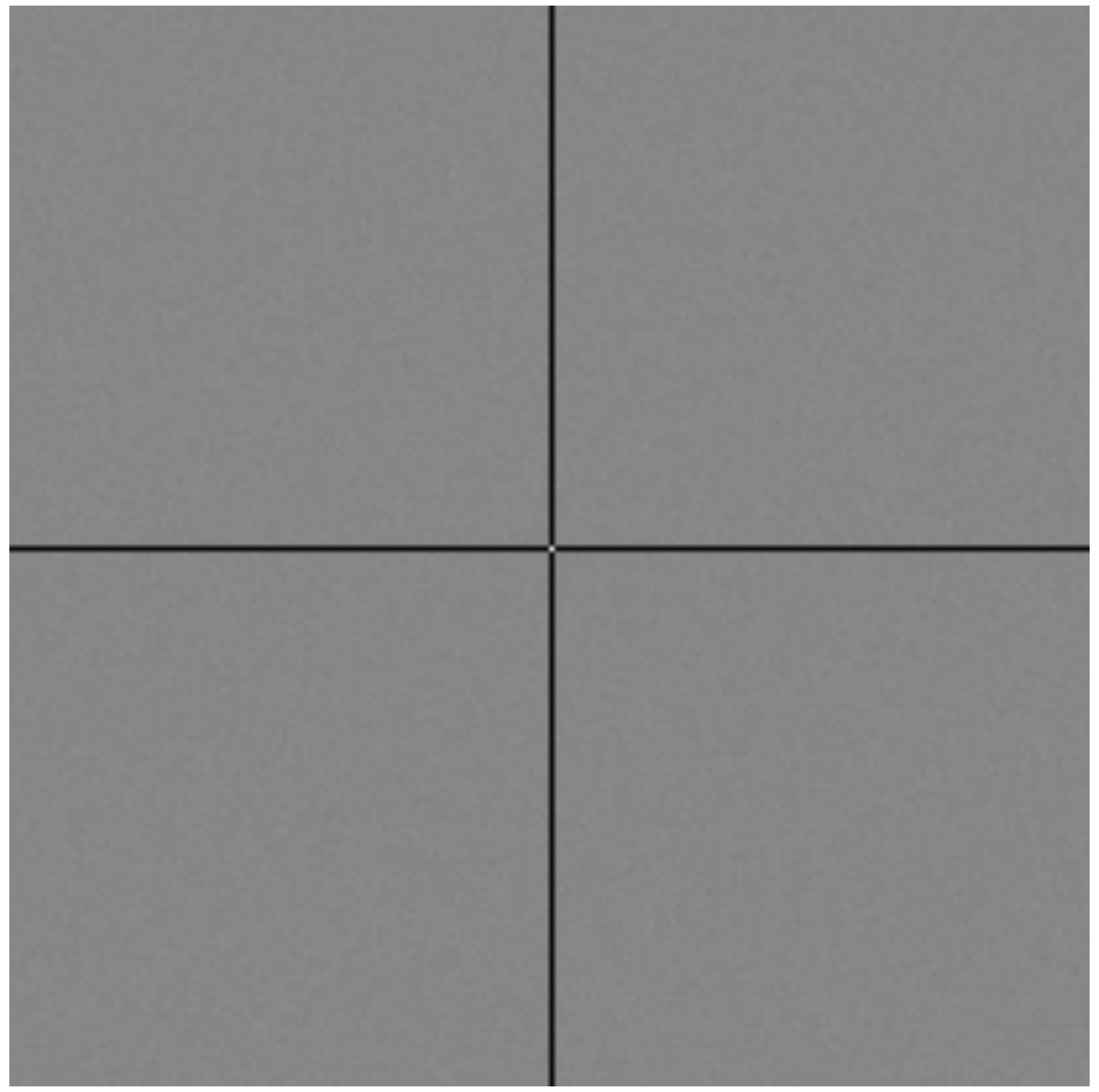


N-rooks
Spectrum

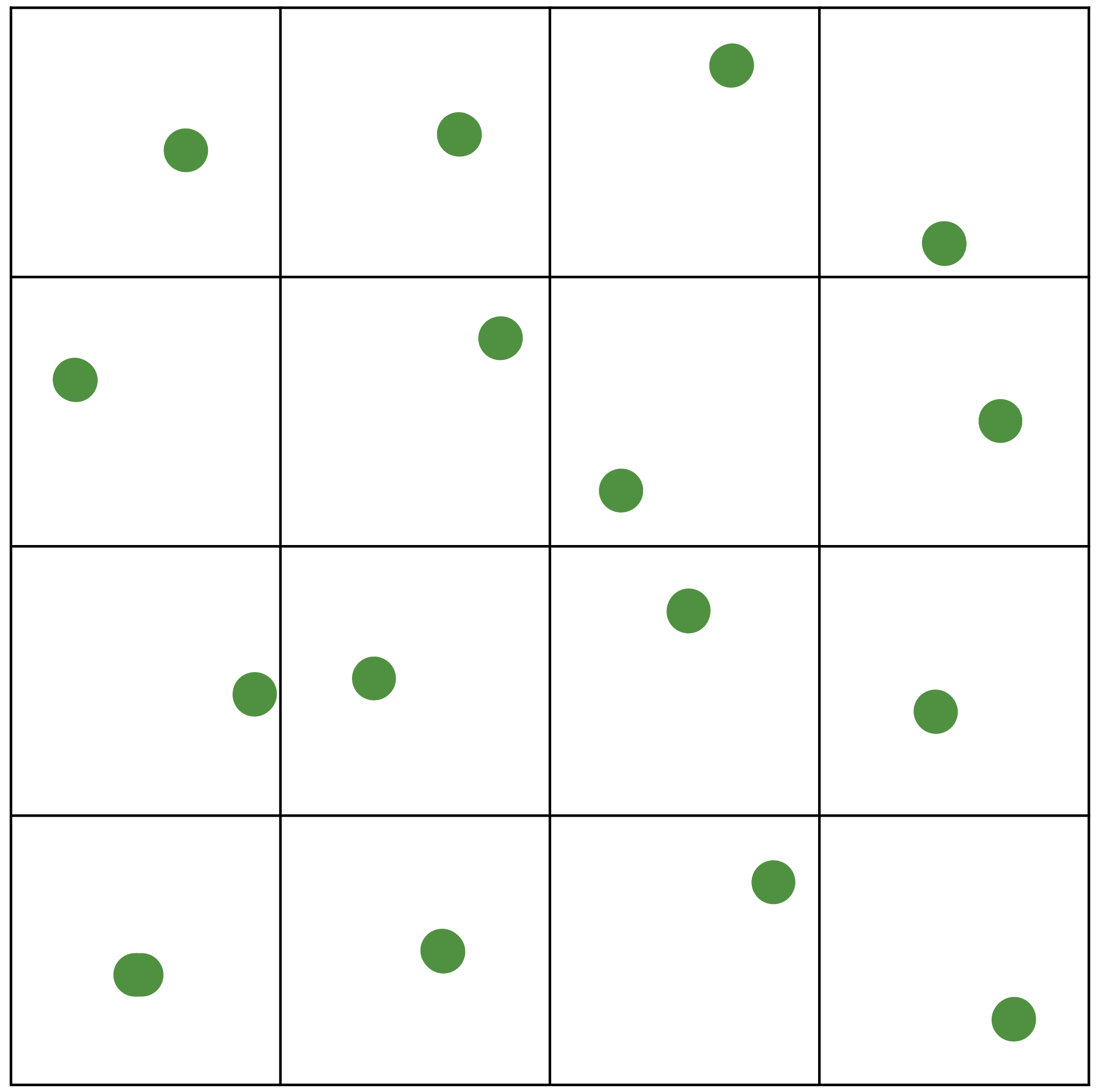
Anisotropic Sampling Power Spectra



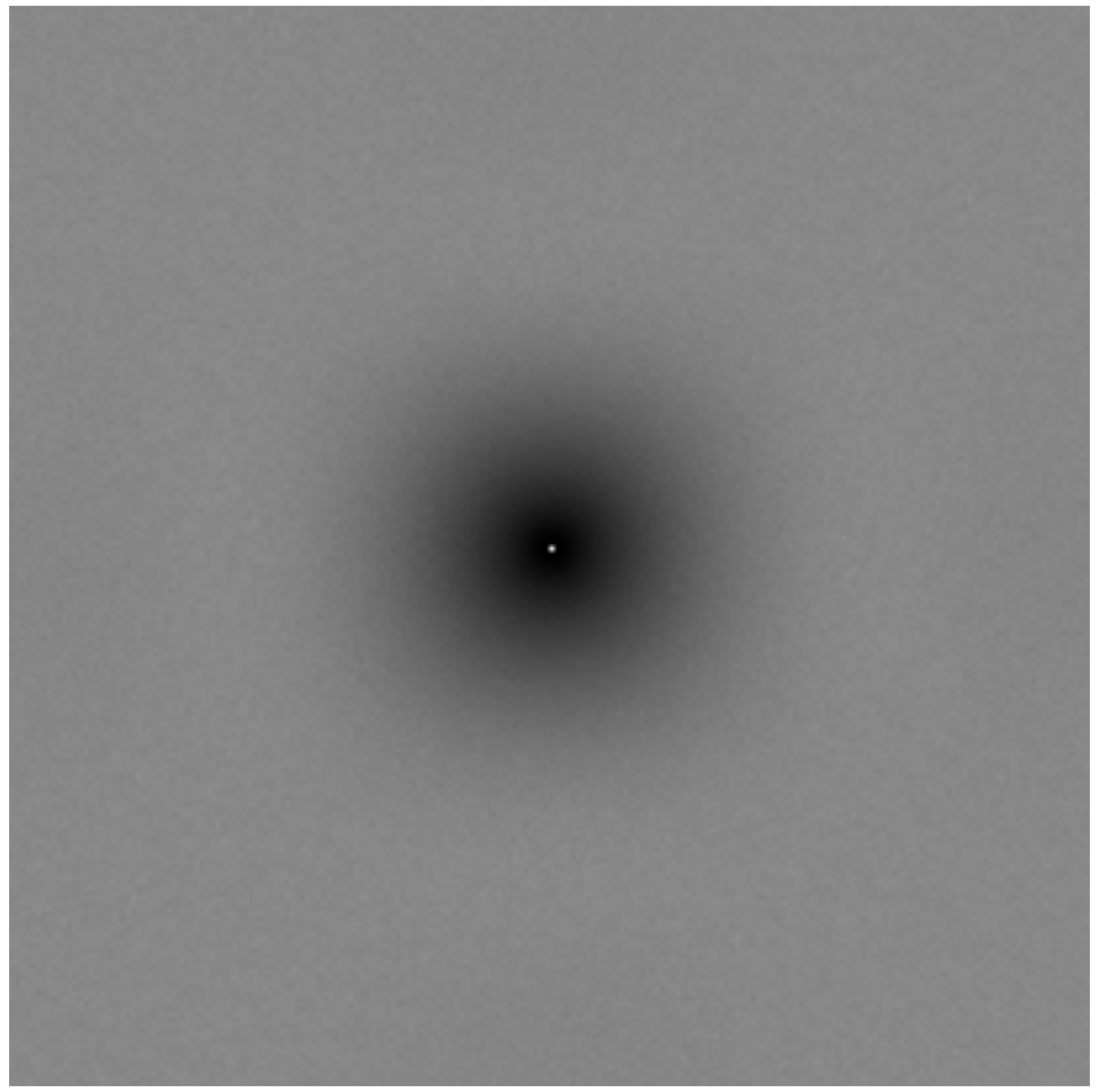
N-rooks /
Latin Hypercube



N-rooks
Spectrum

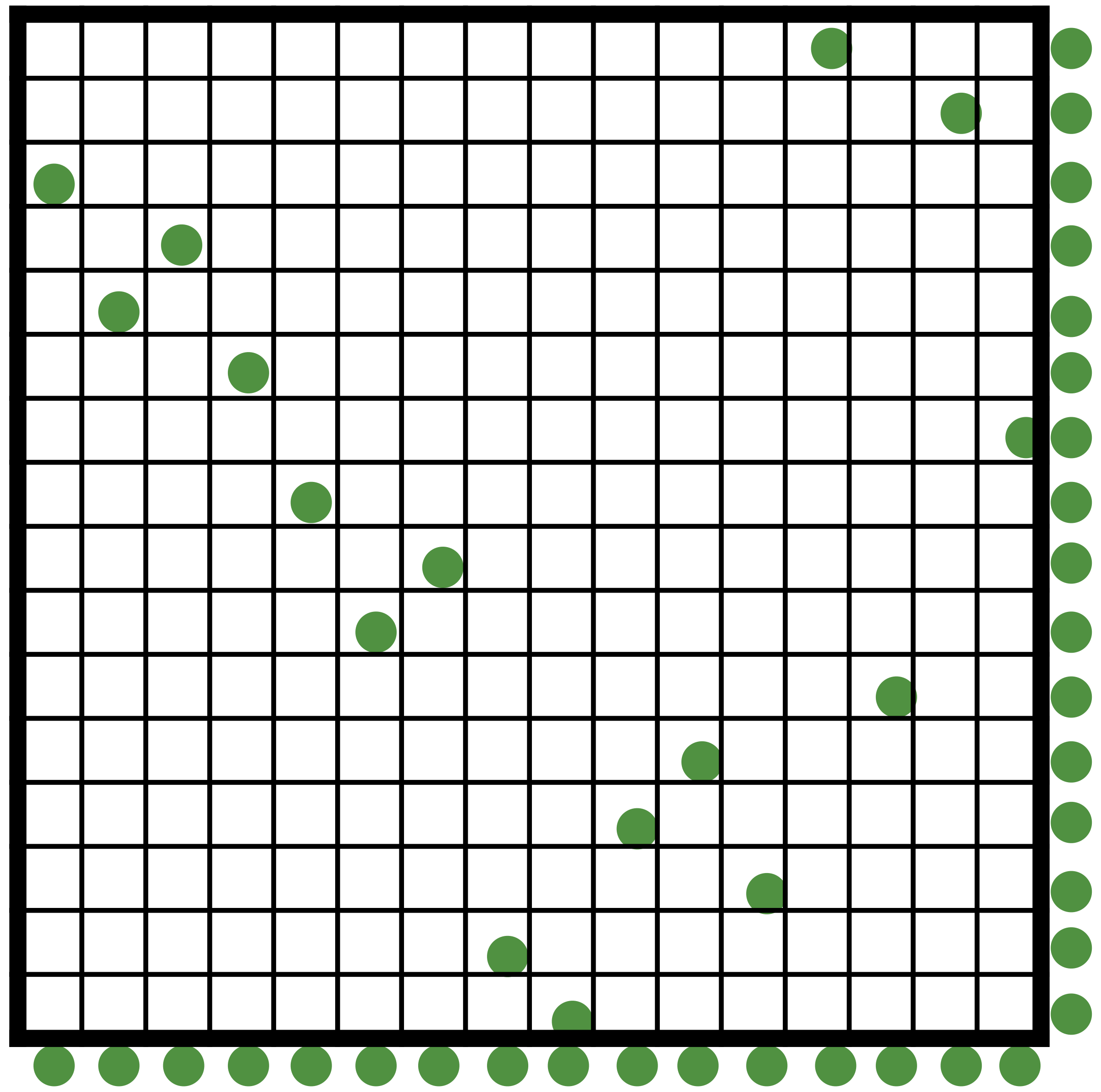


Jitter

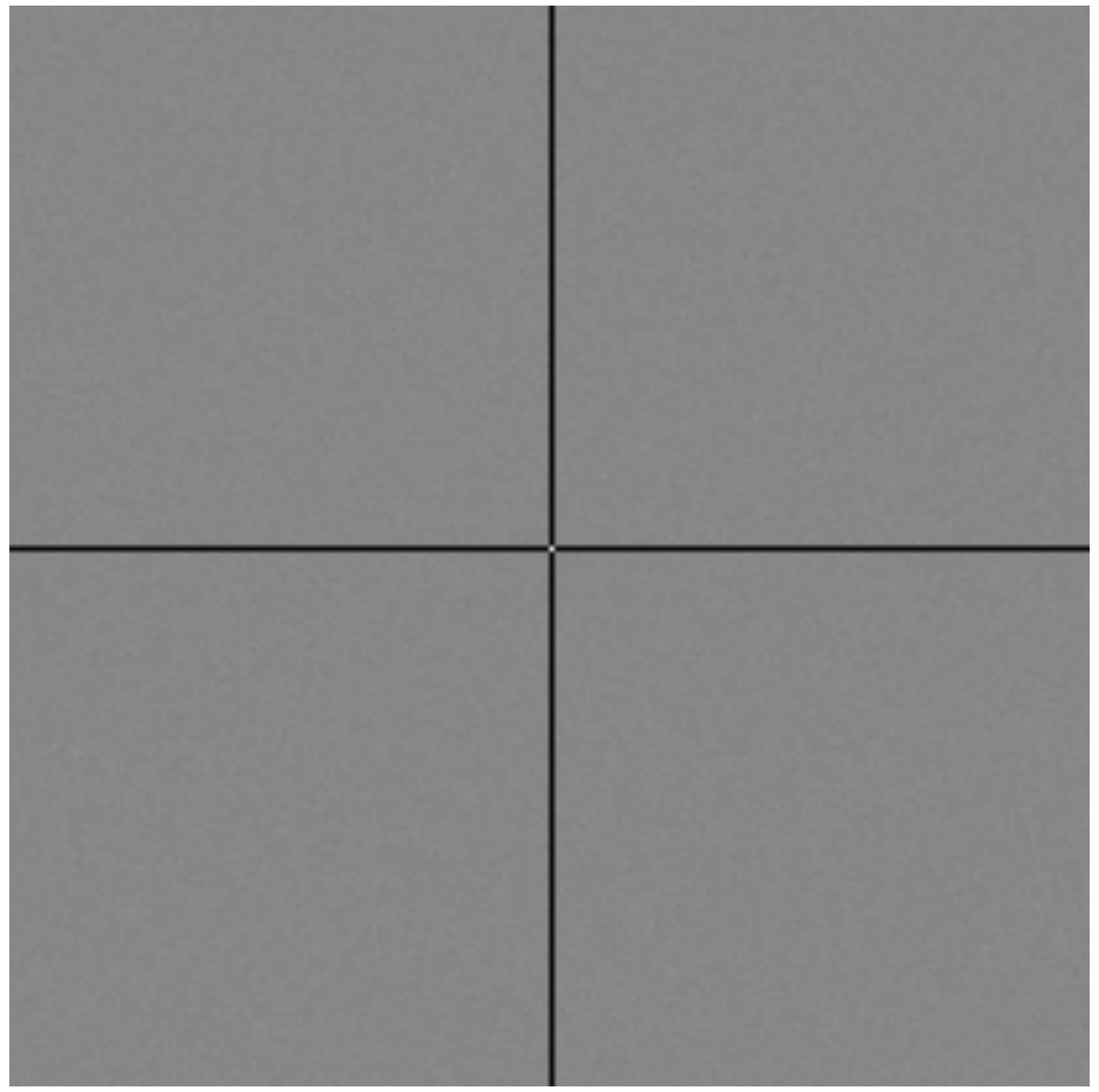


Jitter
Spectrum

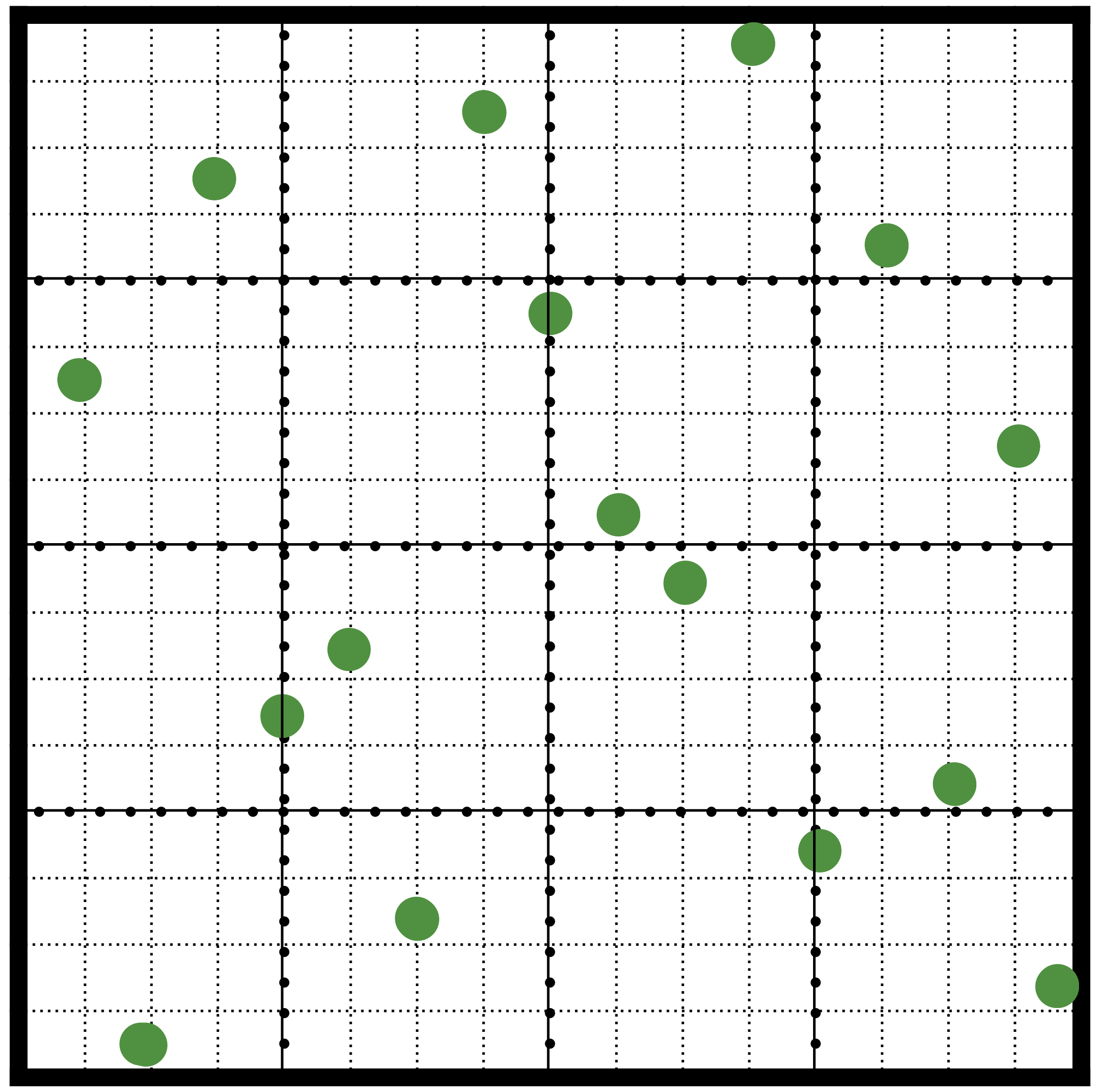
Anisotropic Sampling Power Spectra



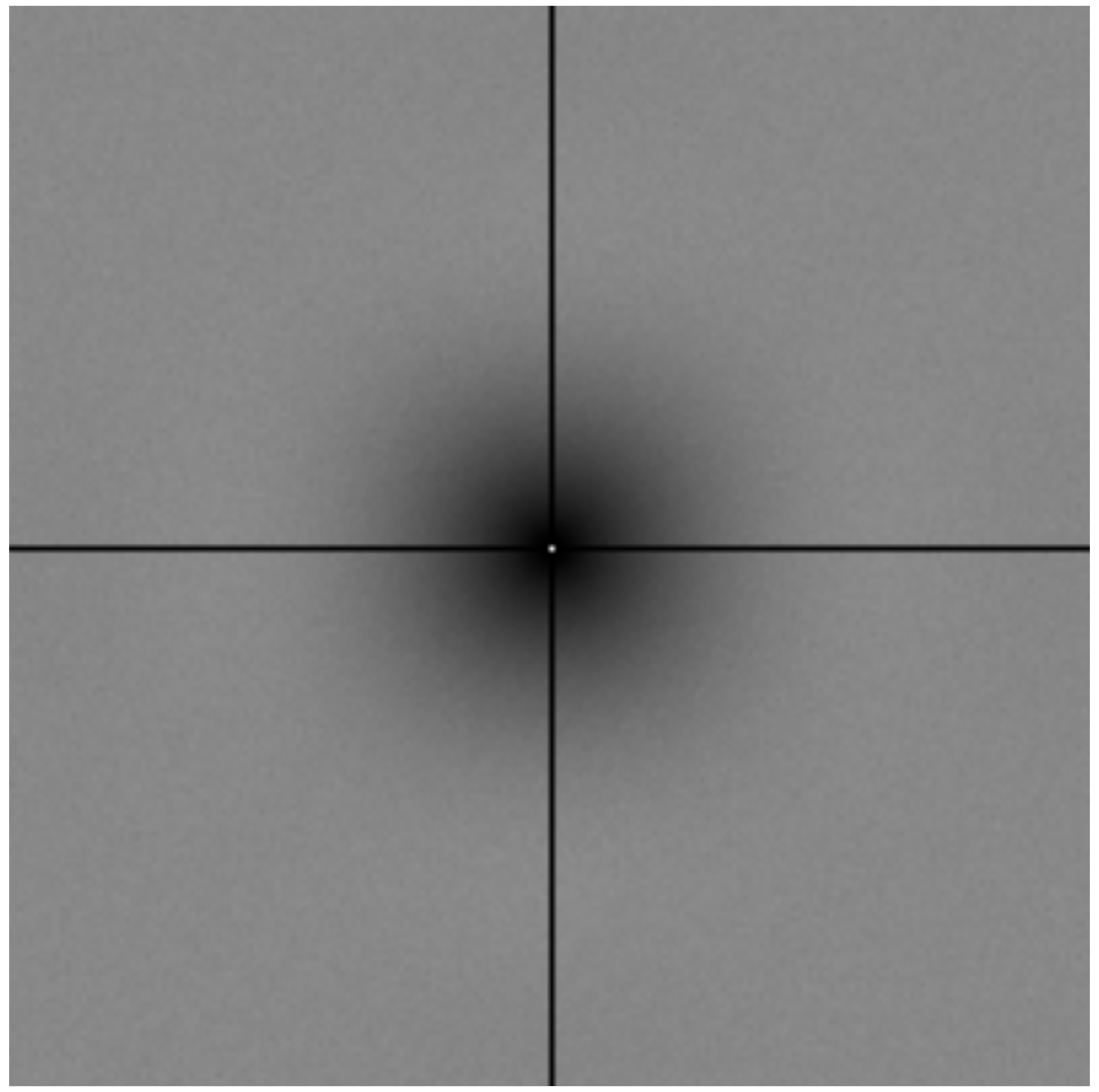
N-rooks / Latin Hypercube



N-rooks Spectrum



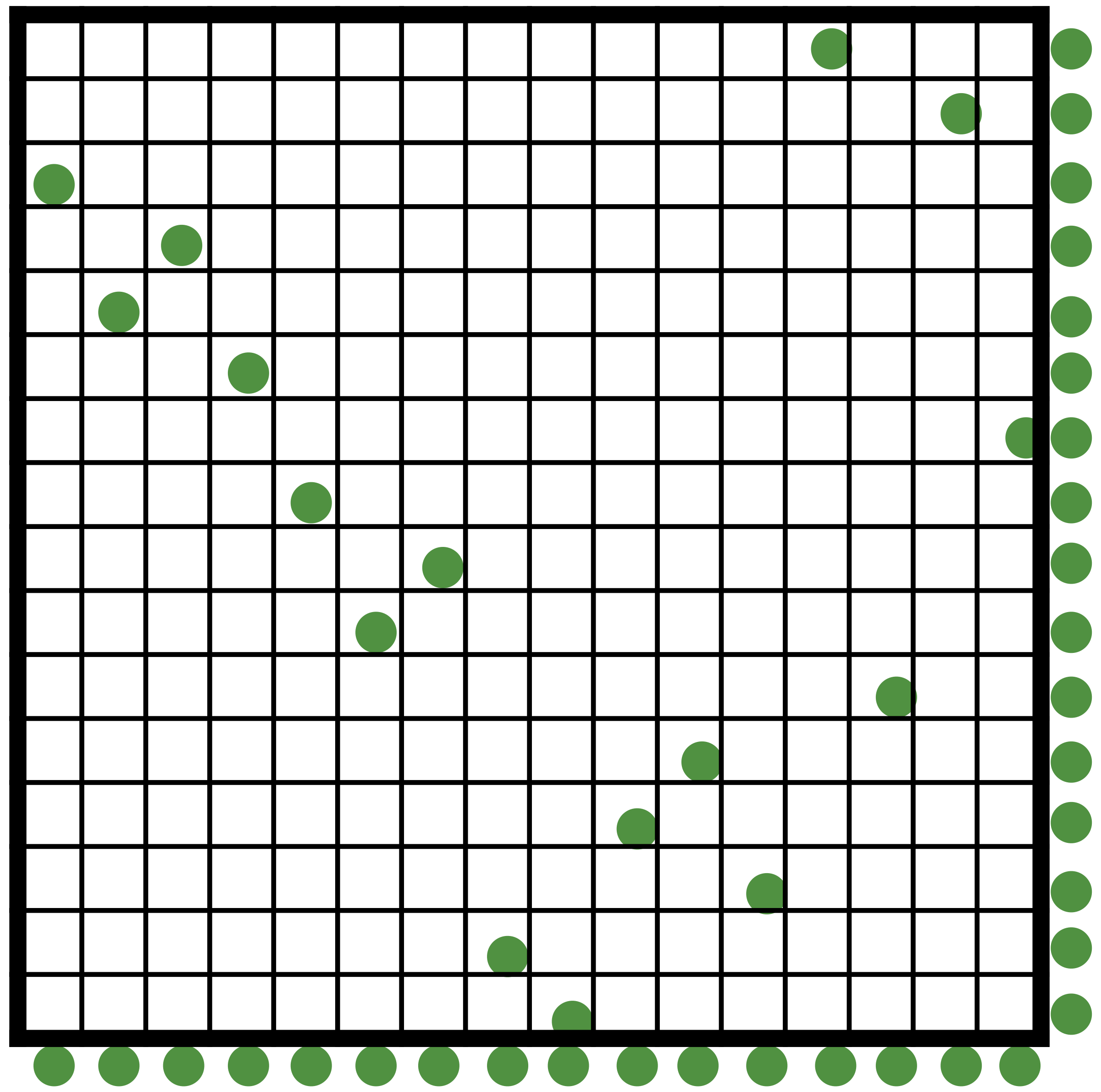
Multi-Jitter



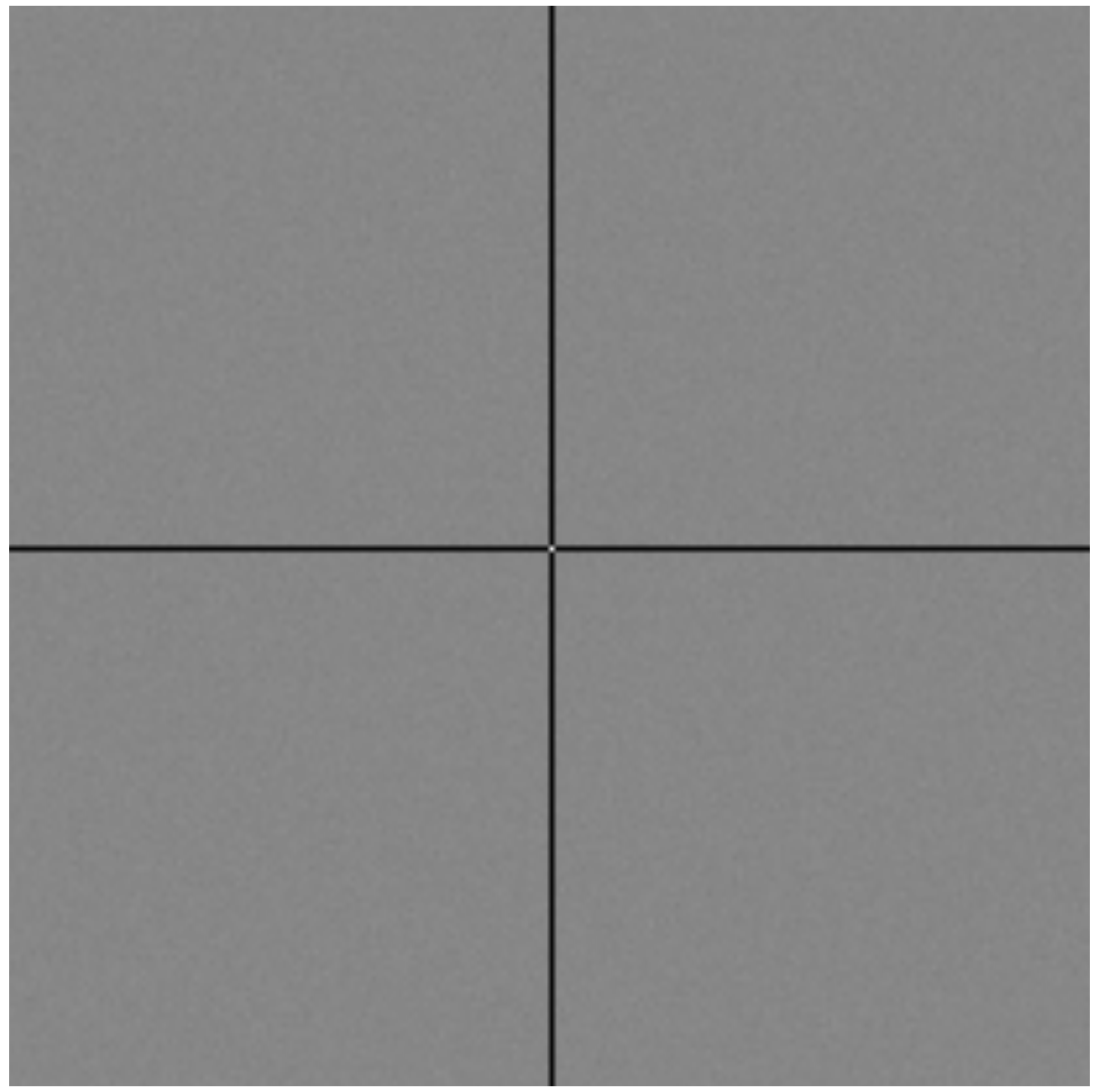
Multi-Jitter Spectrum

Chiu et al. [1993]

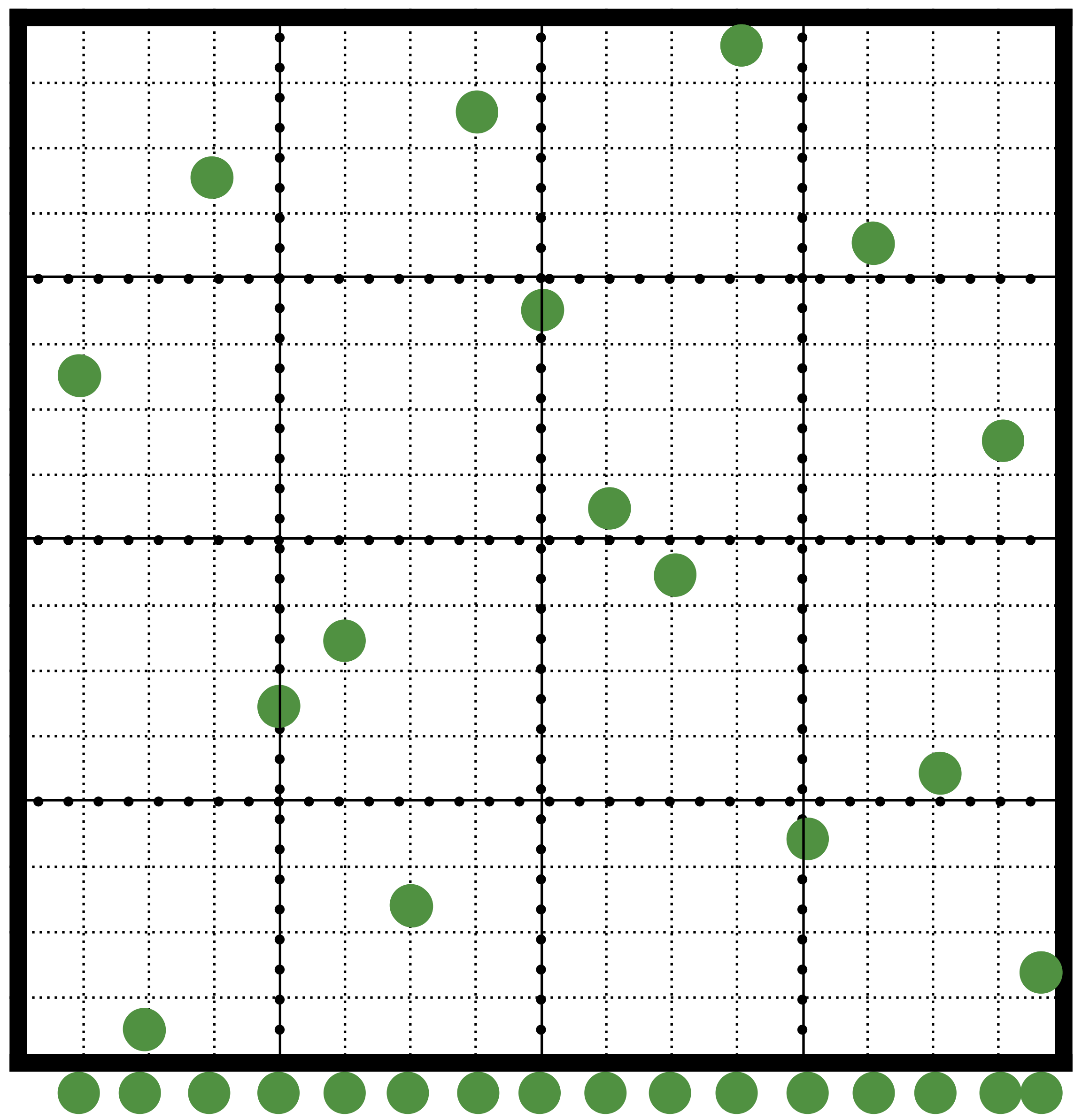
Anisotropic Sampling Power Spectra



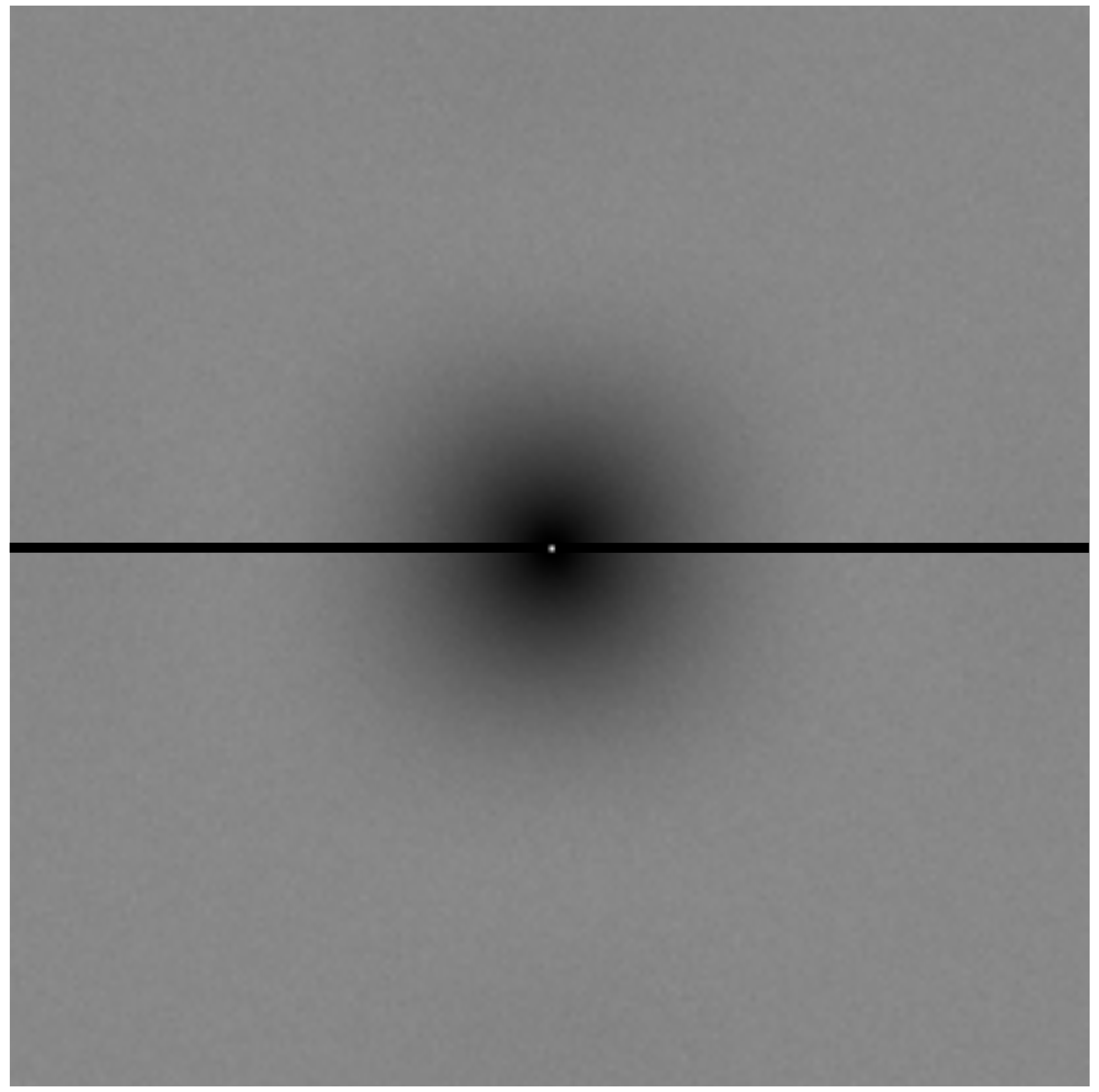
N-rooks / Latin Hypercube



N-rooks Spectrum



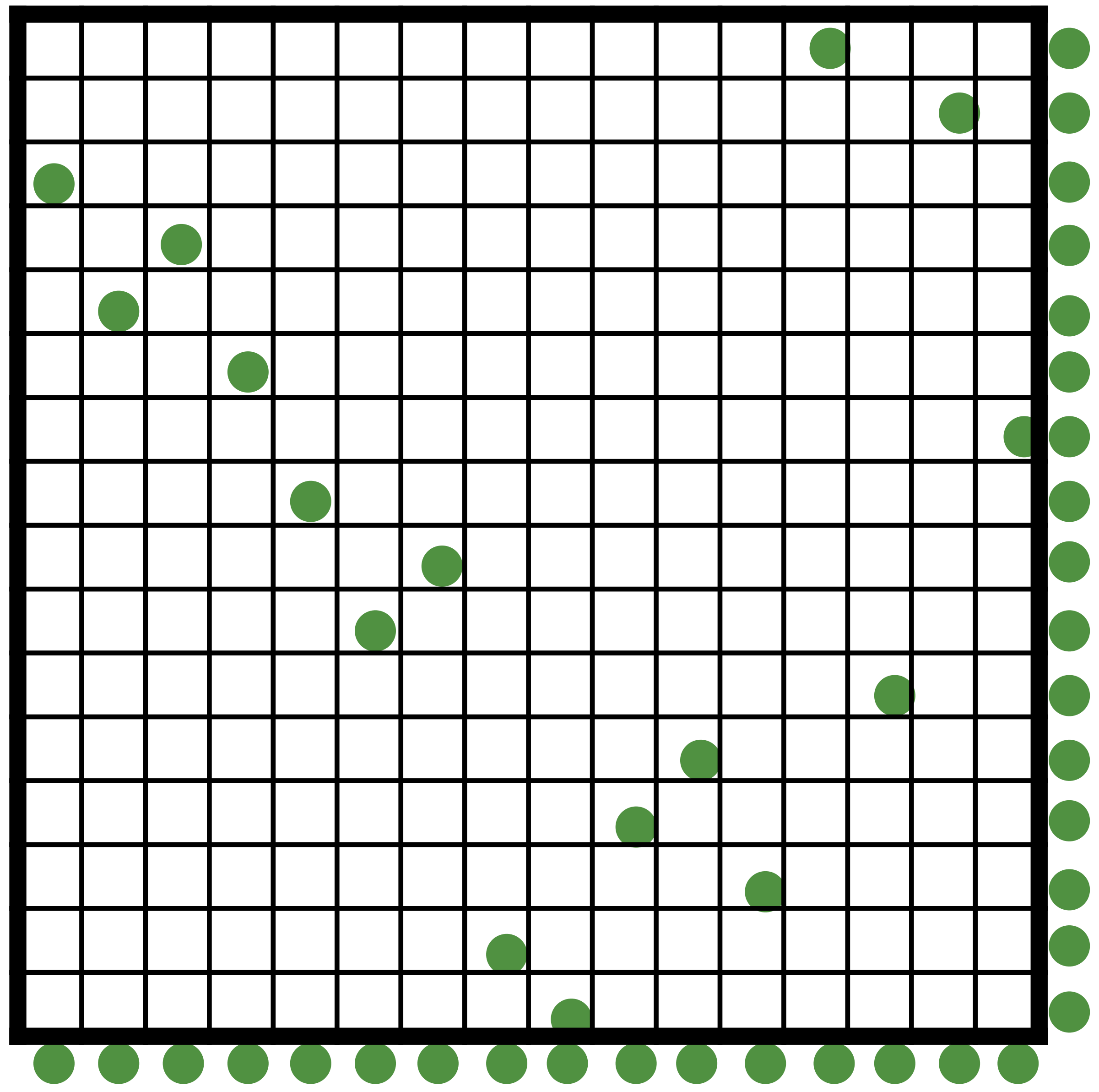
Multi-jitter



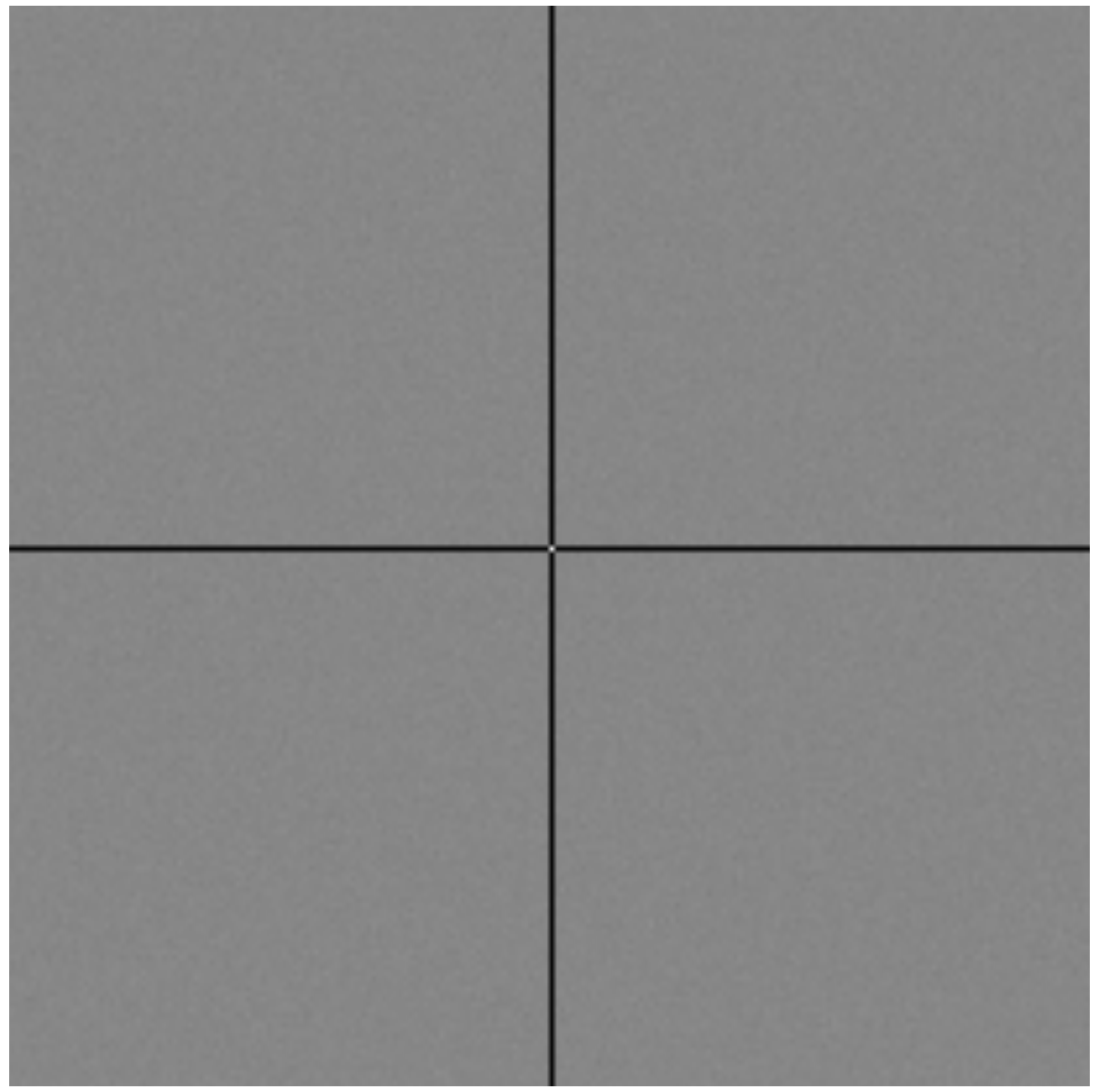
Multi-Jitter Spectrum

Chiu et al. [1993]

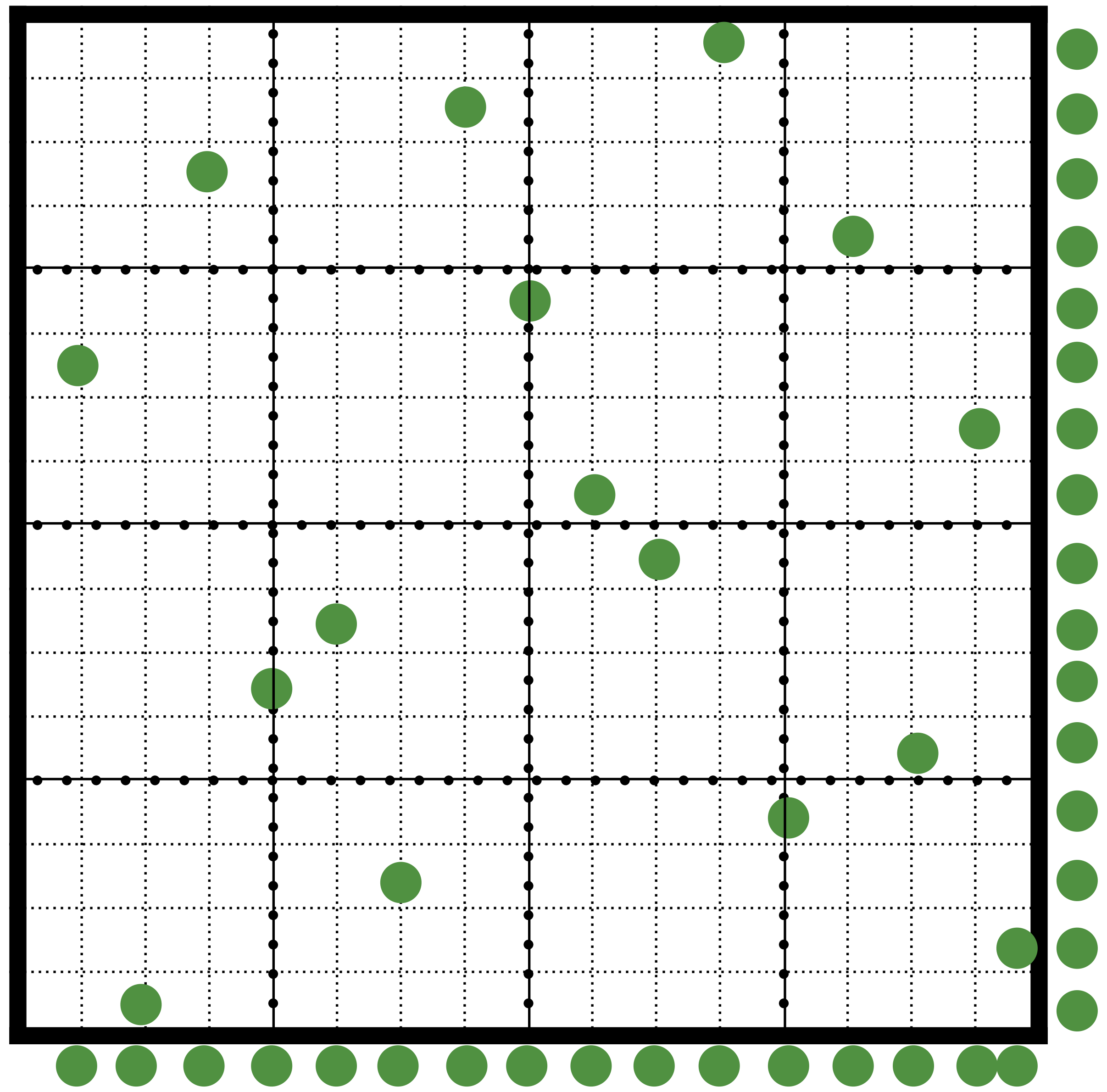
Anisotropic Sampling Power Spectra



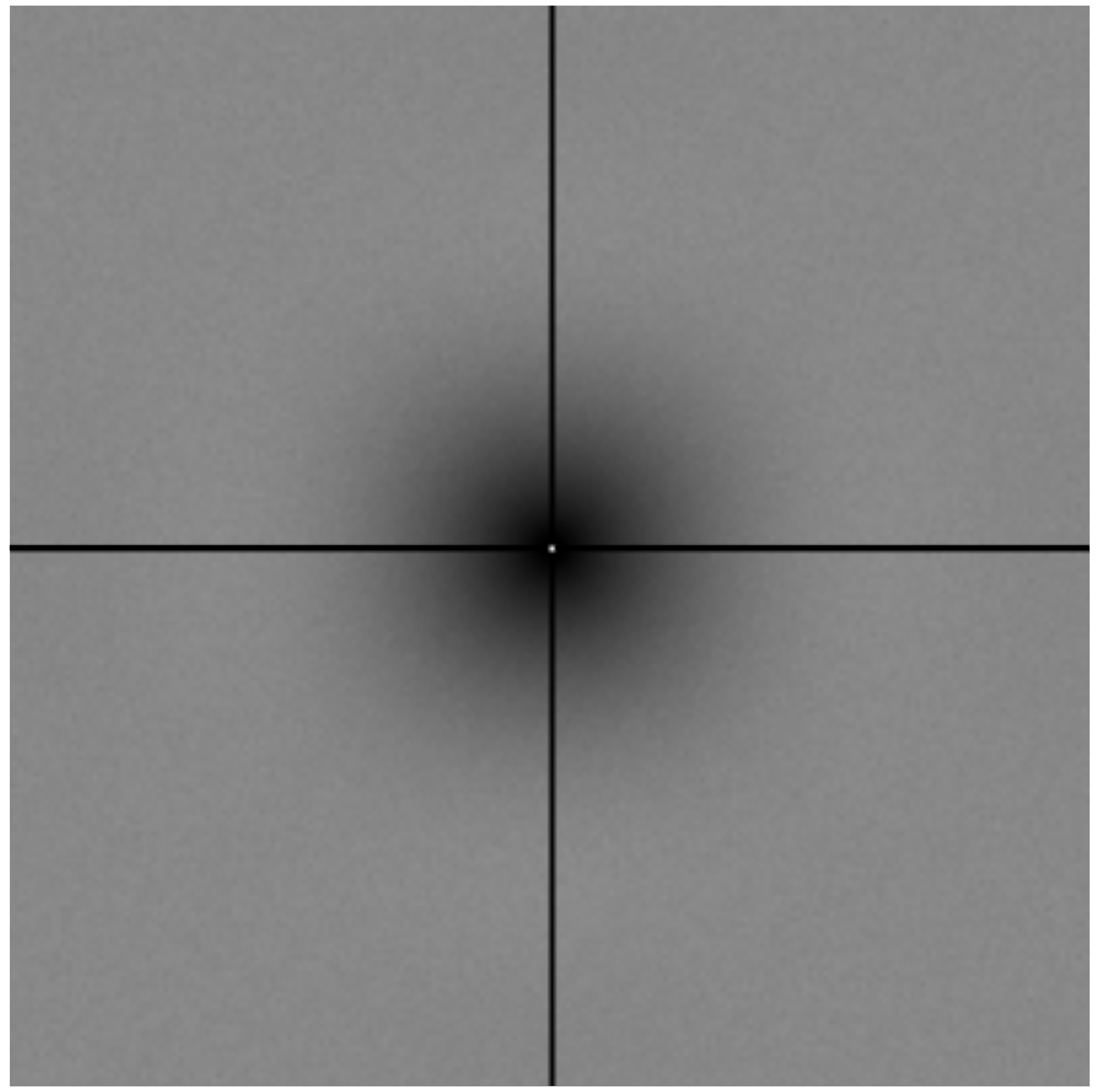
N-rooks /
Latin Hypercube



N-rooks
Spectrum



Multi-jitter

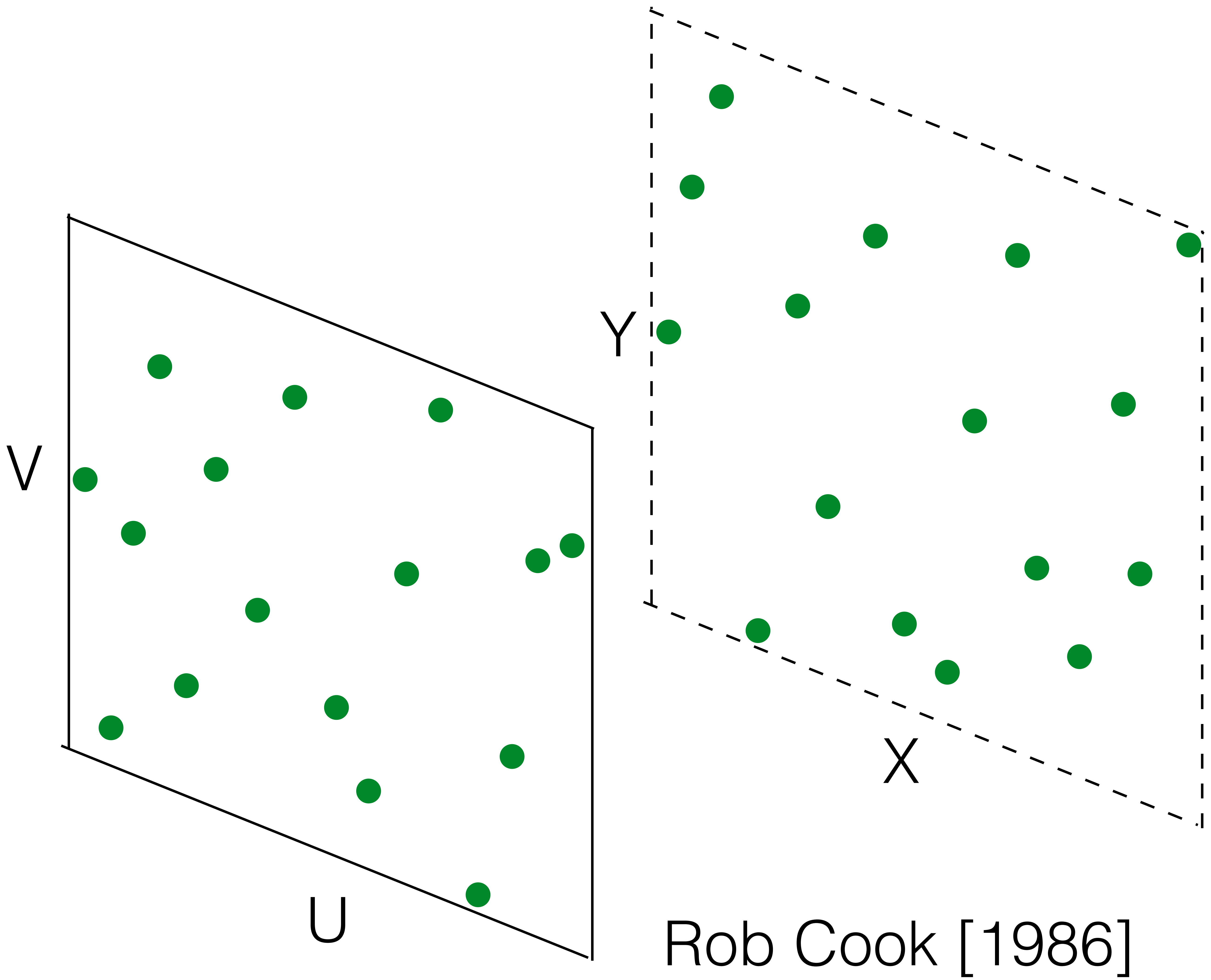


Multi-Jitter
Spectrum

Chiu et al. [1993]

Sampling in Higher Dimensions

4D Sampling



Rob Cook [1986]

2D

$$(x_1, y_1)$$
$$(x_2, y_2)$$
$$(x_3, y_3)$$
$$(x_4, y_4)$$
$$\vdots$$

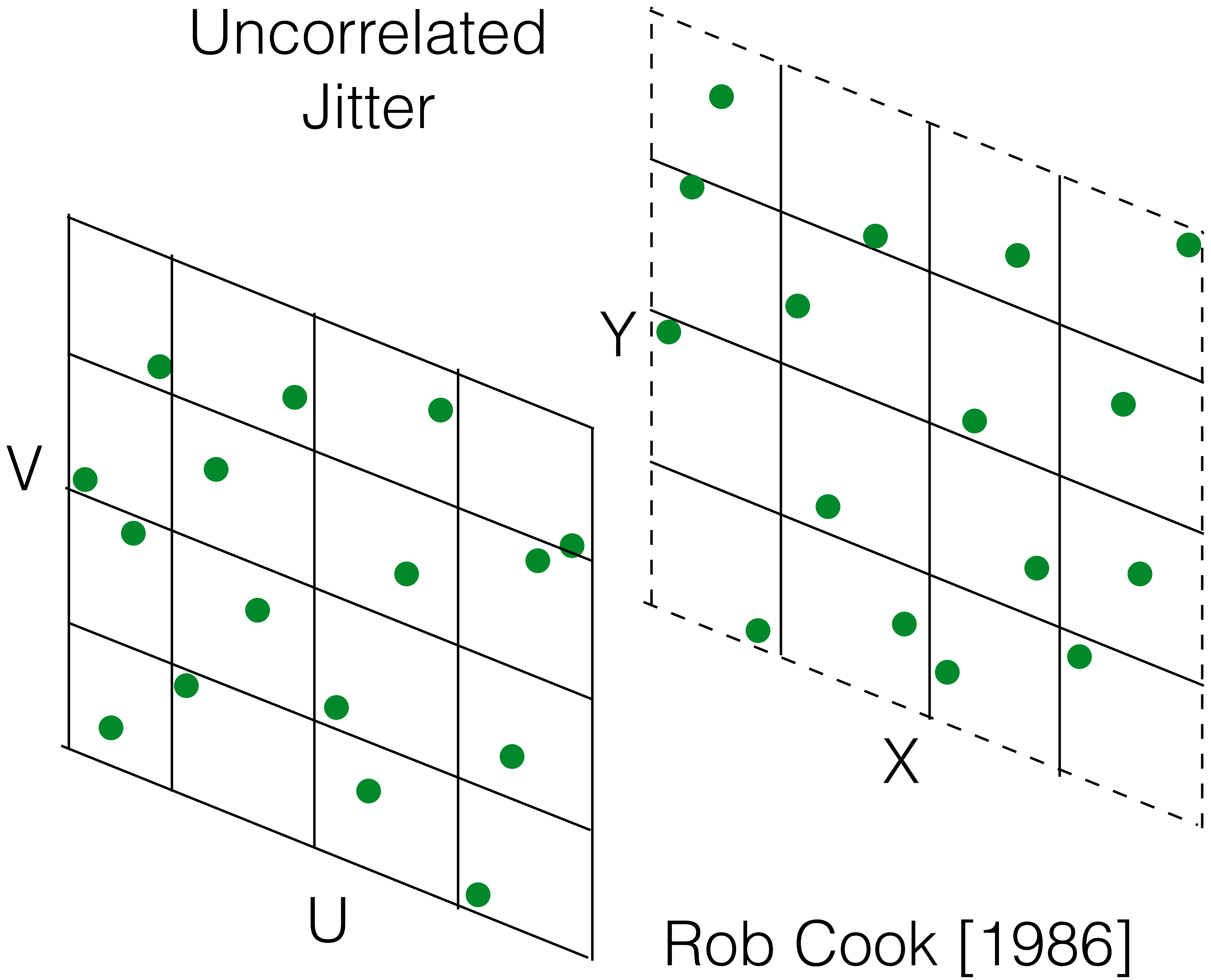
2D

$$(u_1, v_1)$$
$$(u_2, v_2)$$
$$(u_3, v_3)$$
$$(u_4, v_4)$$
$$\vdots$$

4D

$$(x_1, y_1, u_3, v_3)$$
$$(x_2, y_2, u_1, v_1)$$
$$(x_3, y_3, u_4, v_4)$$
$$(x_4, y_4, u_2, v_2)$$
$$\vdots$$

4D Sampling



Rob Cook [1986]

2D

(x_1, y_1)
(x_2, y_2)
(x_3, y_3)
(x_4, y_4)
$\vdots$

2D

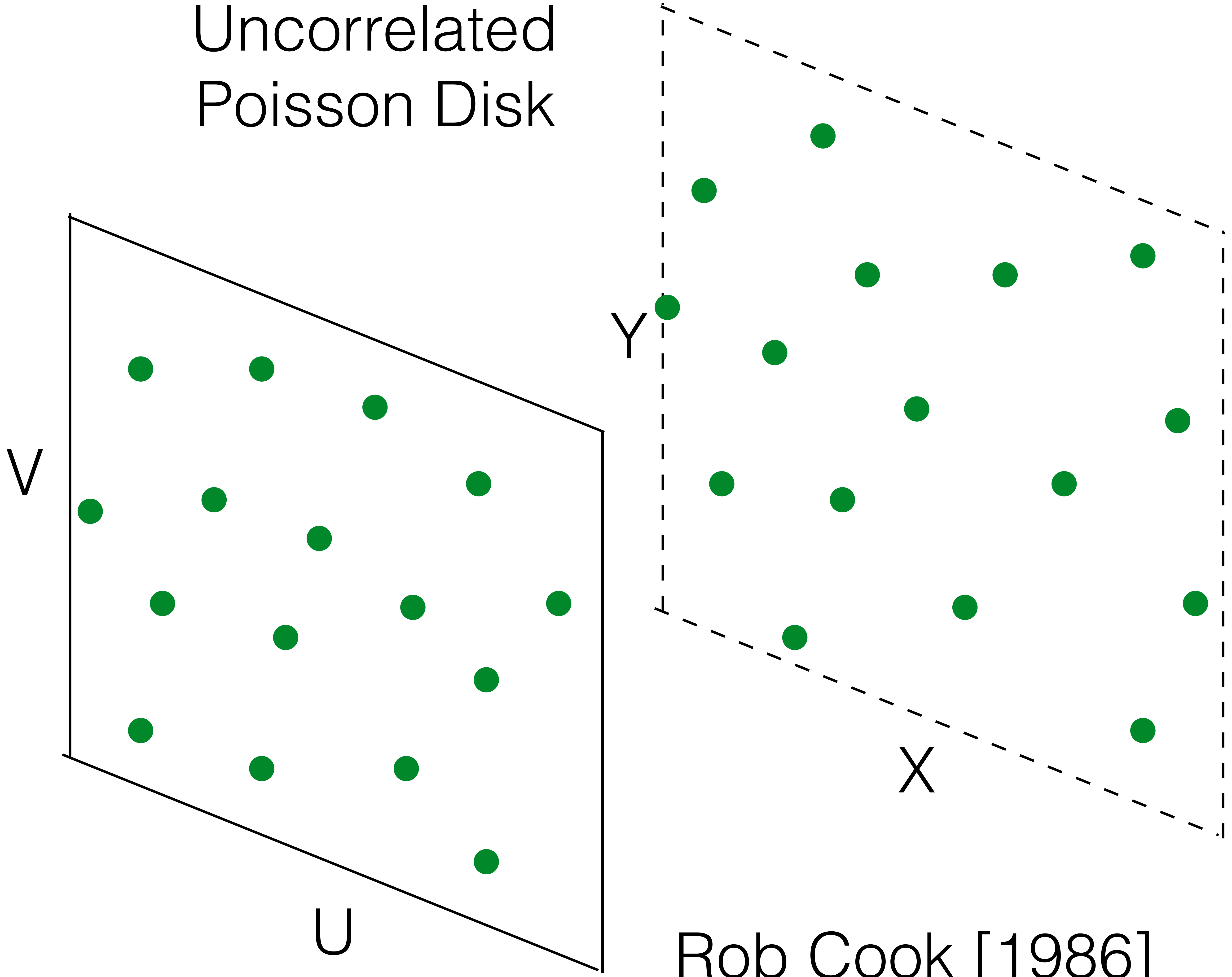
(u_1, v_1)
(u_2, v_2)
(u_3, v_3)
(u_4, v_4)
$\vdots$

4D

(x_1, y_1, u_3, v_3)
(x_2, y_2, u_1, v_1)
(x_3, y_3, u_4, v_4)
(x_4, y_4, u_2, v_2)
$\vdots$

4D Sampling

Uncorrelated
Poisson Disk



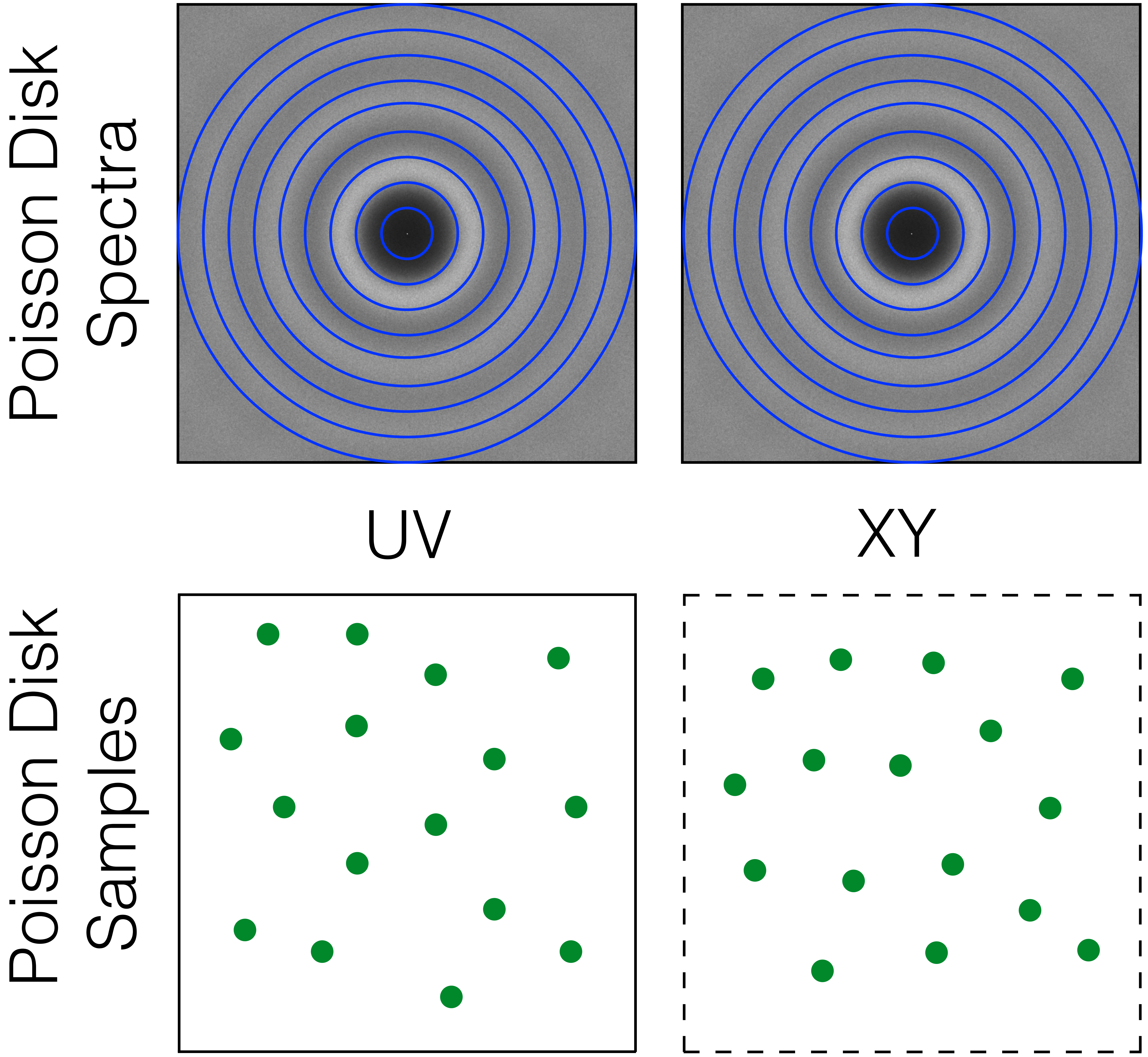
Rob Cook [1986]

2D	2D
(x_1, y_1)	(u_1, v_1)
(x_2, y_2)	(u_2, v_2)
(x_3, y_3)	(u_3, v_3)
(x_4, y_4)	(u_4, v_4)
$\vdots$	$\vdots$

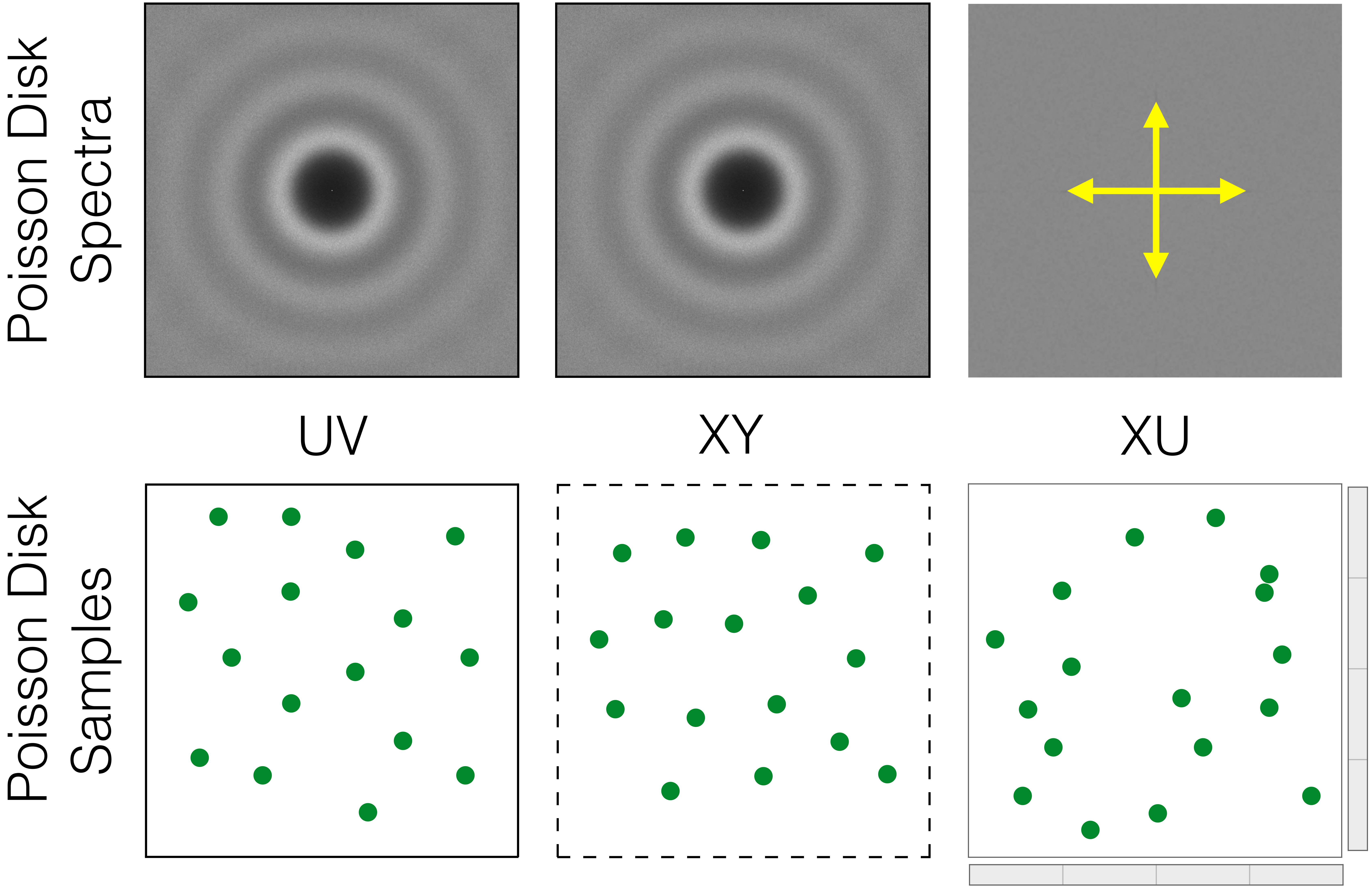
4D

$$(x_1, y_1, u_3, v_3)$$
$$(x_2, y_2, u_1, v_1)$$
$$(x_3, y_3, u_4, v_4)$$
$$(x_4, y_4, u_2, v_2)$$
$$\vdots$$

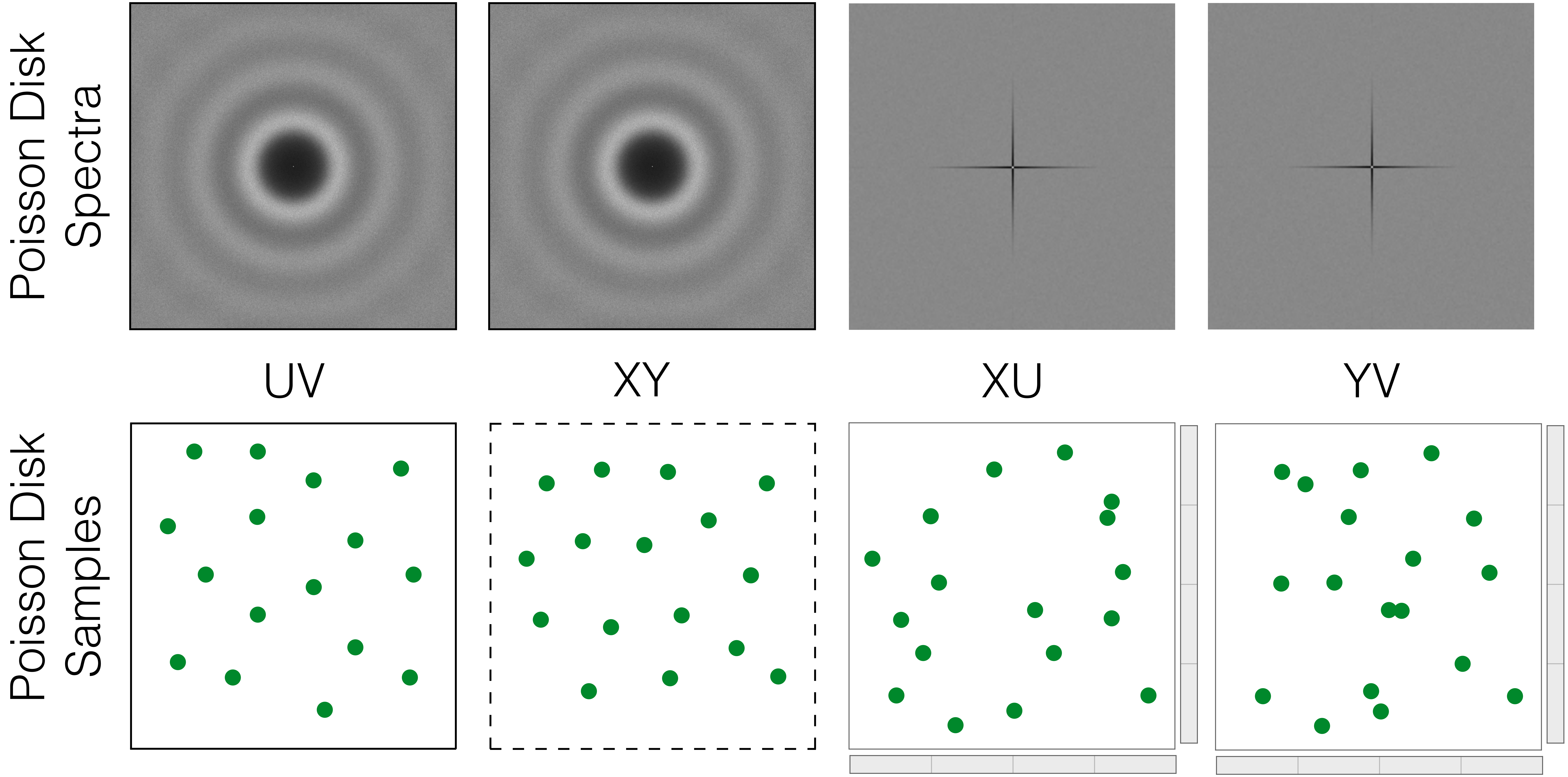
4D Sampling Spectra along Projections



4D Sampling Spectra along Projections



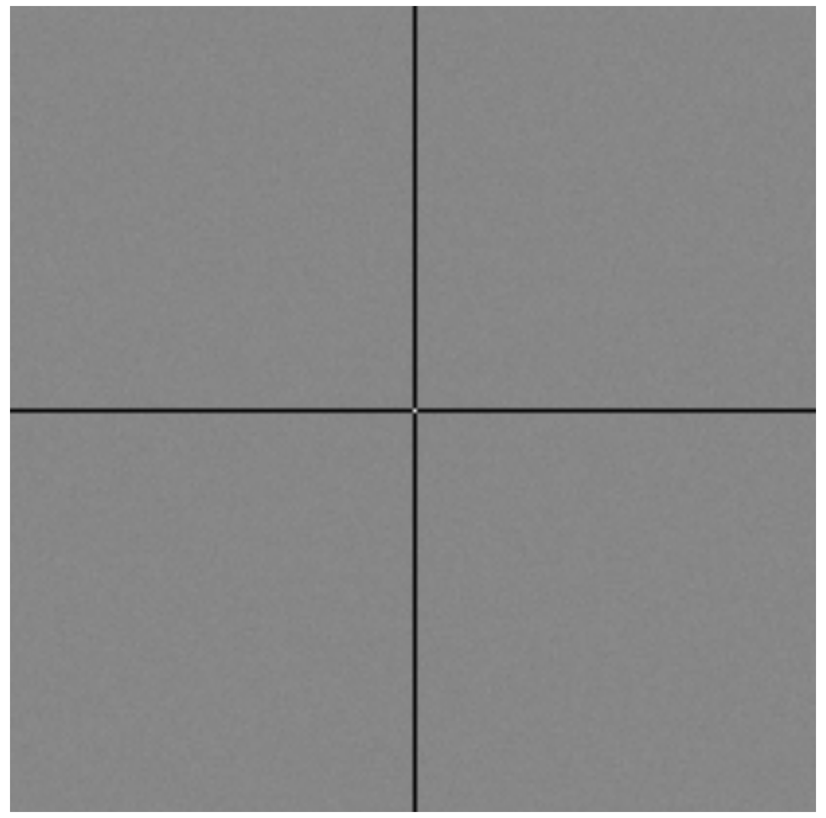
4D Sampling Spectra along Projections



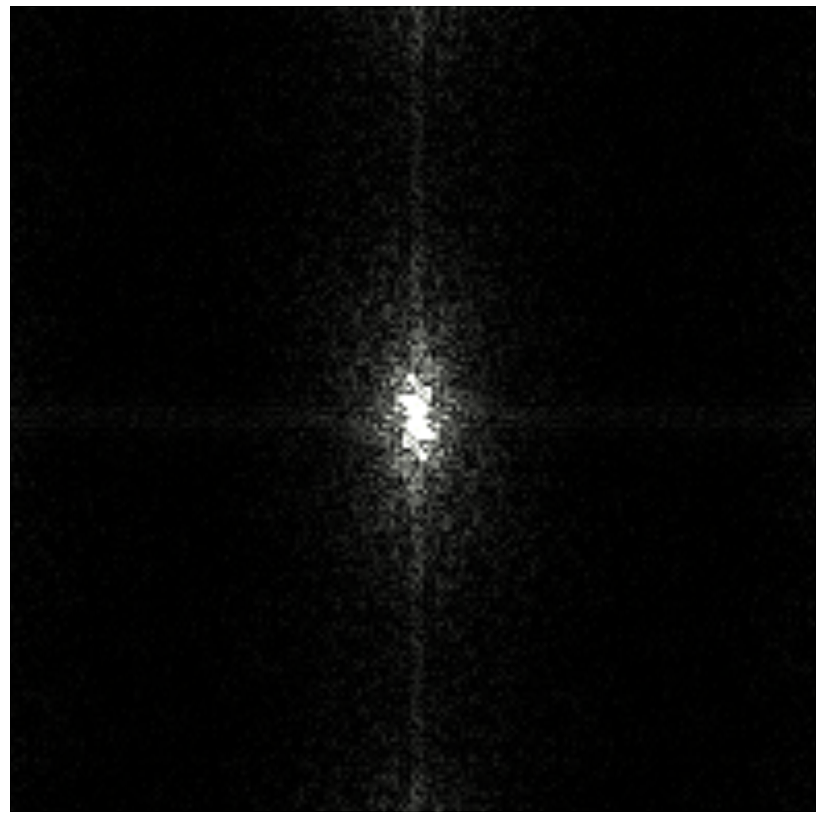
How can we perform Convergence Analysis for Anisotropic Sampling Spectra ?

Variance Formulation for Anisotropic Sampling Spectra

$$Var[\hat{I}] = \int_\Omega \mathbb{E}\left[\mathcal{P}_{S_N}(\nu)\right] \times \mathcal{P}_f(\nu)\, d\nu$$

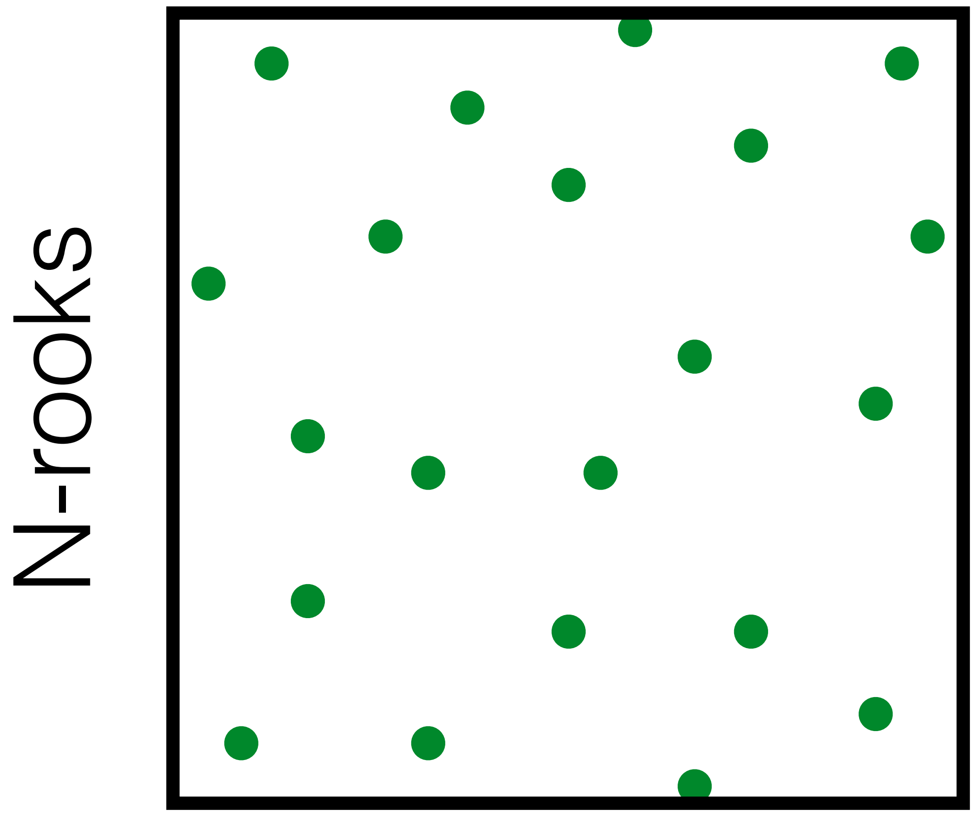


N-rooks spectrum

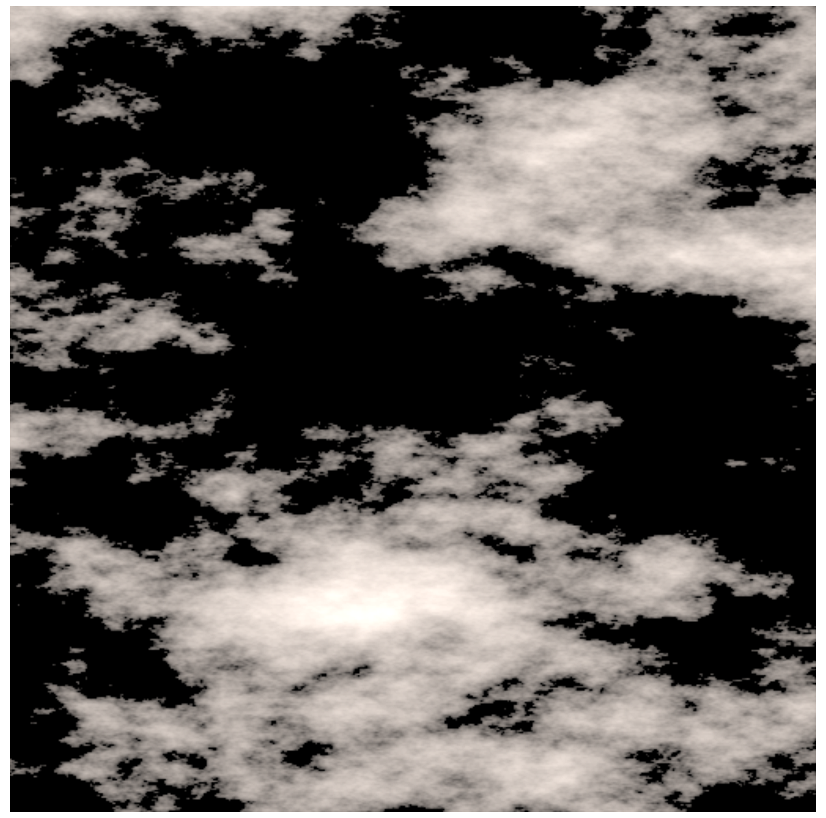


Integrand spectrum

$S_N(\vec{x})$

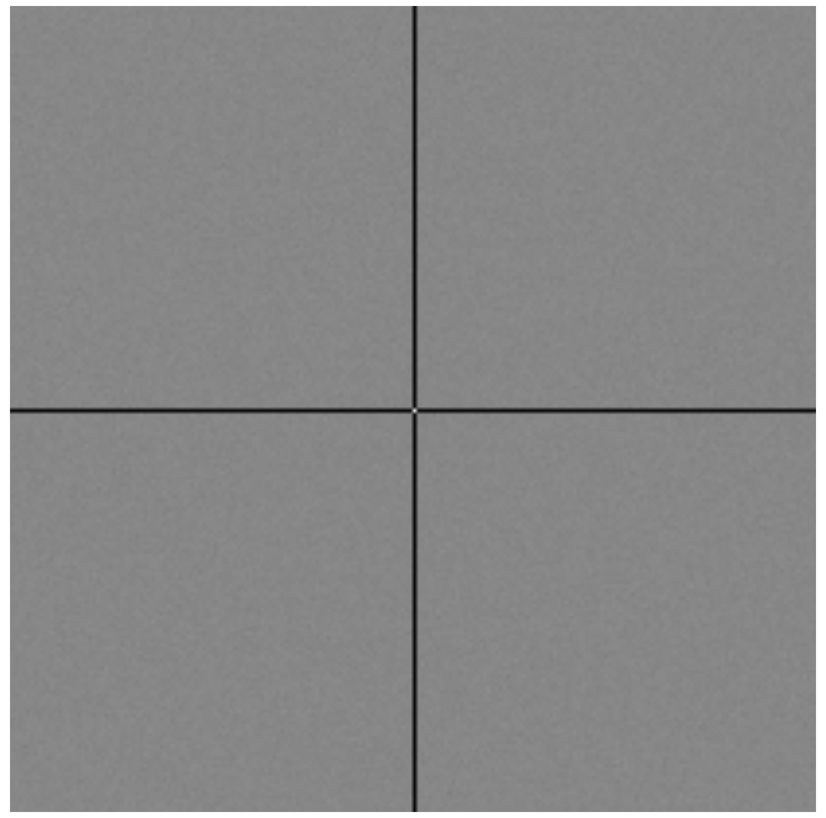


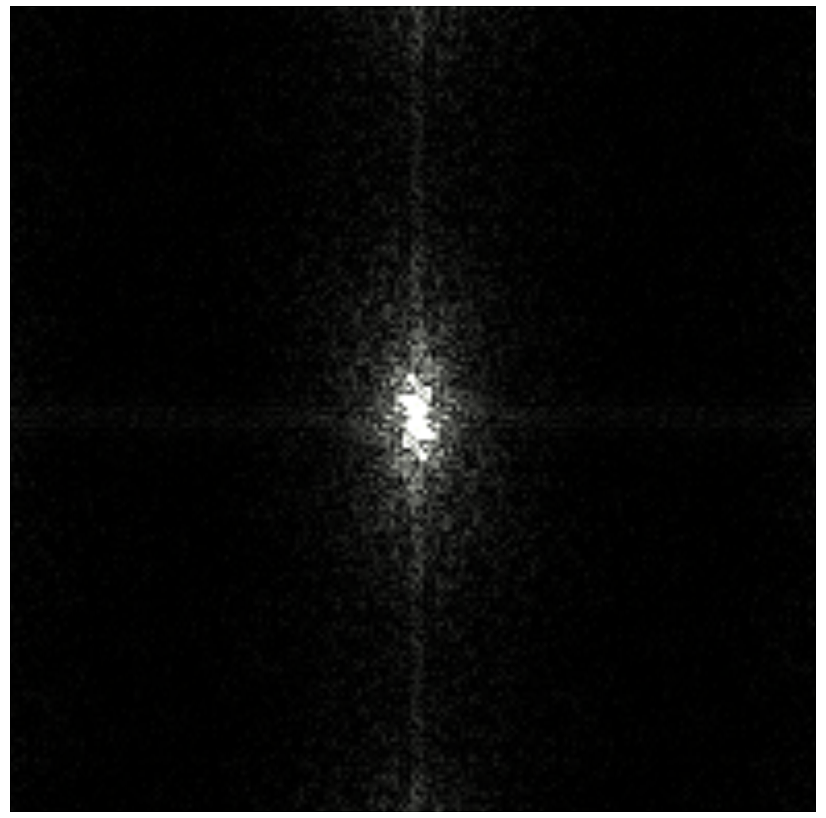
$f(\vec{x})$



Variance Formulation for Anisotropic Sampling Spectra

$$Var[\hat{I}] = \int_0^\infty \rho^{d-1} \int_{\mathcal{S}^{d-1}} \mathbb{E}\left[\tilde{\mathcal{P}}_{S_N}(\rho\mathbf{n})\right] \times \mathcal{P}_f(\rho\mathbf{n}) \, d\mathbf{n} \, d\rho$$

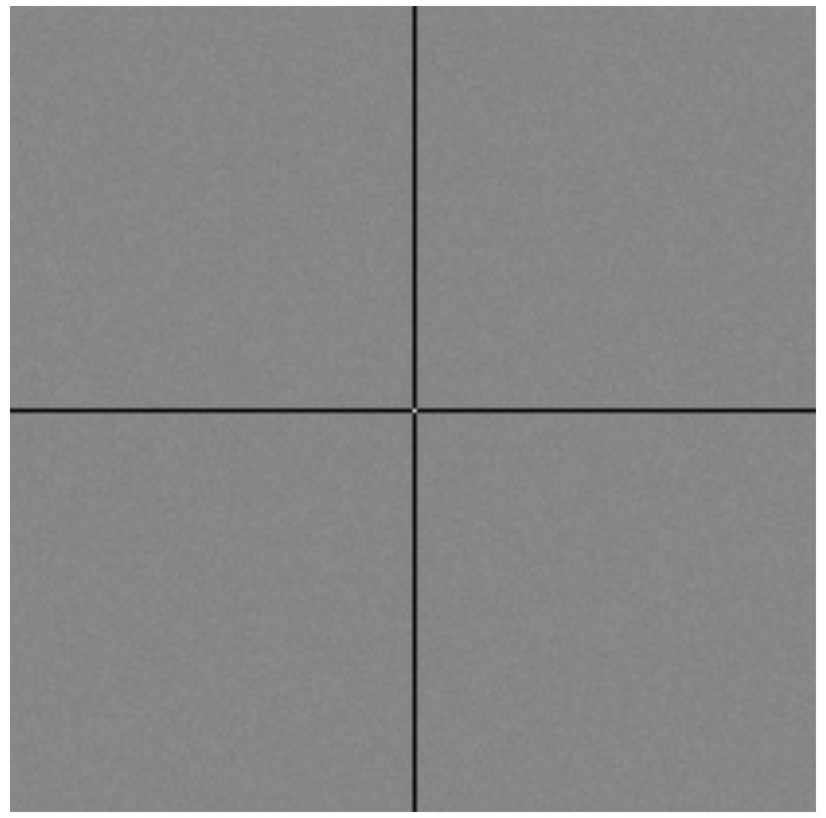




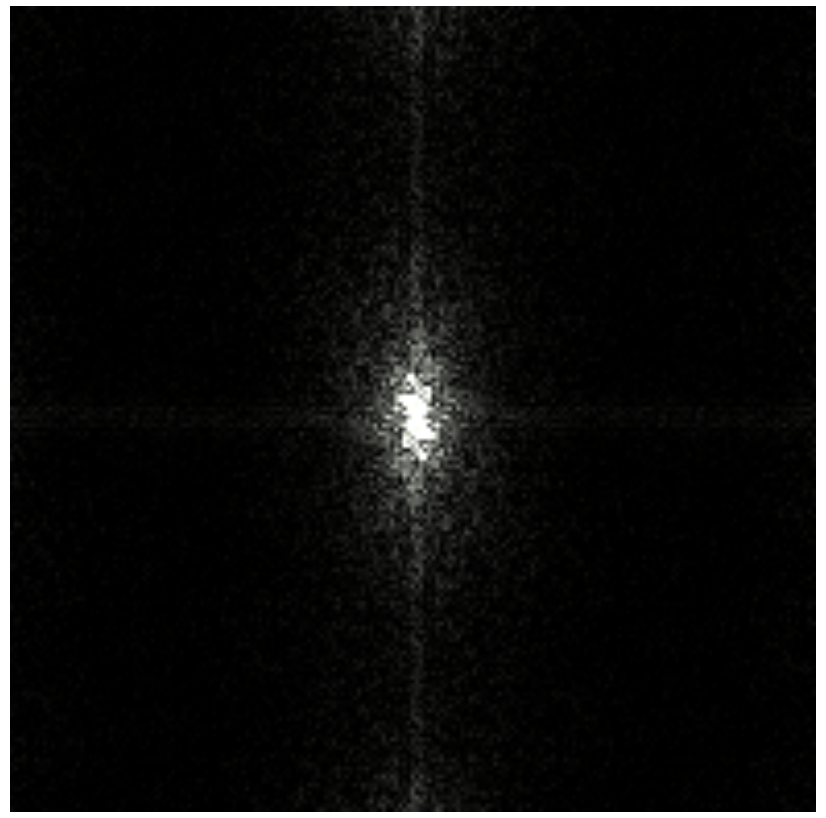
Variance Formulation for Anisotropic Sampling Spectra

$\mathbb{E}\left[\tilde{\mathcal{P}}_{S_N}(\rho\mathbf{n})\right]$ $\qquad$ $\mathcal{P}_f(\rho\mathbf{n})$

$$Var[\hat{I}] = \int_{\mathcal{S}^{d-1}} \int_0^\infty \rho^{d-1}$$



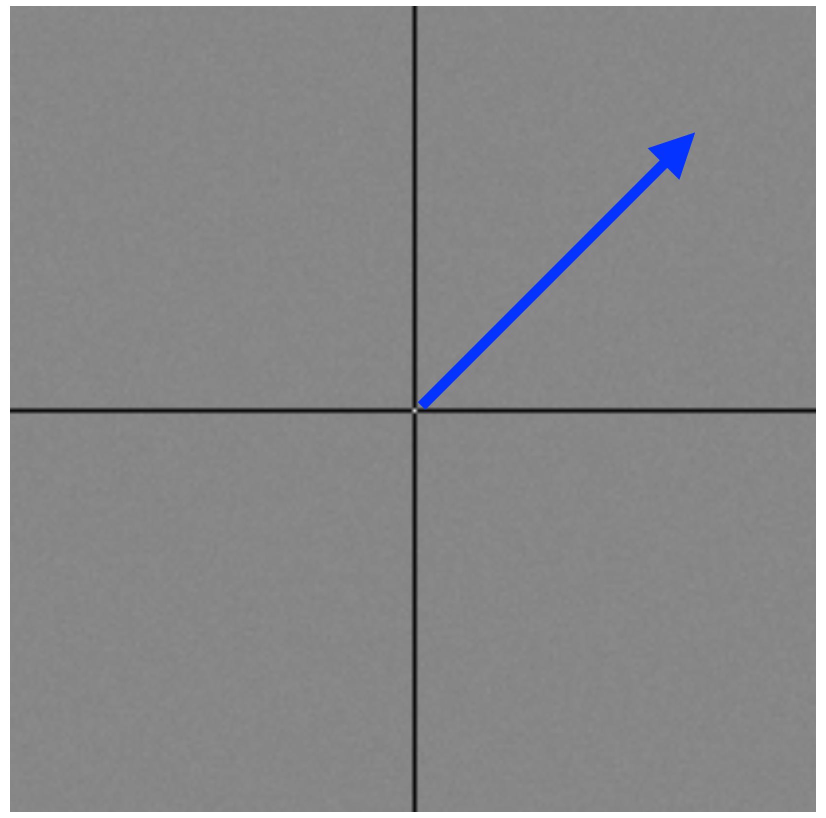
$\times$

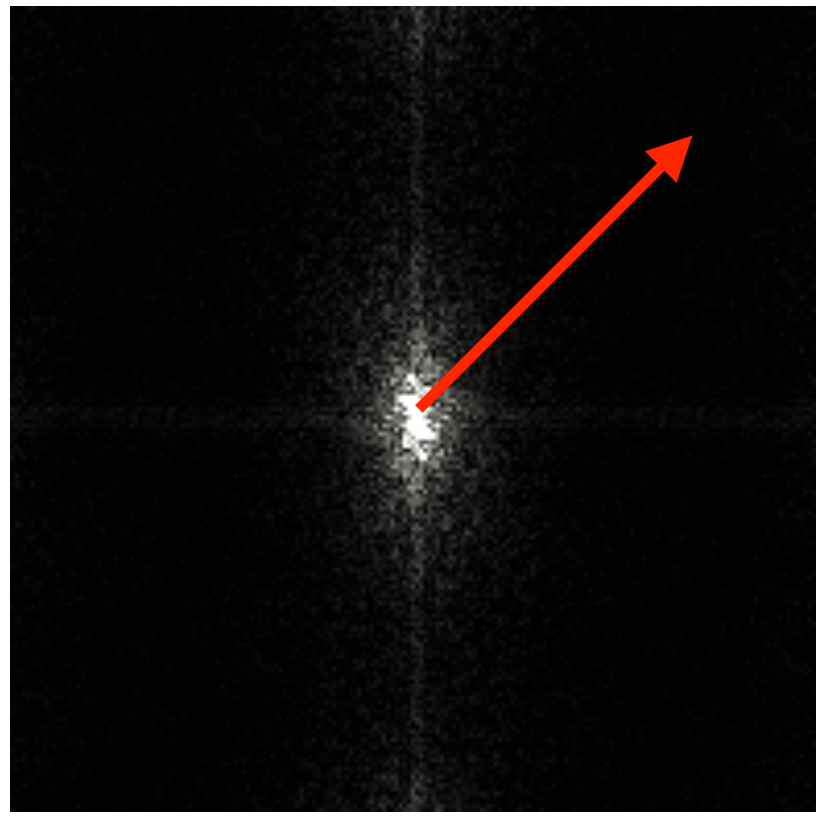


$d\rho\, d\mathbf{n}$

Variance Formulation for Anisotropic Sampling Spectra

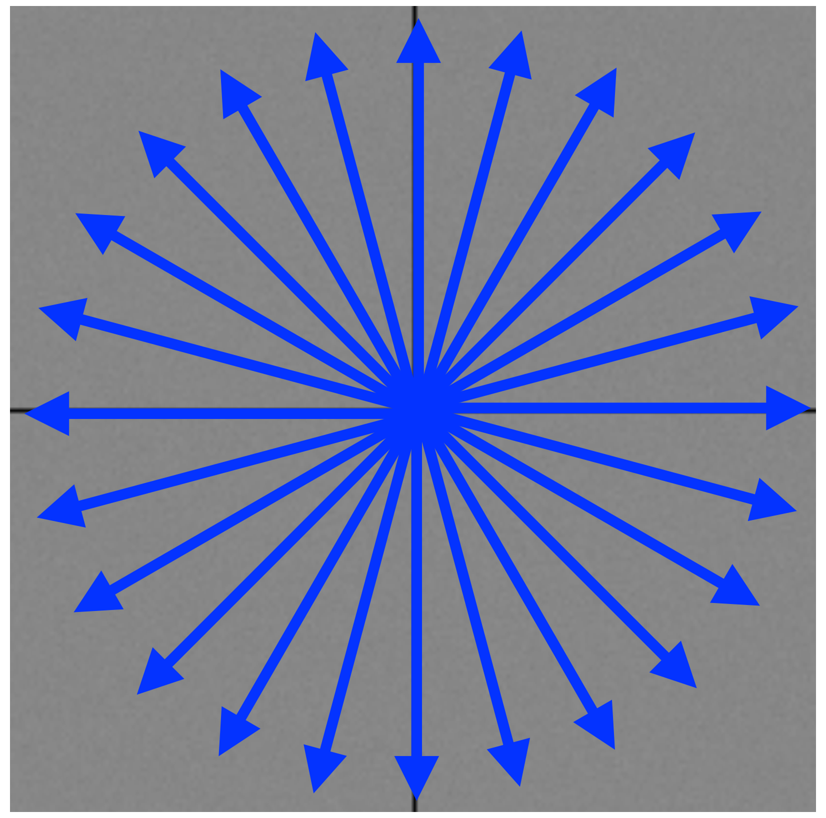
$$Var[\hat{I}] = \int_{\mathcal{S}^{d-1}} \int_0^\infty \rho^{d-1} \, \mathbb{E}\left[\mathcal{P}_{S_N}(\rho_k \mathbf{n_k})\right] \times \mathcal{P}_f(\rho_k \mathbf{n}_k) \, d\rho \, d\mathbf{n}$$

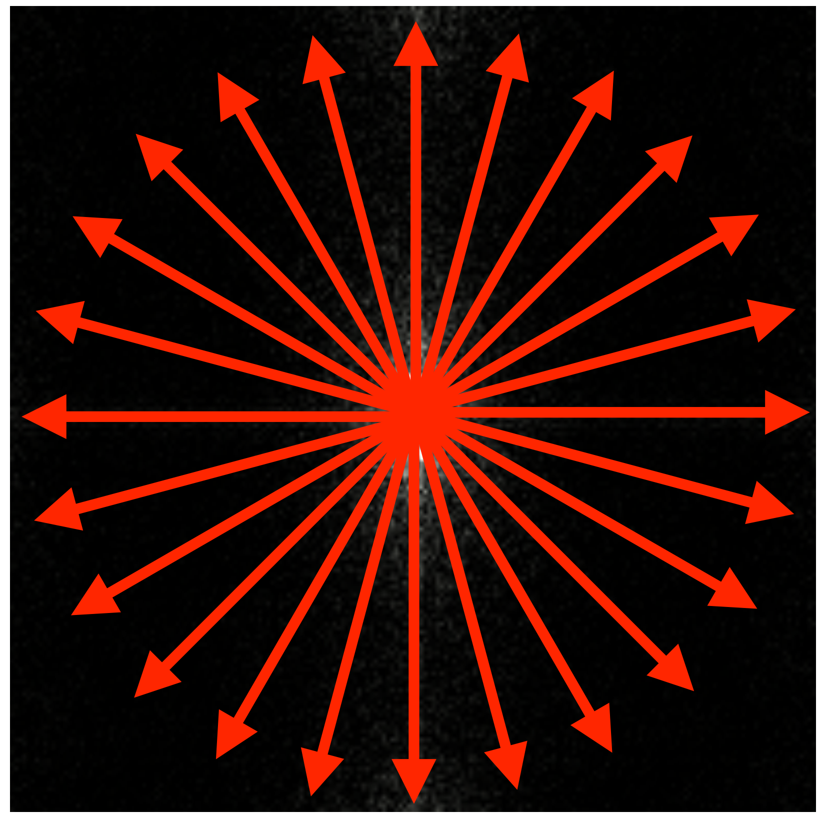




Variance Formulation for Anisotropic Sampling Spectra

$$Var[\hat{I}] = \int_{\mathcal{S}^{d-1}} \int_0^{\infty} \rho^{d-1} \, \mathbb{E}\left[\mathcal{P}_{S_N}(\rho_k \mathbf{n_k})\right] \times \mathcal{P}_f(\rho_k \mathbf{n}_k) \, d\rho \, d\mathbf{n}$$



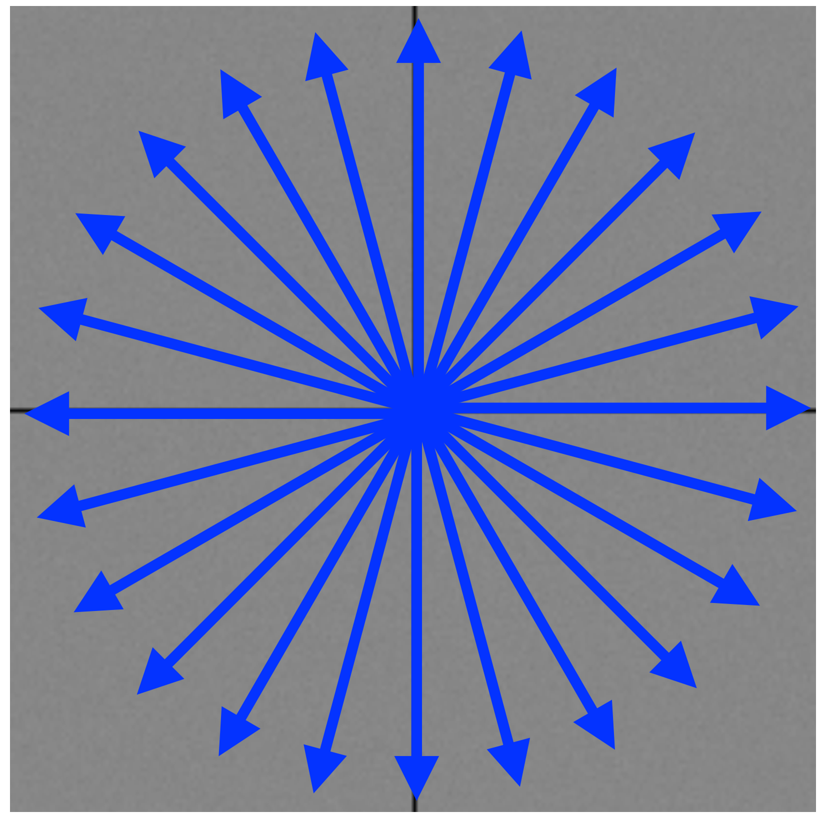


Variance Formulation for Anisotropic Sampling Spectra

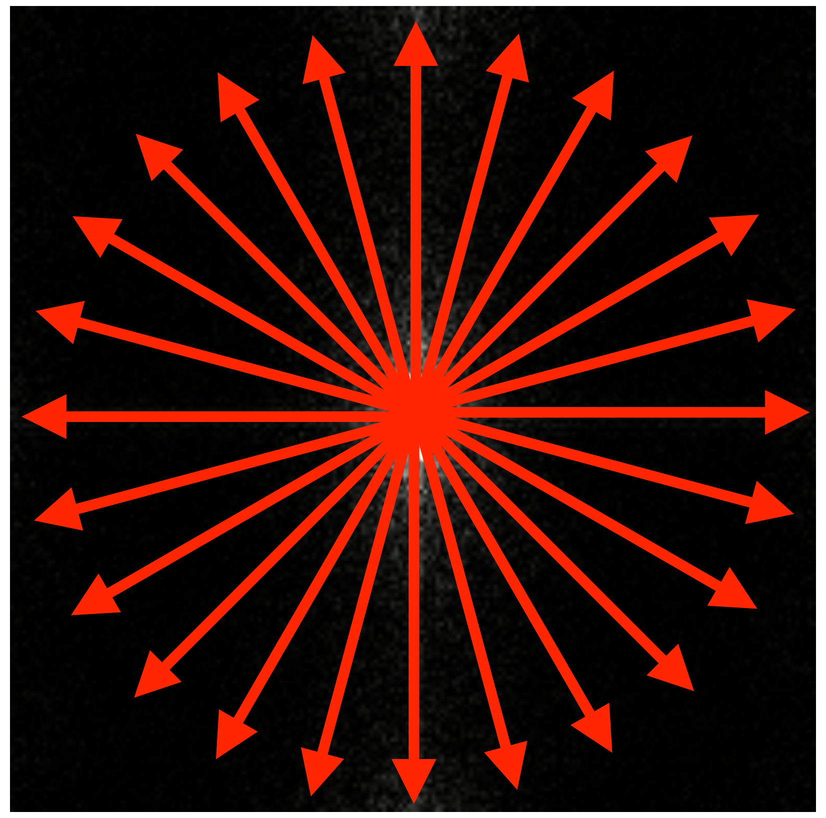
$\mathbb{E}\left[\mathcal{P}_{S_N}(\rho_k \mathbf{n_k})\right]$

$\mathcal{P}_f(\rho_k \mathbf{n}_k)$

$$Var[\hat{I}] = \lim_{m \to \infty} \sum_{k=1}^{m} \int_0^\infty \rho^{d-1}$$



$\times$



$$d\rho \; \Delta \mathbf{n}_k$$

Variance Formulation for Anisotropic Sampling Spectra

$$Var[\hat{I}] = \lim_{m \to \infty} \sum_{k=1}^{m} \int_0^{\infty} \rho^{d-1} \; \mathbb{E}\left[\mathcal{P}_{S_N}(\rho_k \mathbf{n_k})\right] \quad \times \qquad \mathcal{P}_f(\rho_k \mathbf{n}_k) \quad d\rho \; \Delta \mathbf{n}_k$$

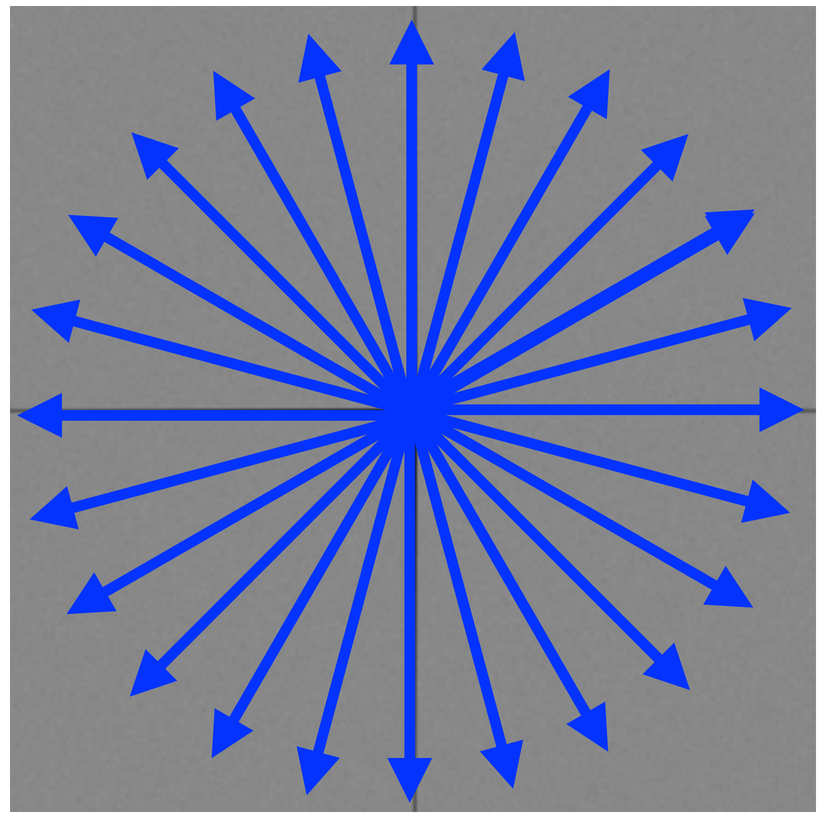
Variance Formulation for Anisotropic Sampling Spectra

$$Var[\hat{I}] = \lim_{m\to\infty} \sum_{k=1}^{m} \int_0^{\infty} \rho^{d-1}\, \mathbb{E}\left[\mathcal{P}_{S_N}(\rho_k \mathbf{n_k})\right] \mathcal{P}_f(\rho_k \mathbf{n}_k)\, d\rho\, \Delta\mathbf{n}_k$$

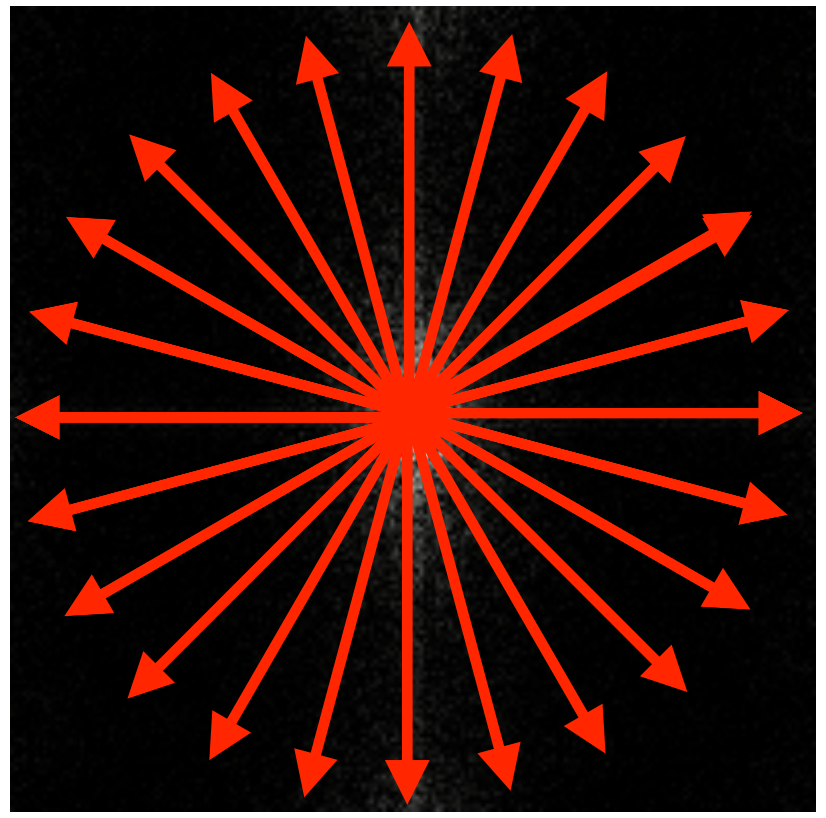
Variance Formulation for Anisotropic Sampling Spectra

$$Var[\hat{I}] = \lim_{m\to\infty} \sum_{k=1}^{m} \int_0^{\infty} \rho^{d-1}\, \mathbb{E}\left[\mathcal{P}_{S_N}(\rho_k \mathbf{n_k})\right] \mathcal{P}_f(\rho_k \mathbf{n}_k)\, d\rho\, \Delta \mathbf{n}_k$$

$\left\langle \mathcal{P}_{S_N}(\rho_k \mathbf{n_k}) \right\rangle$

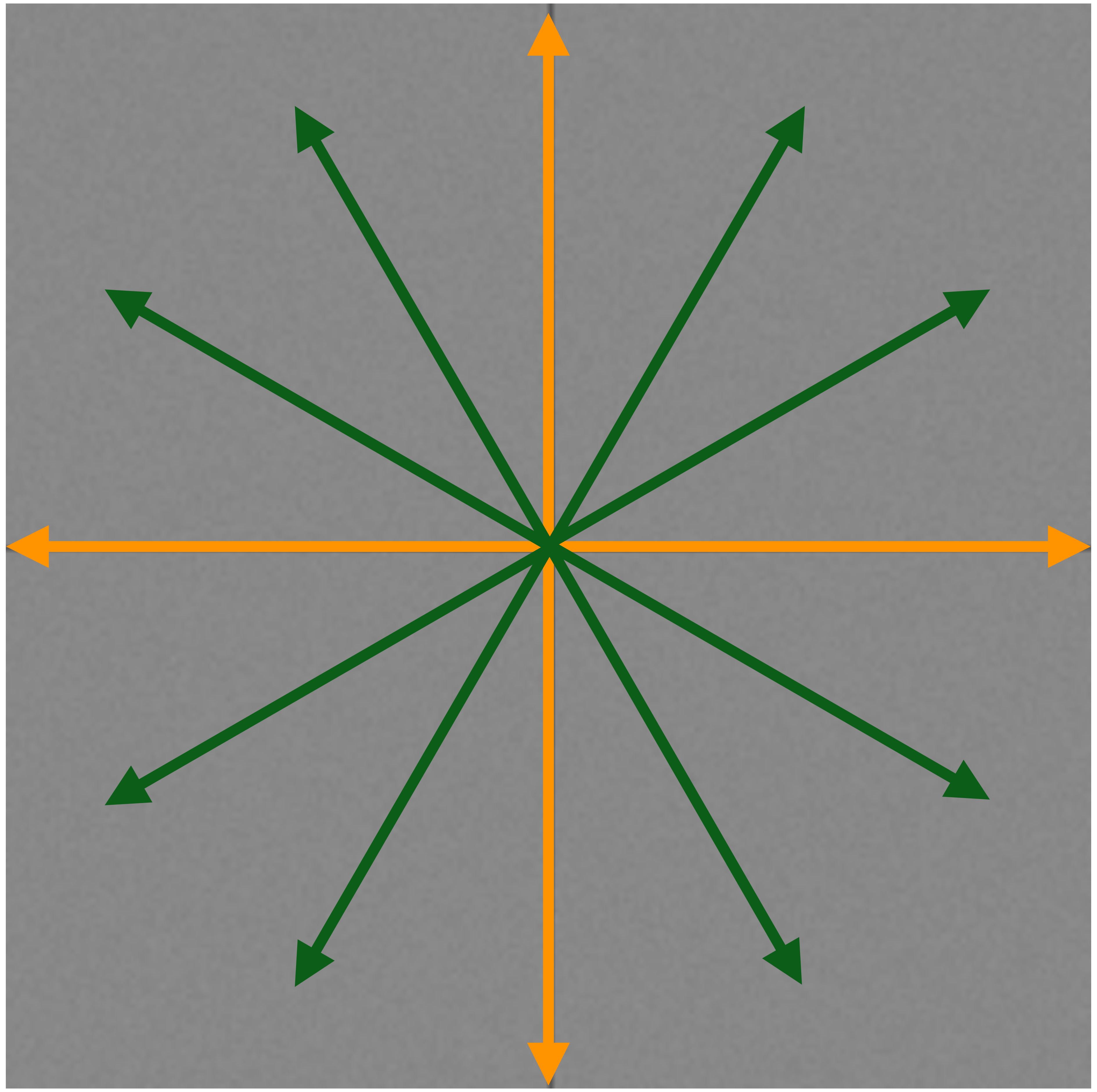


$\mathcal{P}_f(\rho_k \mathbf{n}_k)$

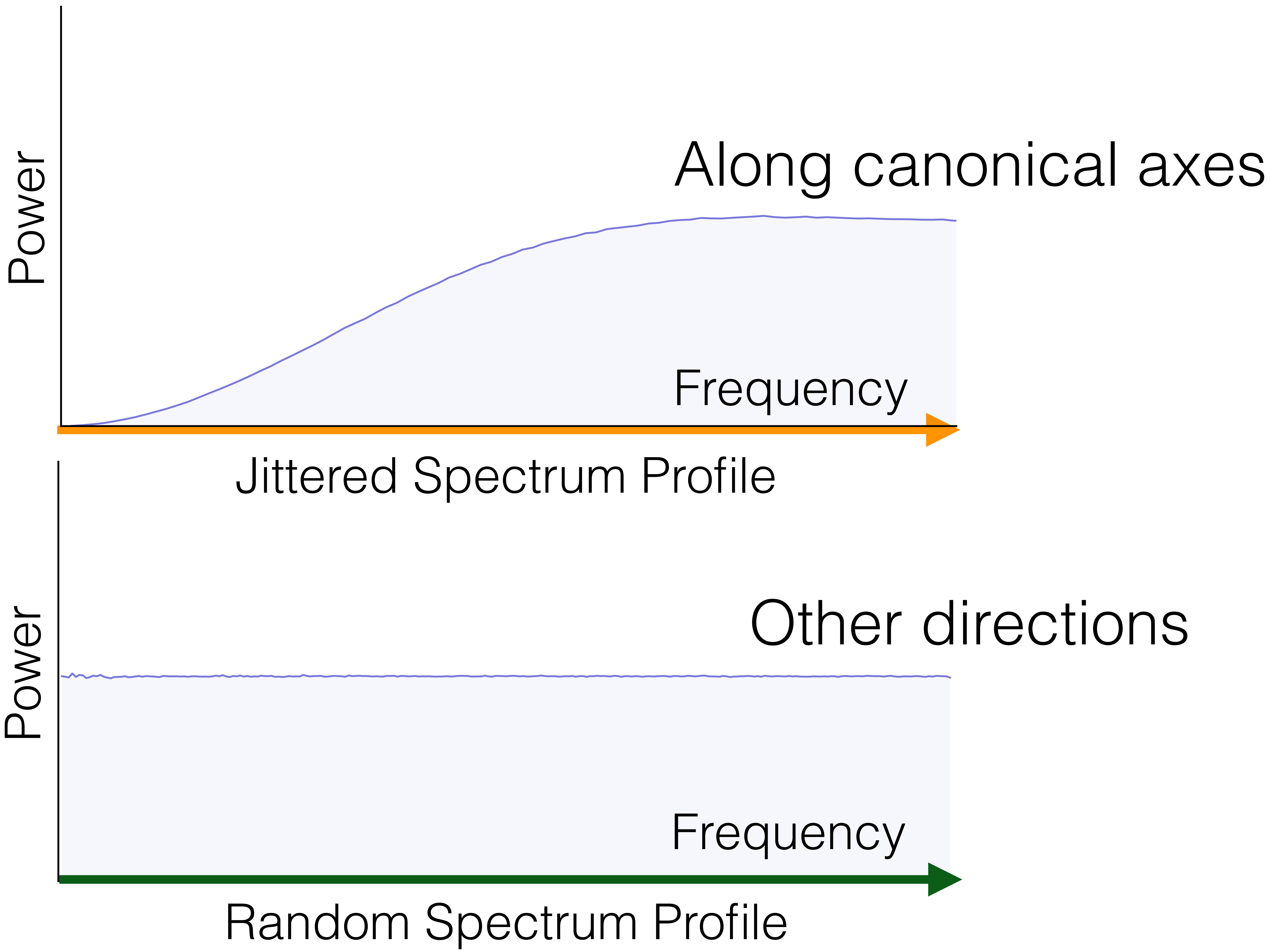


Convergence Analysis for Anisotropic Sampling Spectra

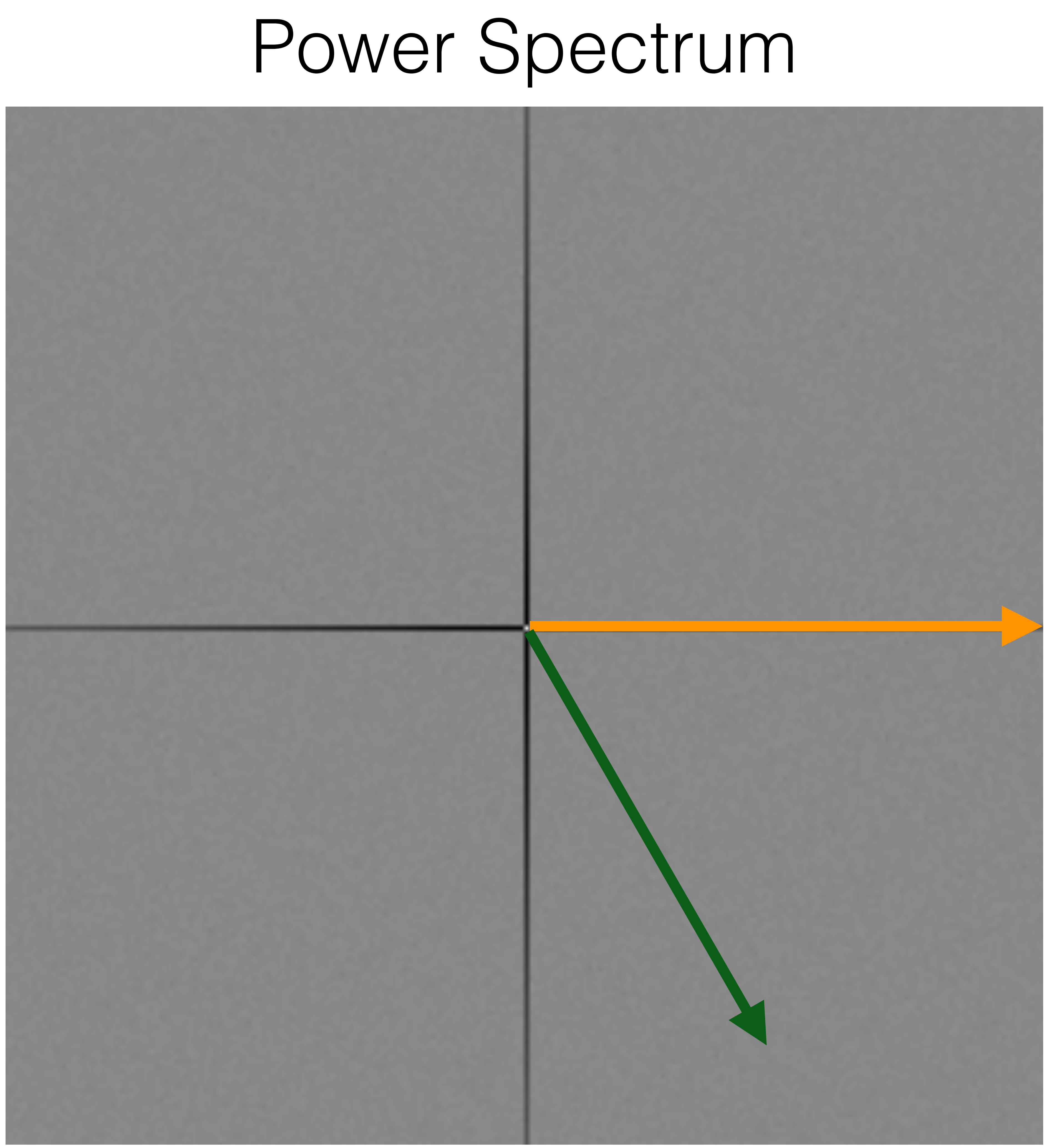
Power Spectrum

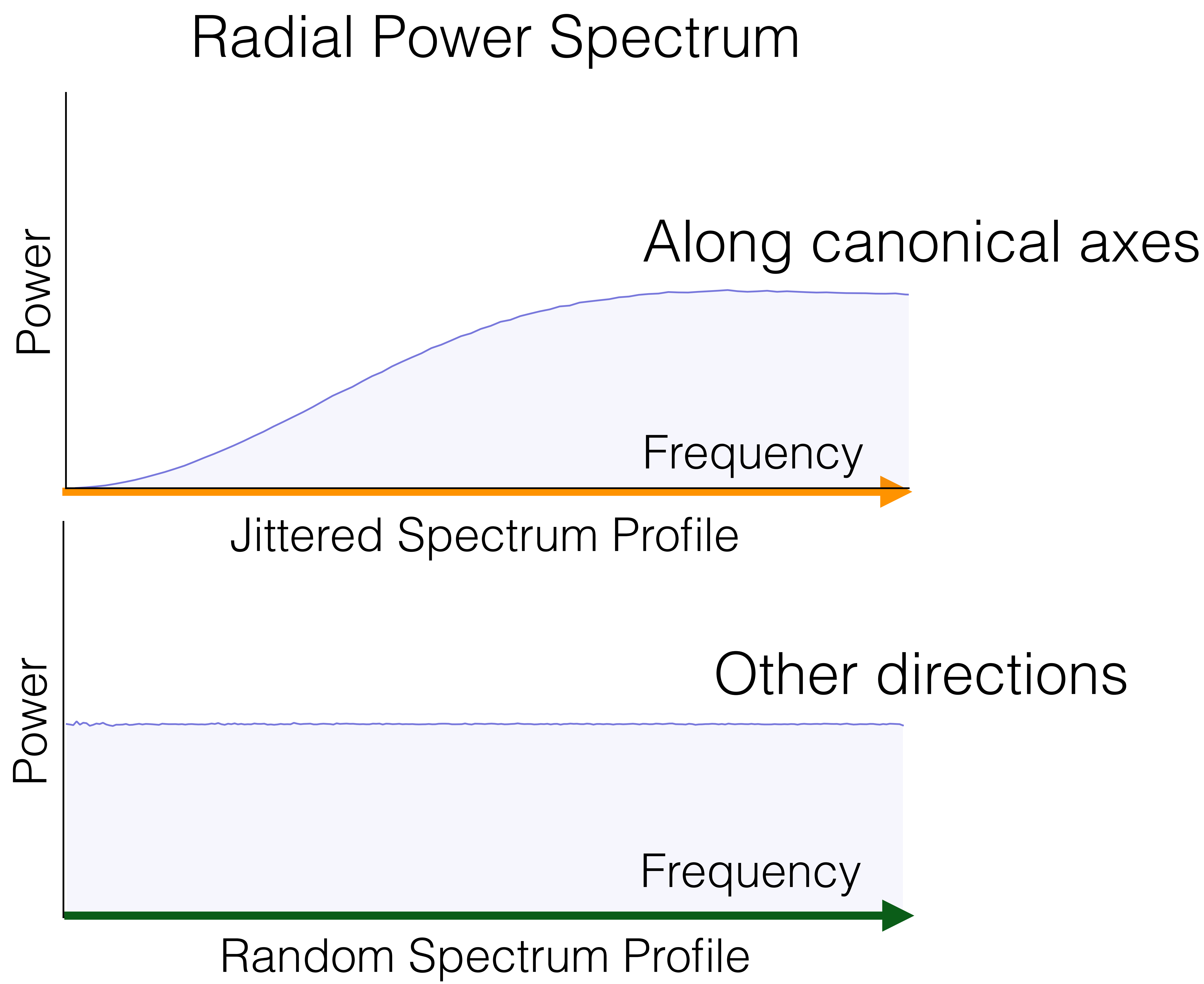


Radial Power Spectrum



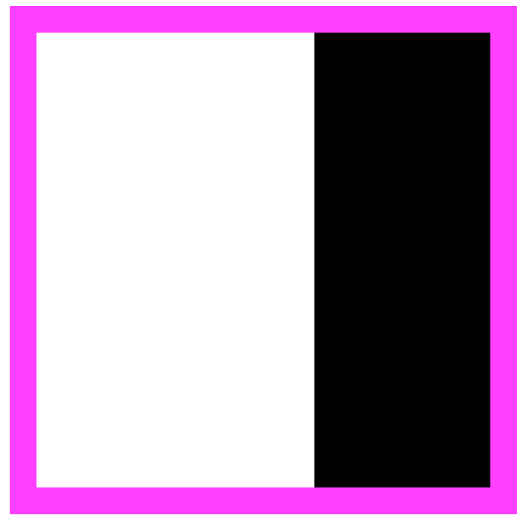
Convergence Analysis for Anisotropic Sampling Spectra





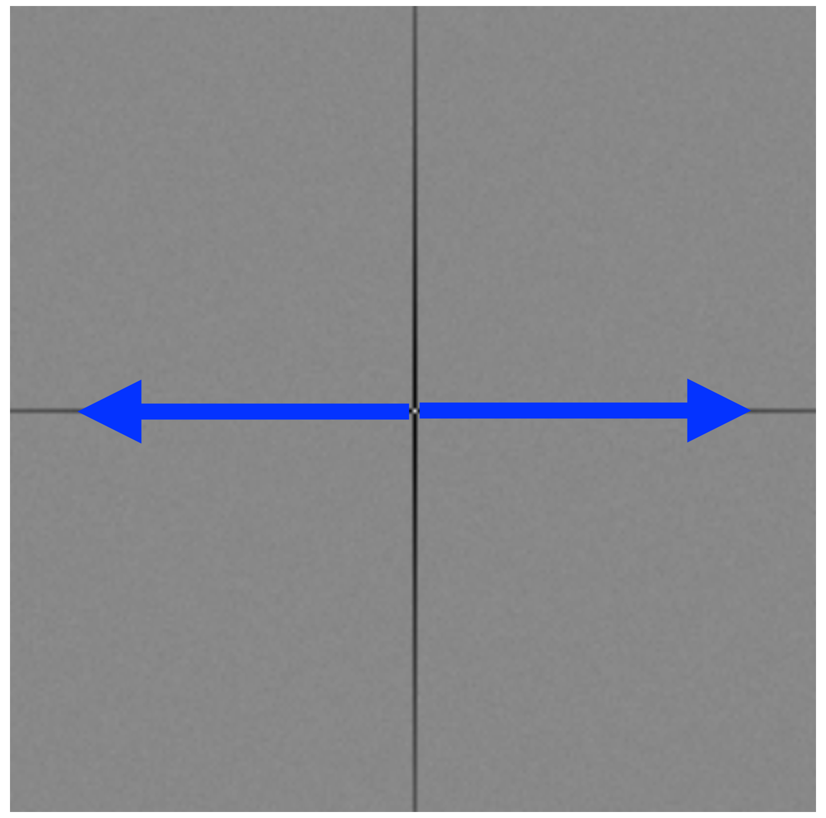
Variance due to N-rooks Sampler

$f(\vec{x})$



$Var[\hat{I}] = \int_\Omega$

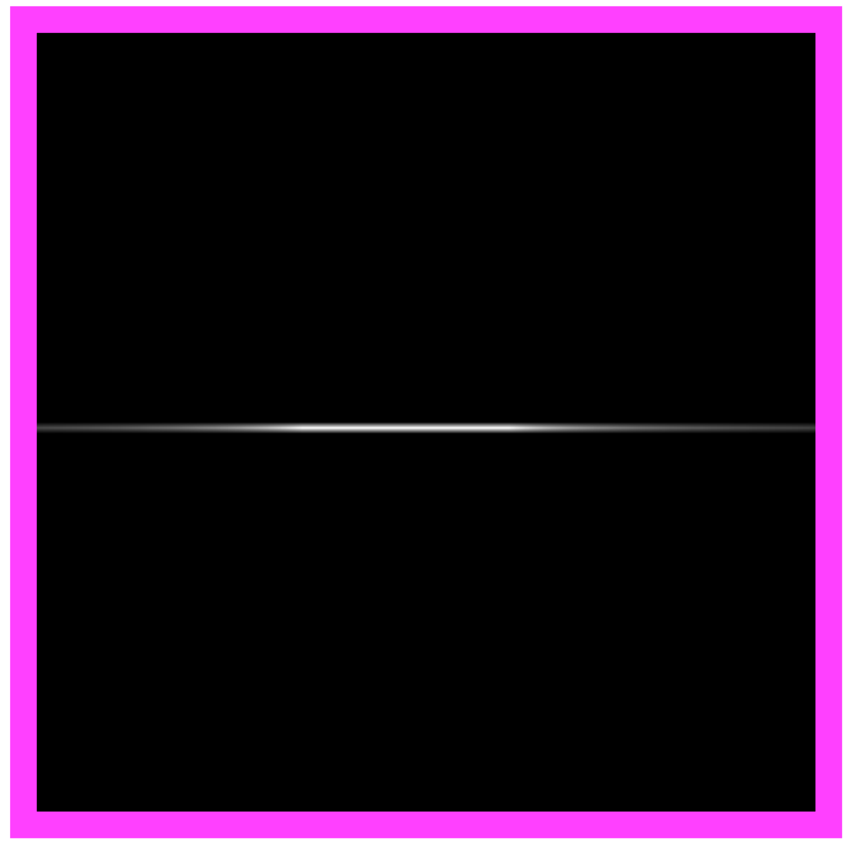
$\langle \mathcal{P}_{S_N}(\nu) \rangle$



N-rooks spectrum

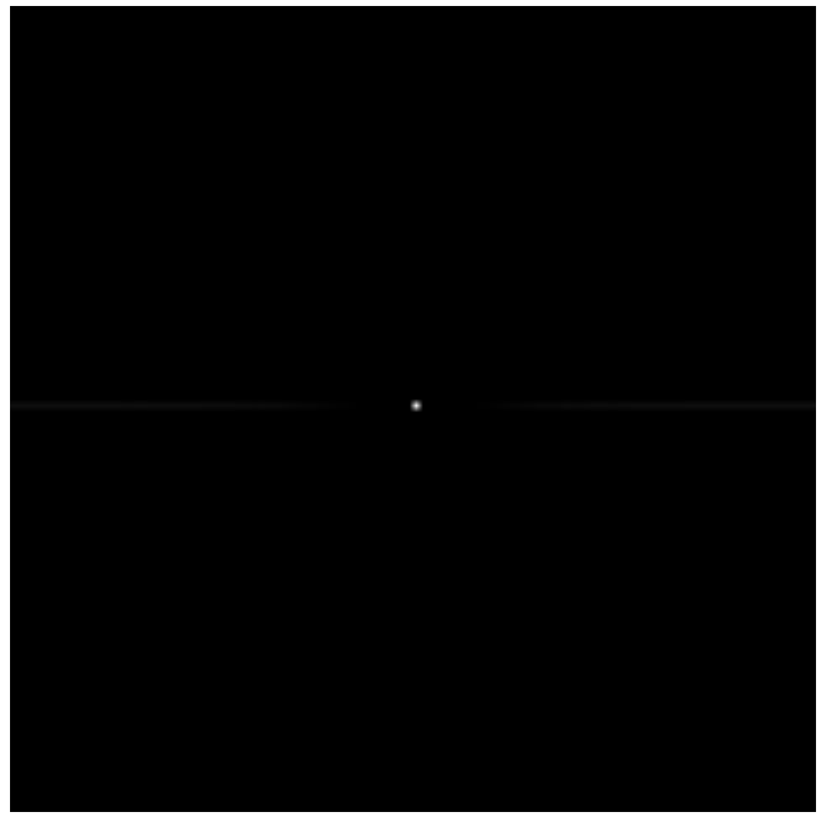
X

$\mathcal{P}_f(\nu)$



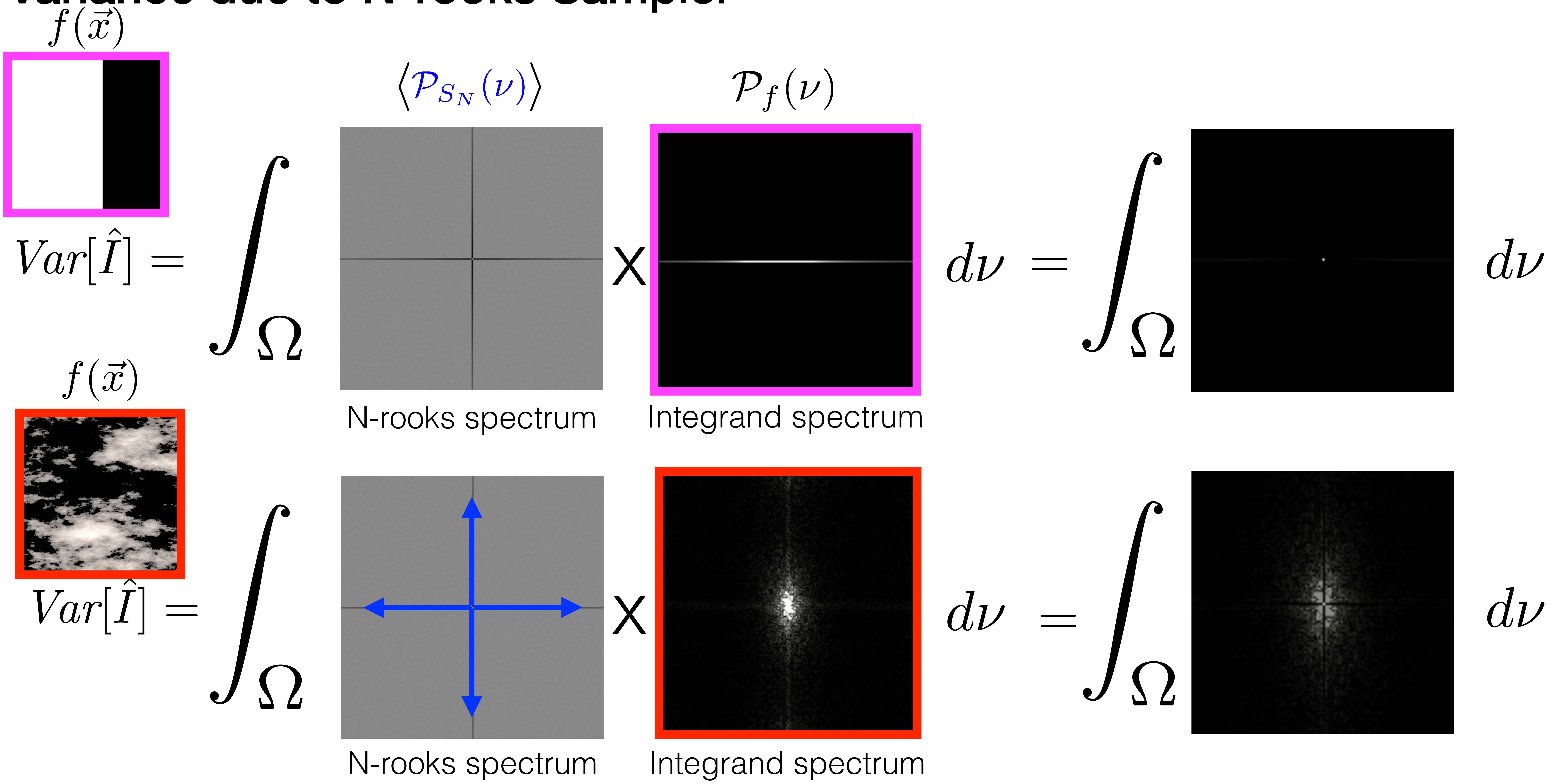
Integrand spectrum

$d\nu = \int_\Omega$

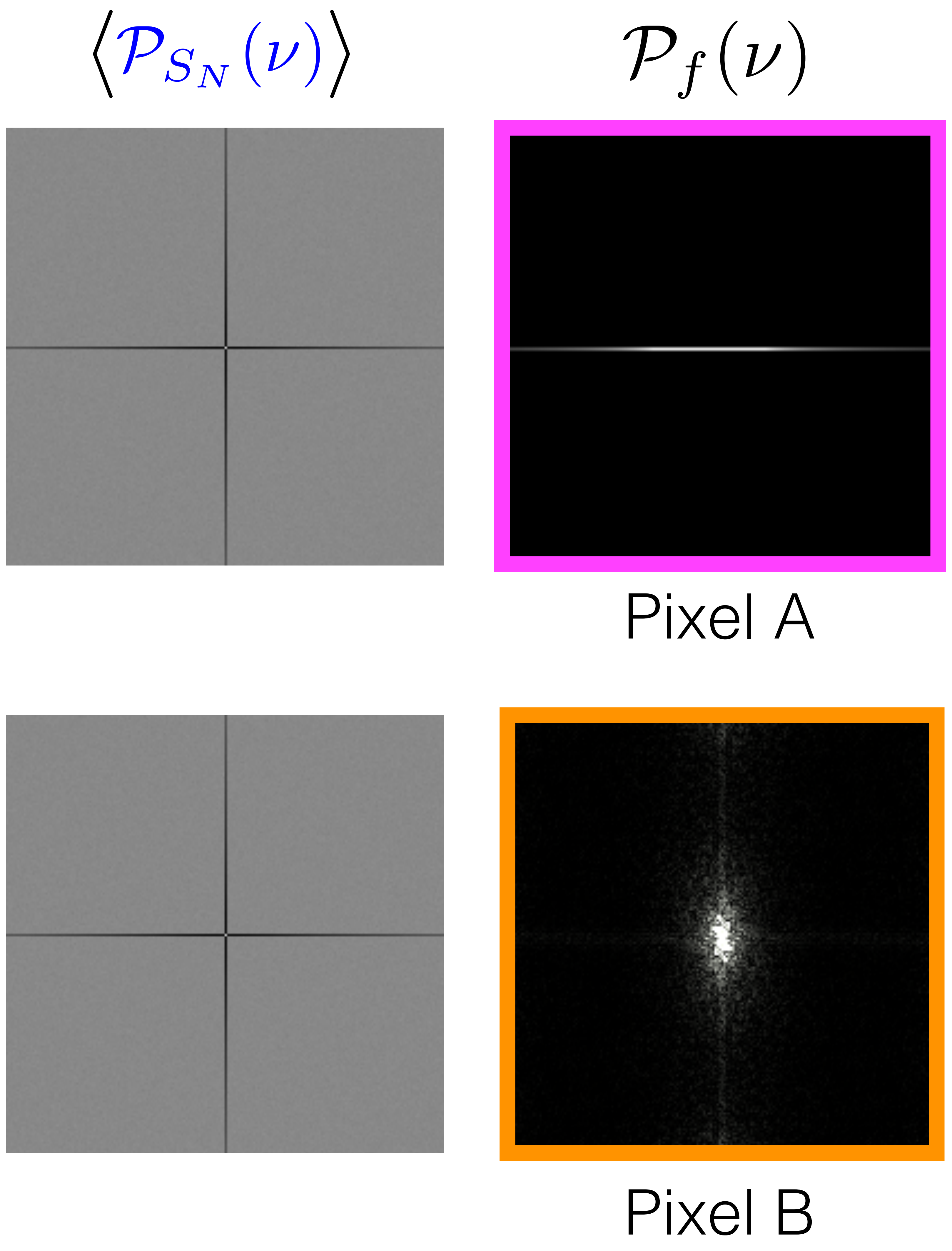


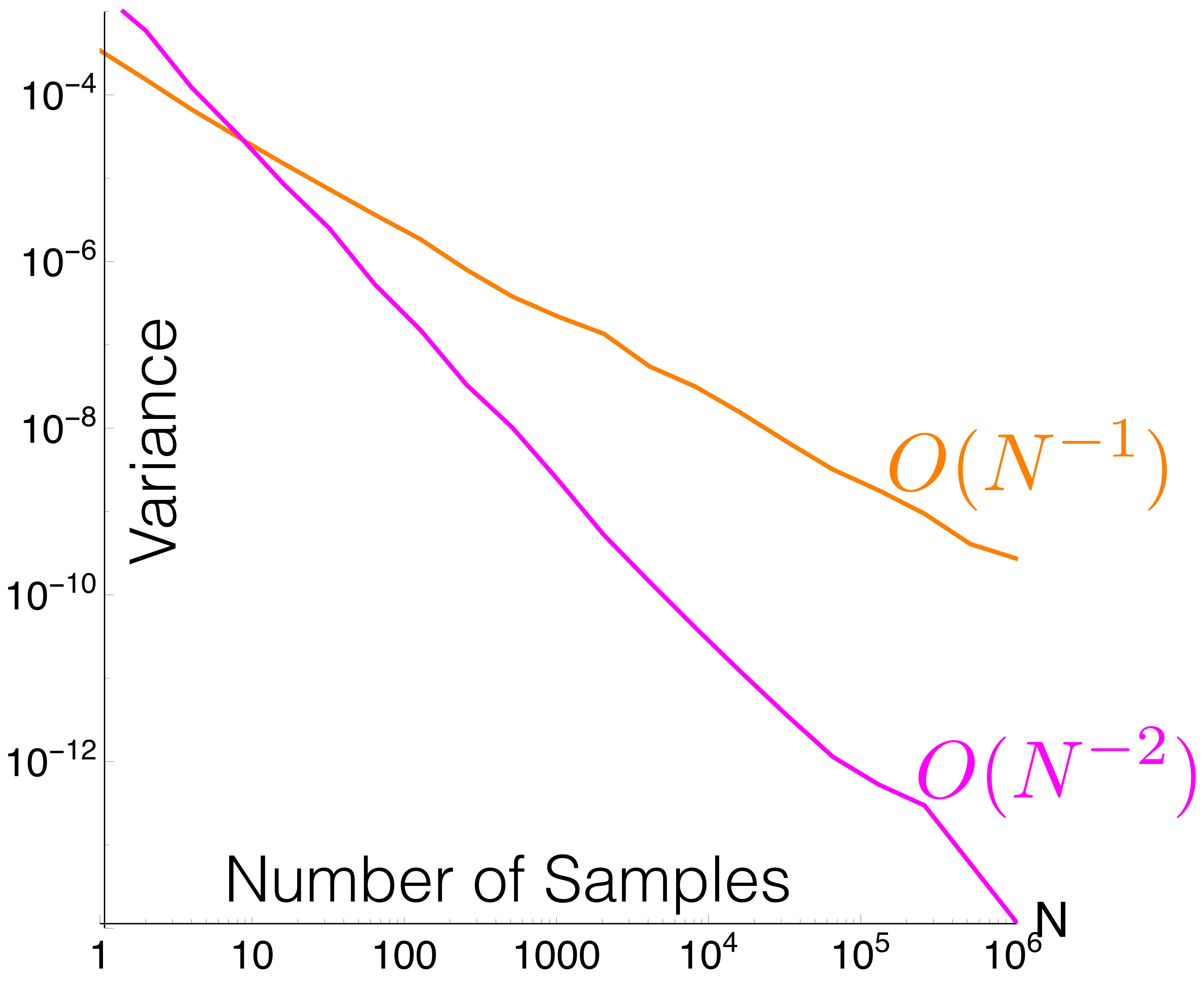
$d\nu$

Variance due to N-rooks Sampler



Variance Convergence of Latin Hypercube (N-rooks)



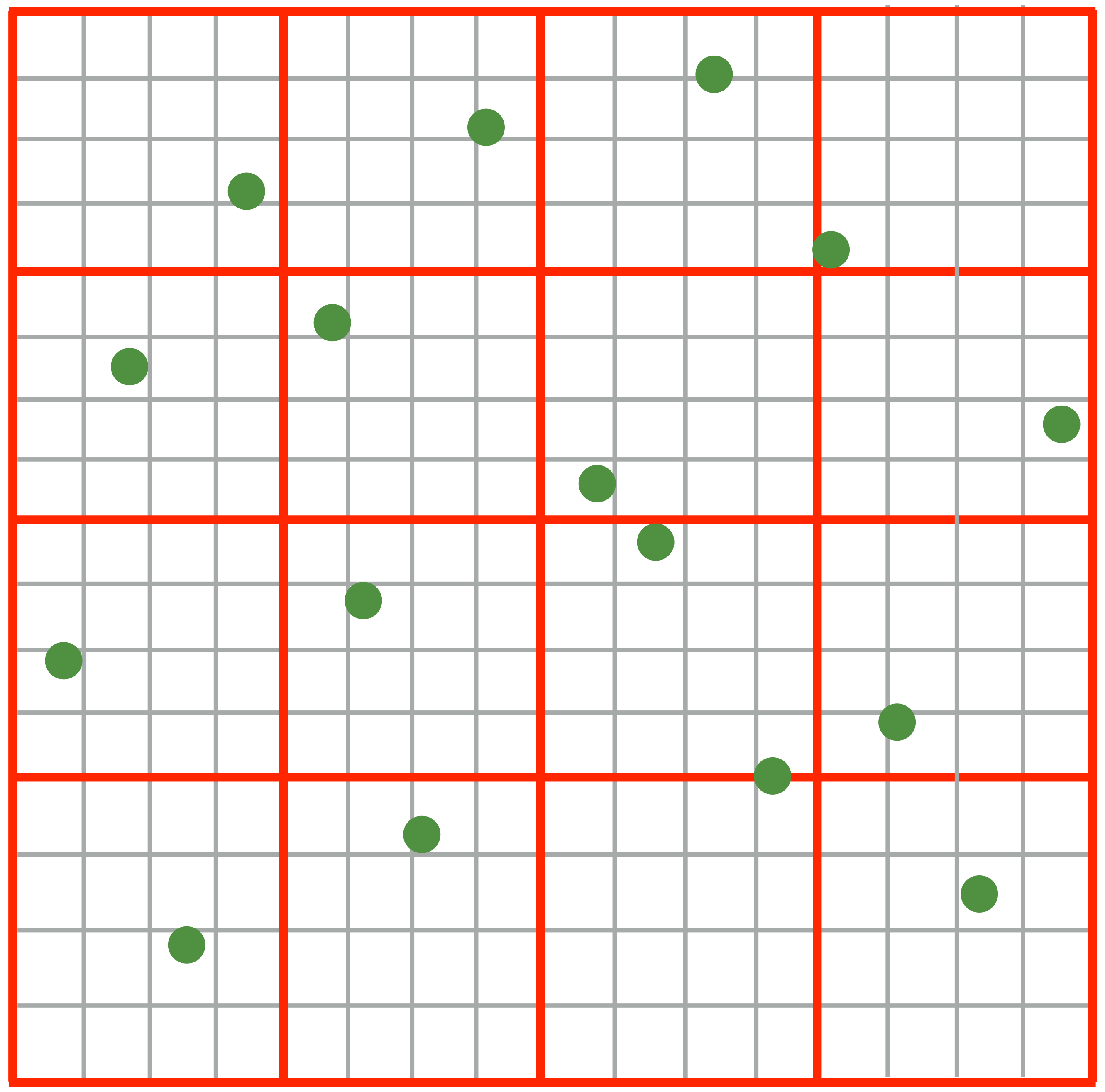


Non-Axis Aligned Integrand Spectra

$\mathcal{P}_f(\nu)$

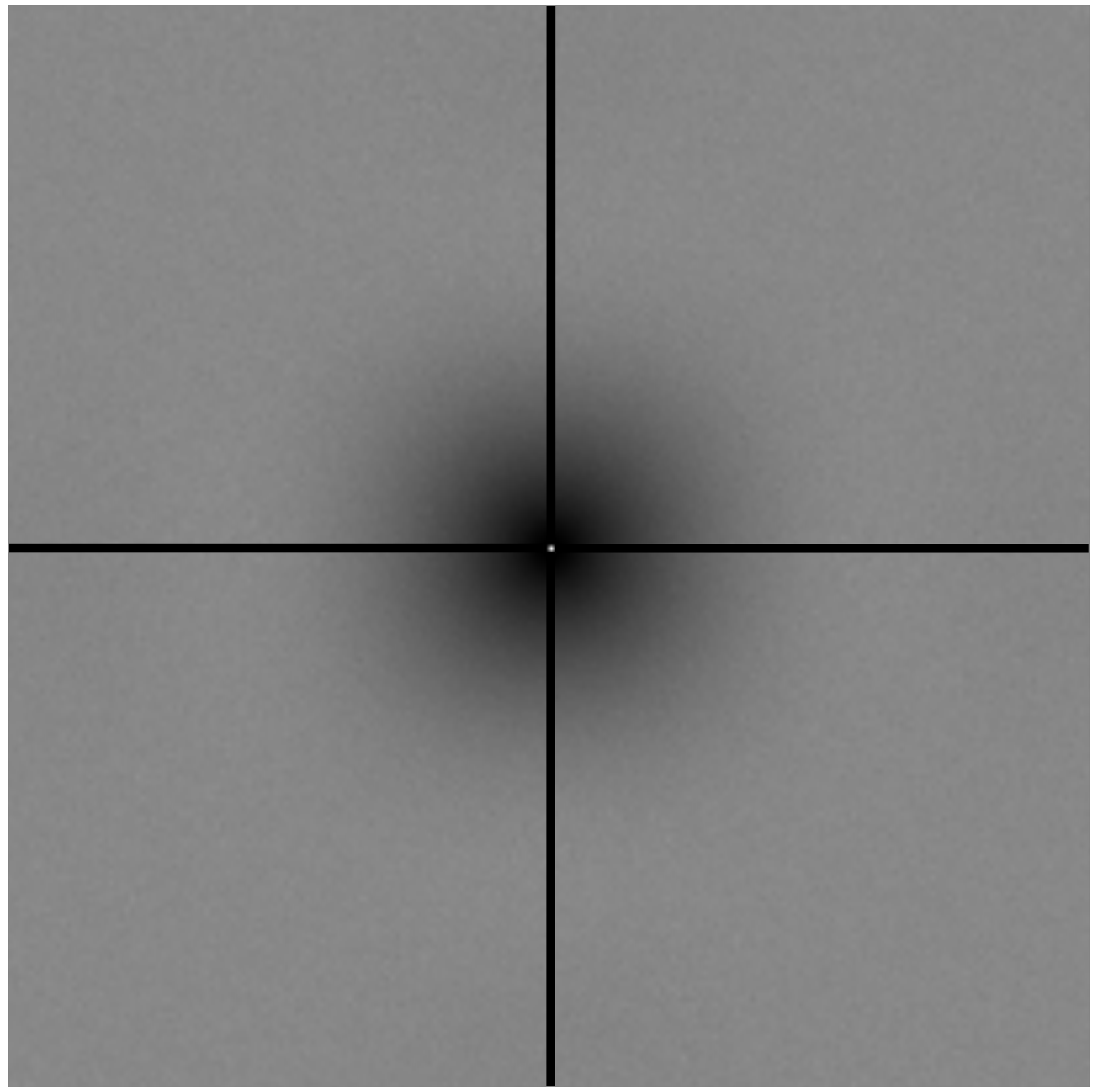
Integrand Spectrum

Non-Axis Aligned Integrand Spectra



Multi-jittered Samples

$\langle \mathcal{P}_{S_N}(\nu) \rangle$

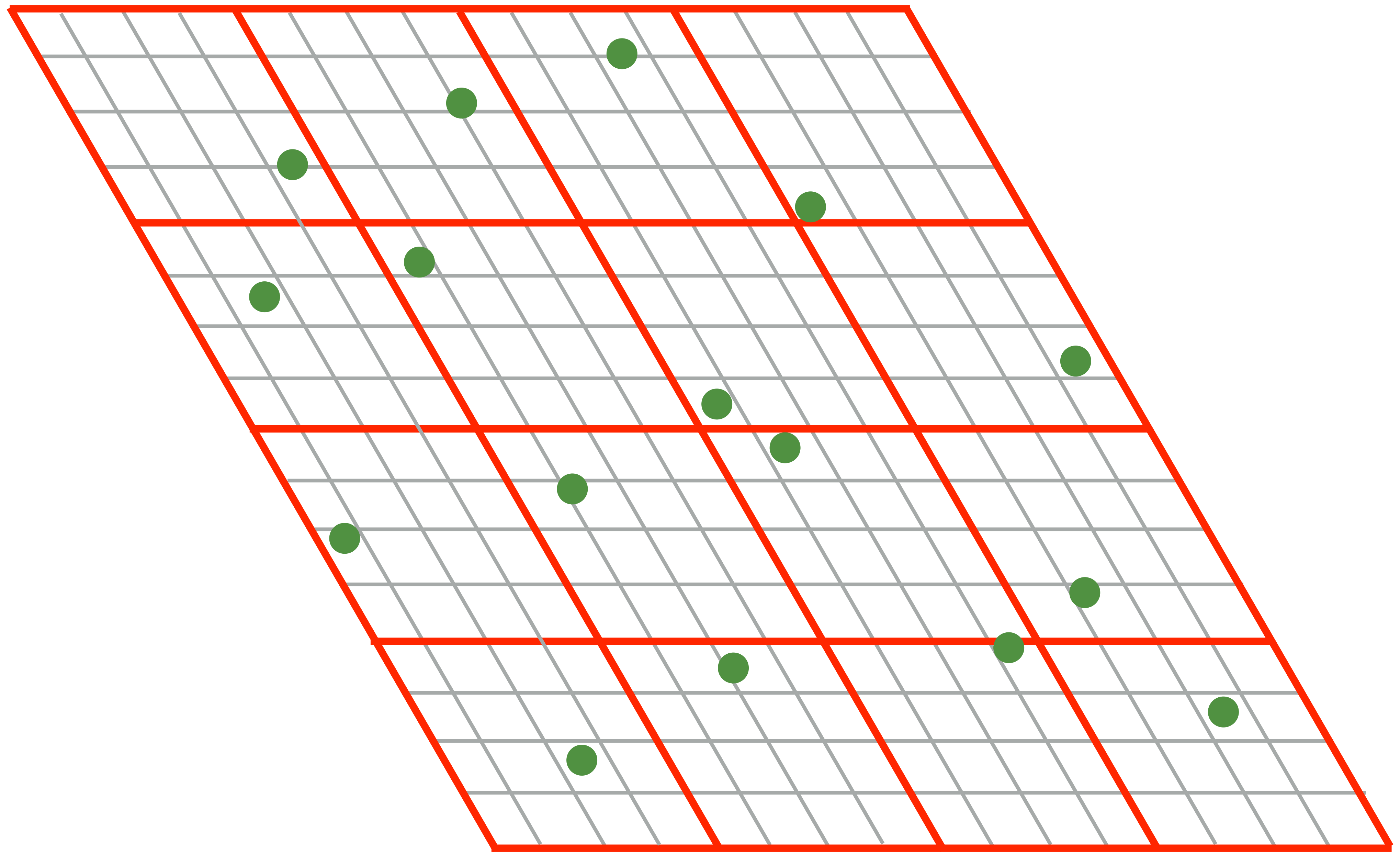


Sampling Spectrum

$\mathcal{P}_f(\nu)$

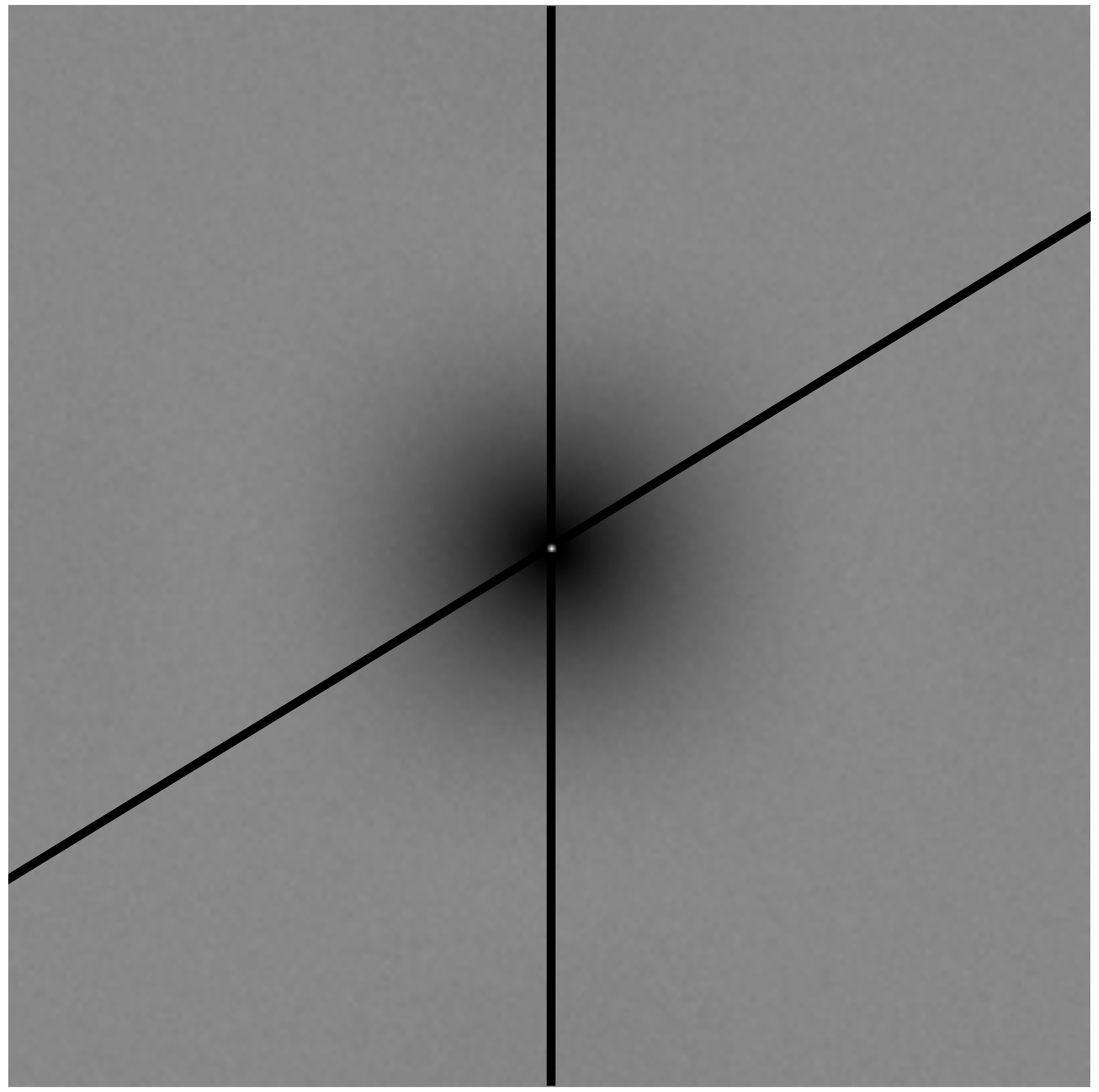
Integrand Spectrum

Shearing Multi-Jittered Samples



Sheared Samples

$\langle \mathcal{P}_{S_N}(\nu) \rangle$

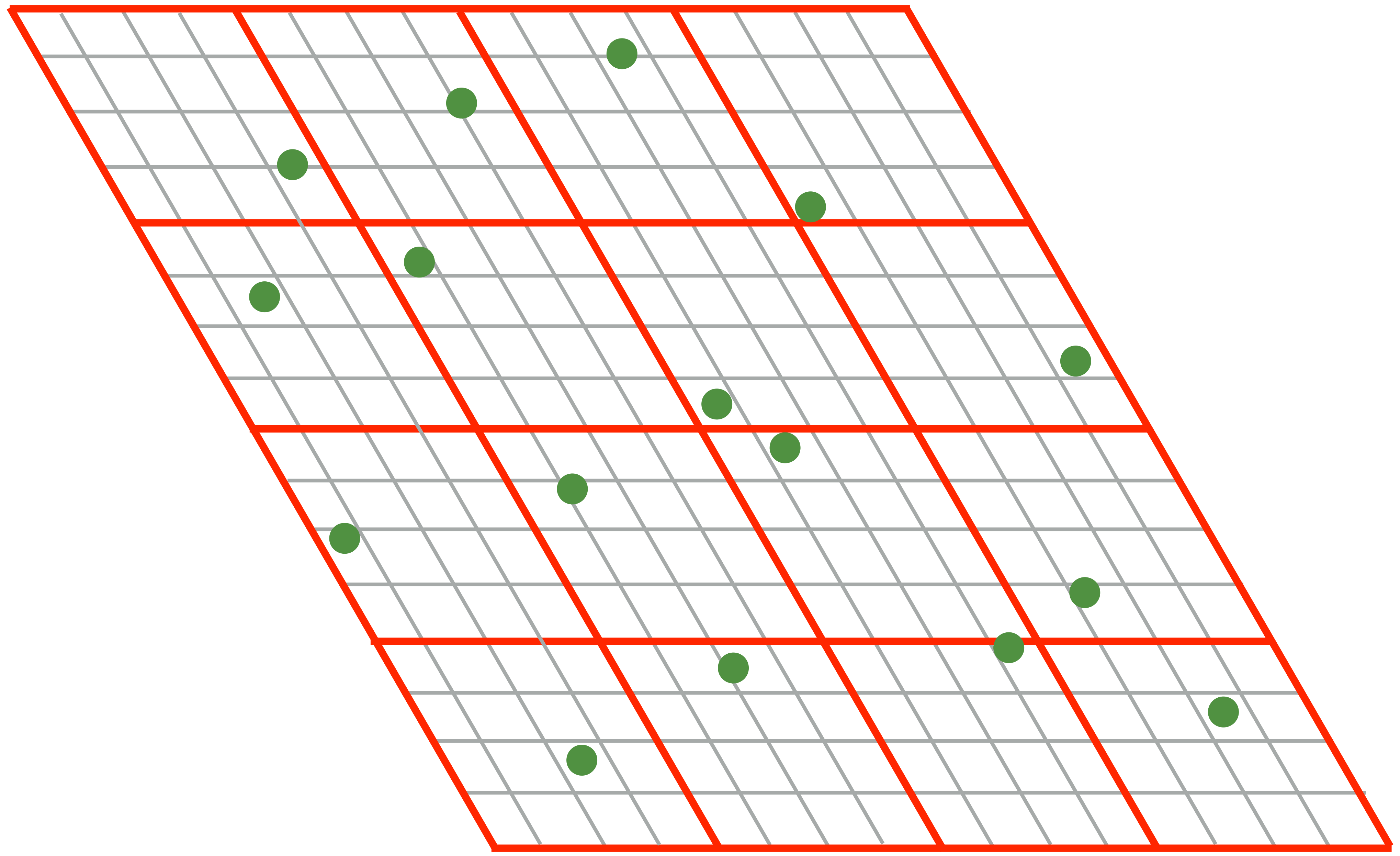


Sheared Spectrum

$\mathcal{P}_f(\nu)$

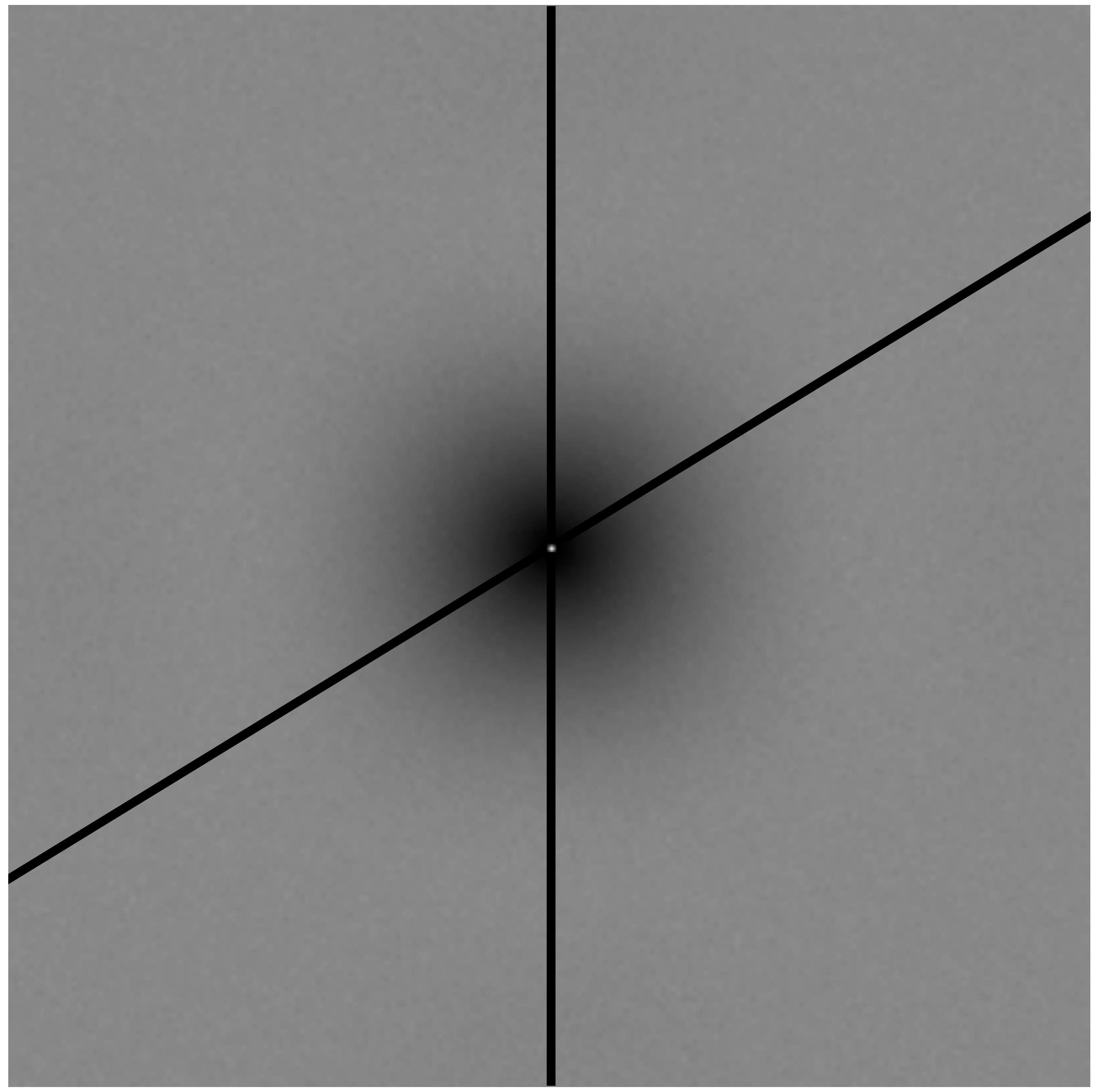
Integrand Spectrum

How can we determine the sample shearing parameters ?



Sheared Samples

$\langle \mathcal{P}_{S_N}(\nu) \rangle$



Sheared Spectrum

$\mathcal{P}_f(\nu)$

Integrand Spectrum

Frequency Analysis of Light Transport

Related Work

- Frequency Analysis of Light Transport Durand et al. [2005]
- Depth of Field Soler et al. [2009]
- Motion Blur Egan et al. [2009]
- Ambient Occlusion Egan et al. [2011] and more…

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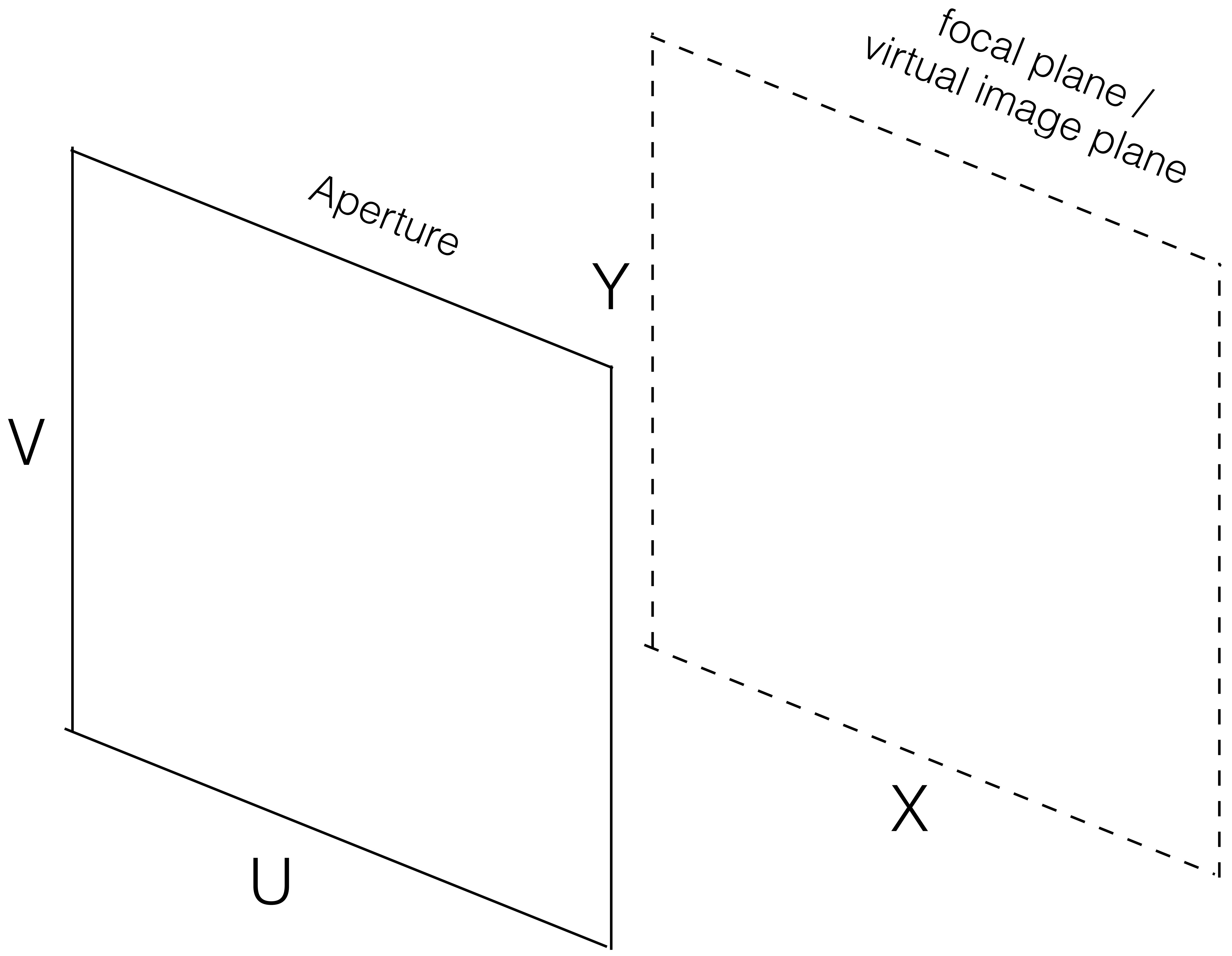
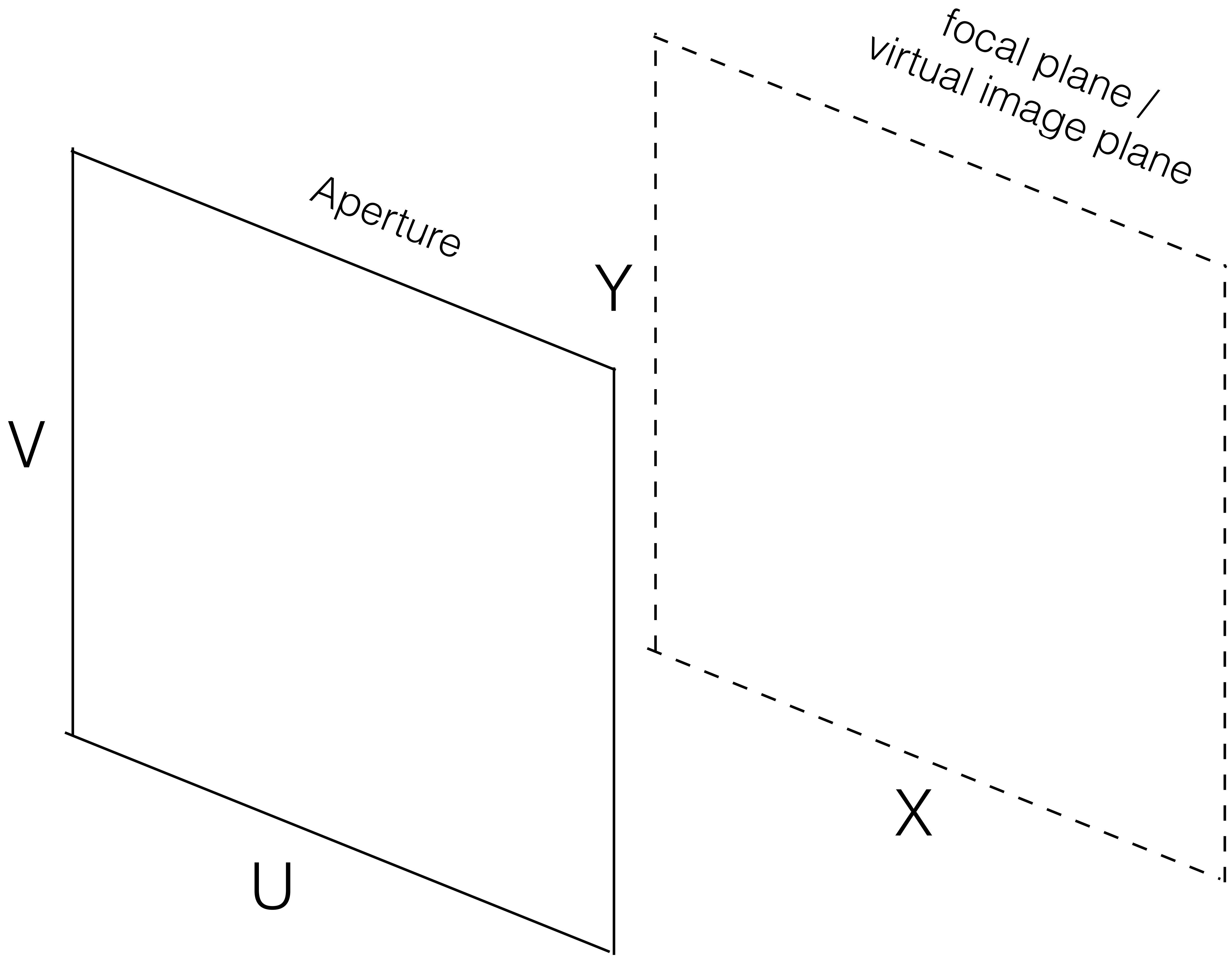
Reconstruction

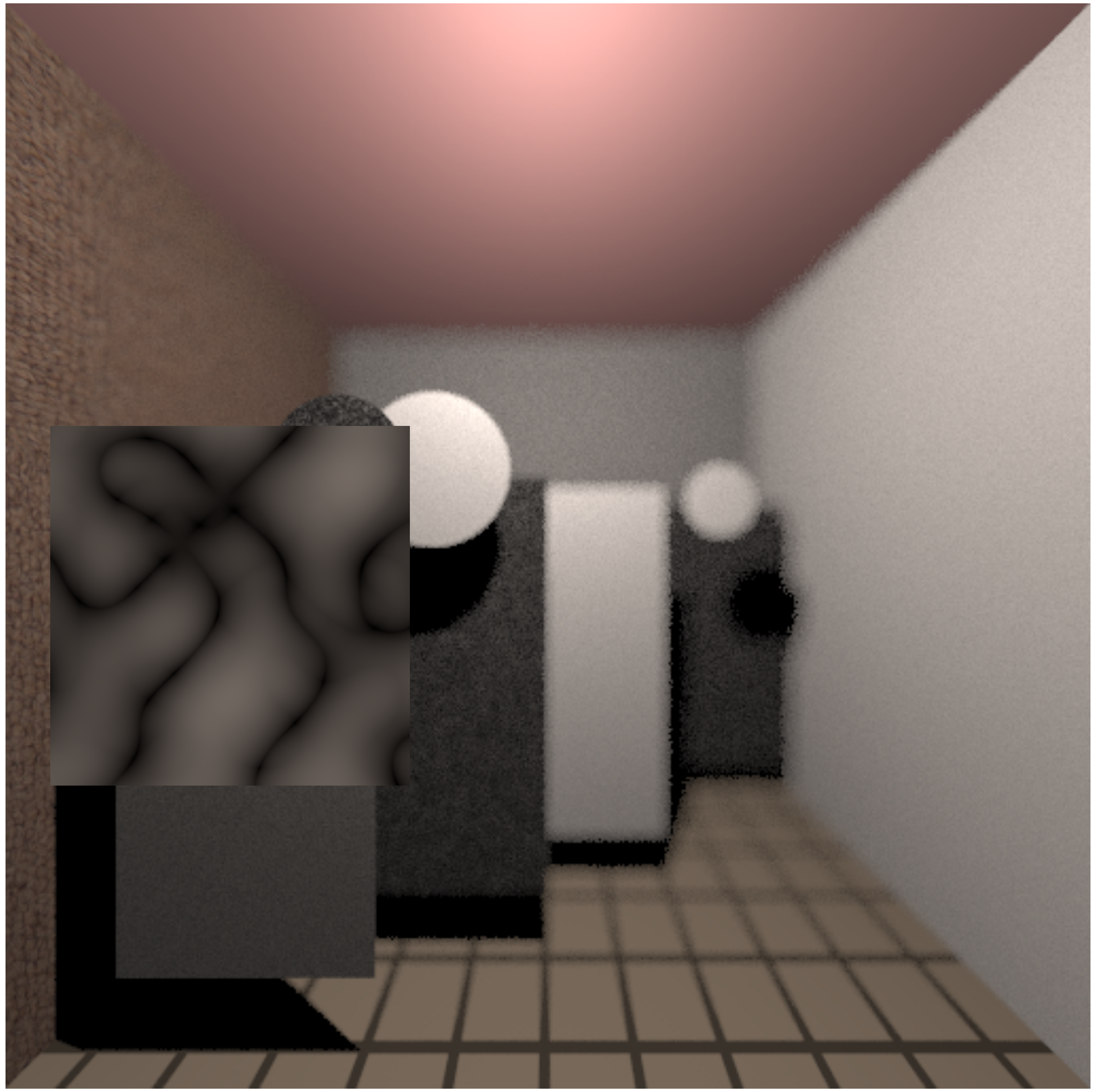
Integration

Depth of Field Analysis

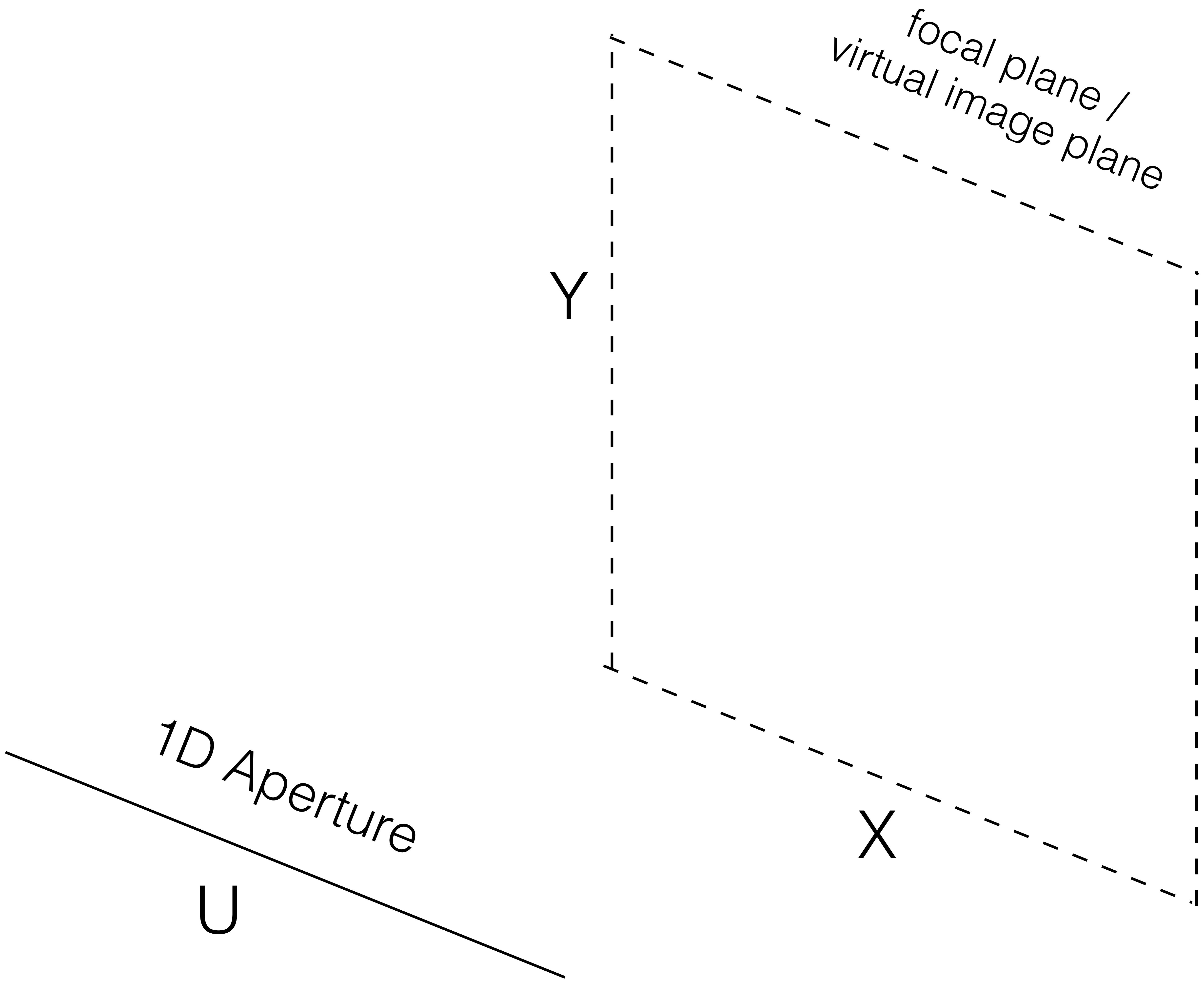


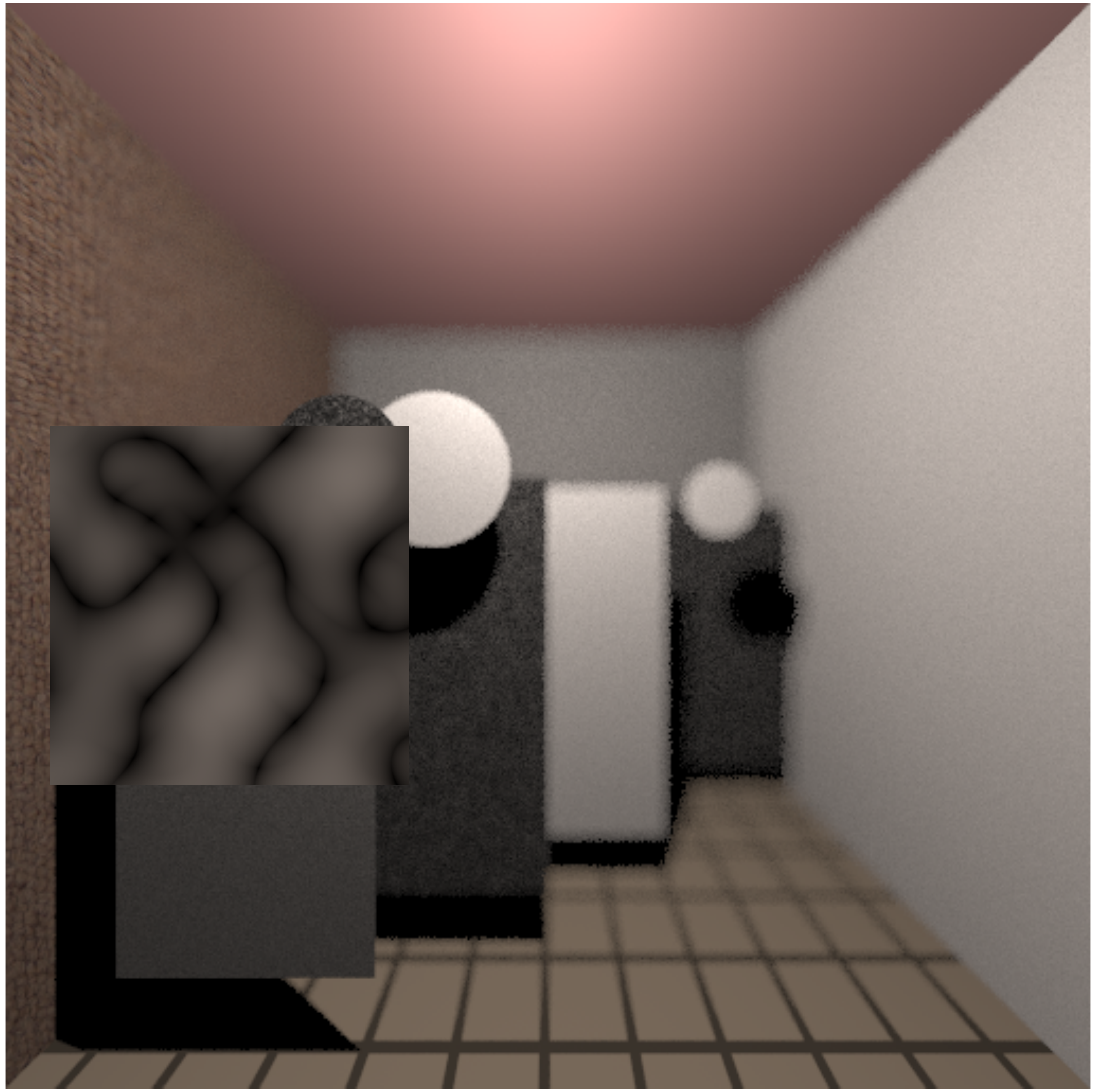
Depth of Field Analysis



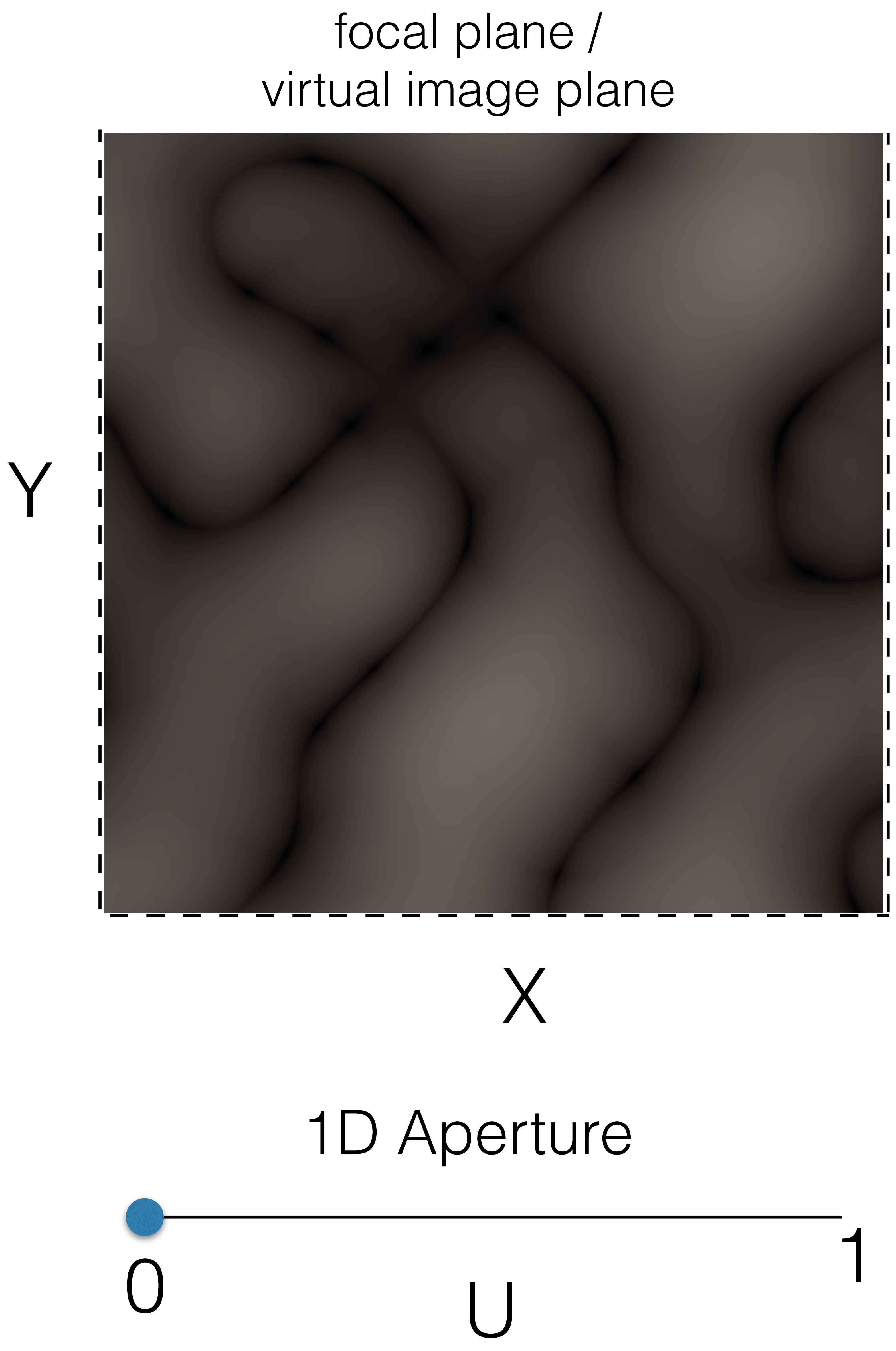


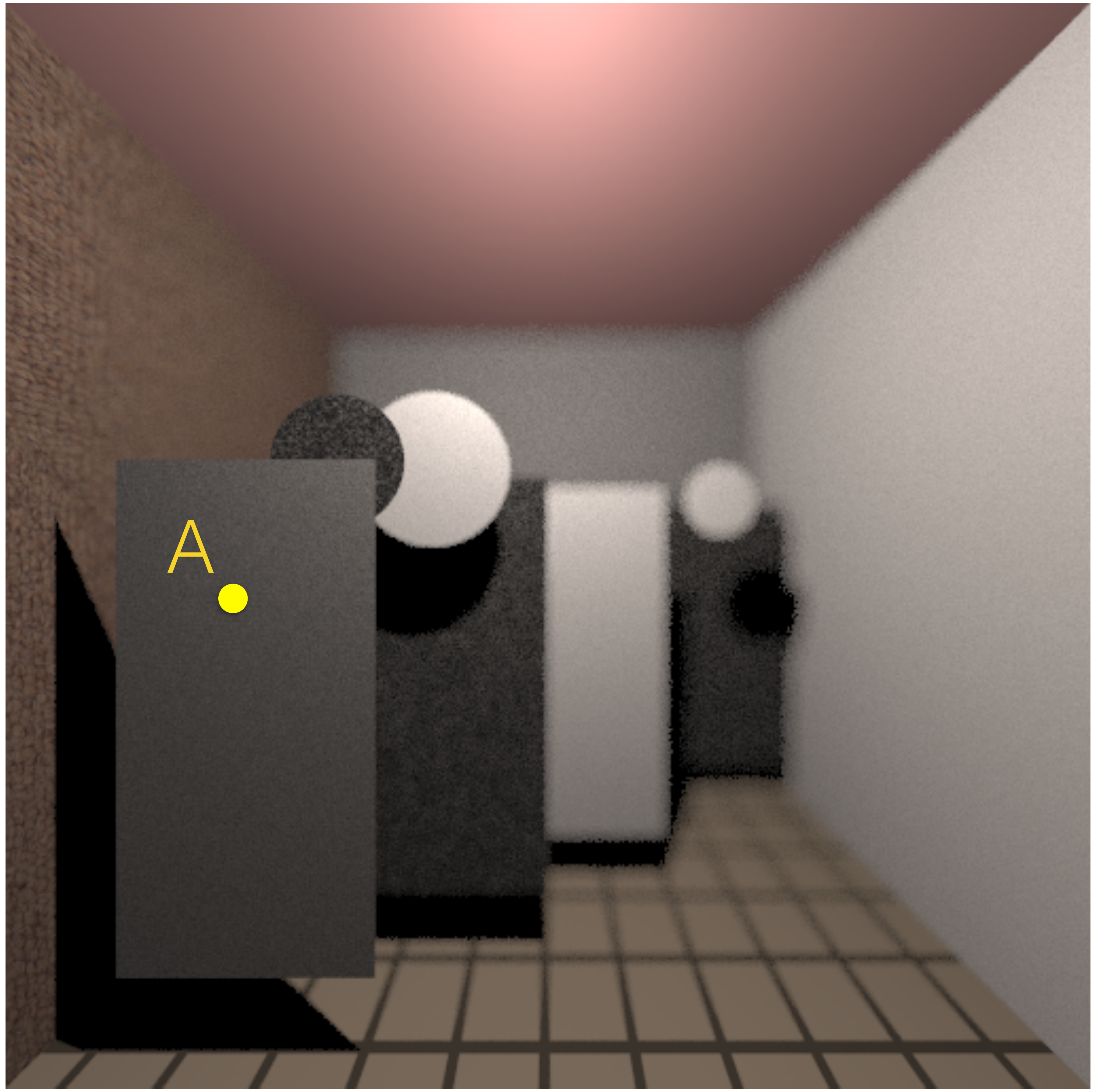
Depth of Field Analysis



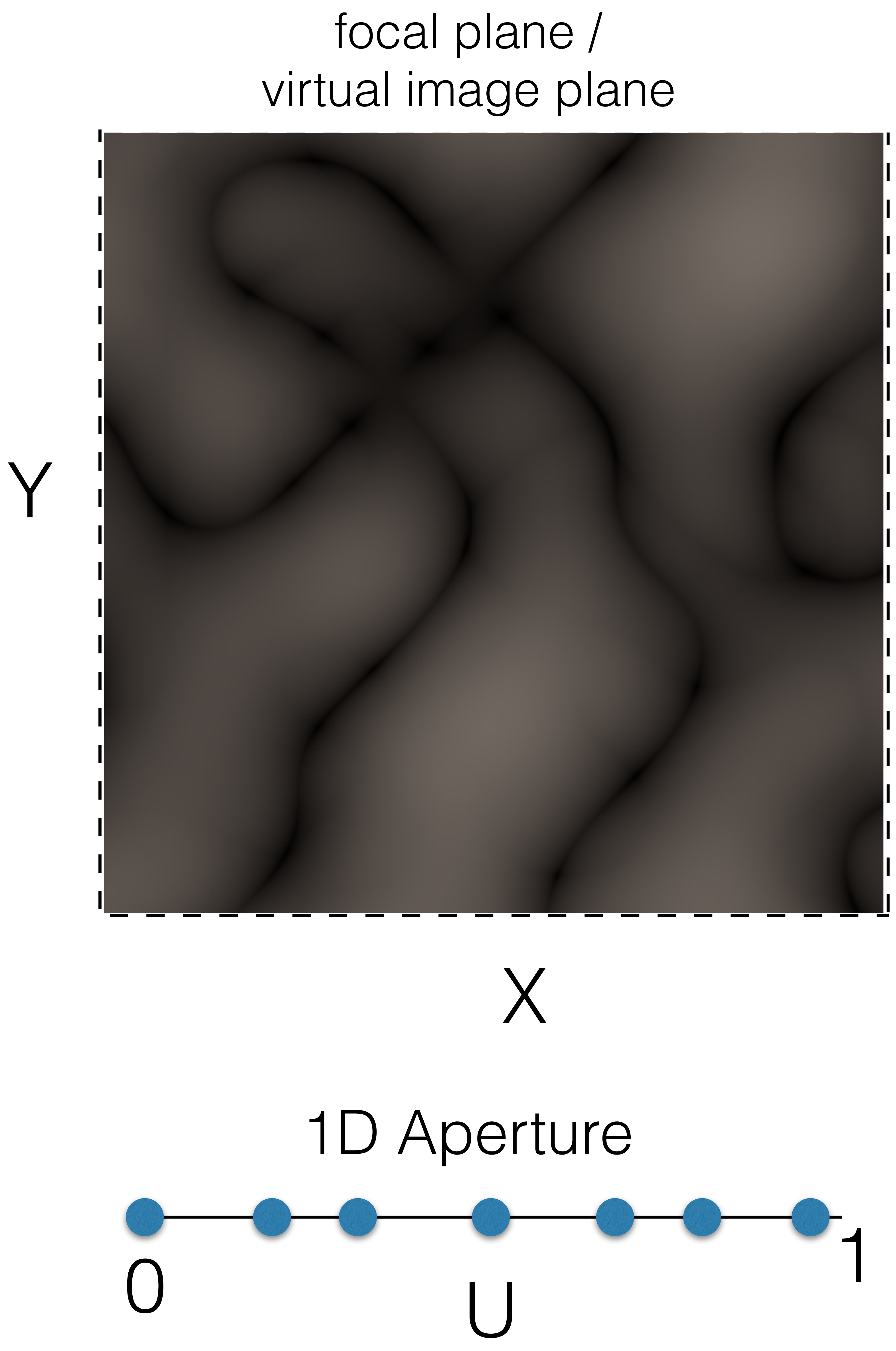


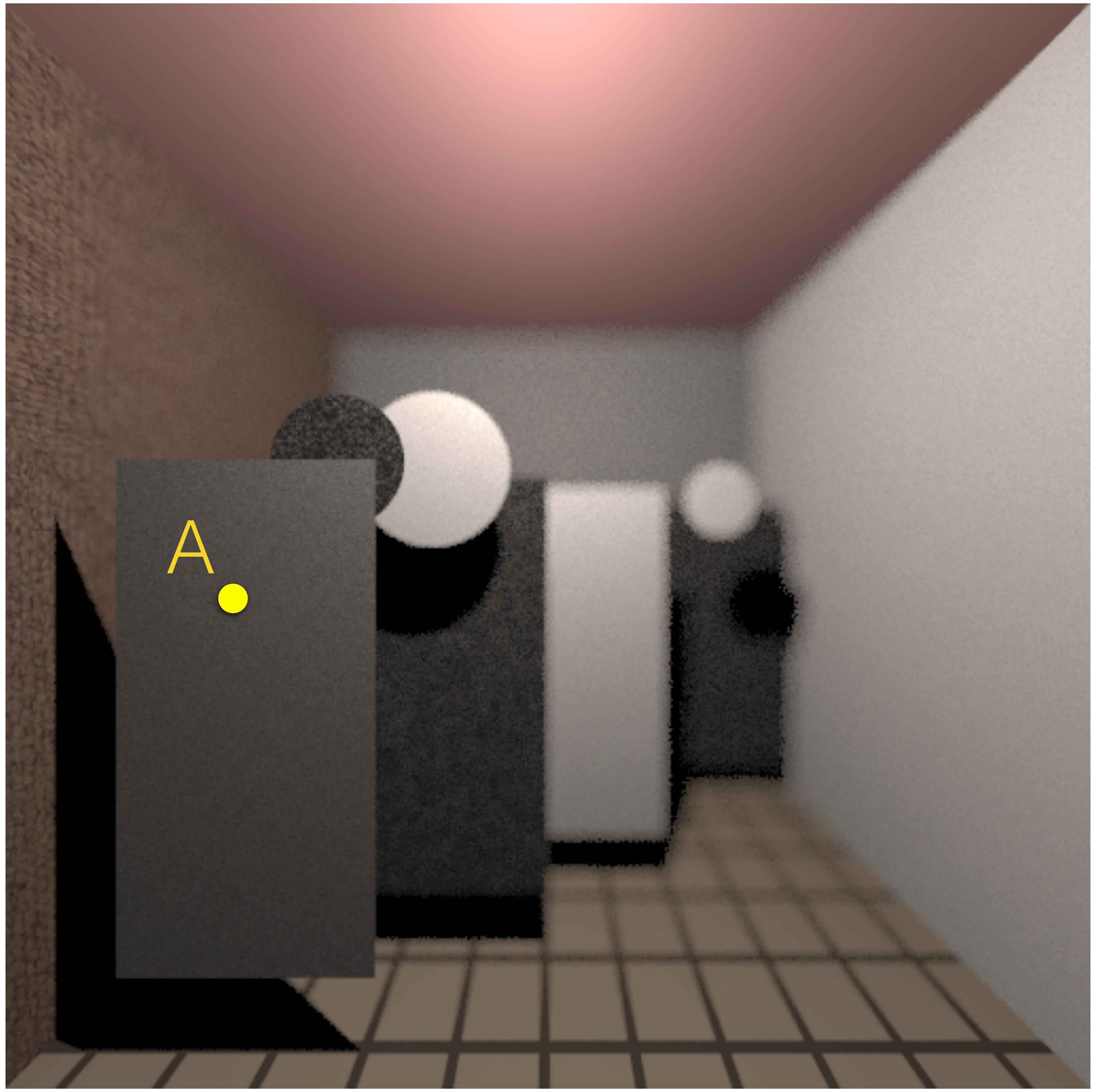
Depth of Field Analysis



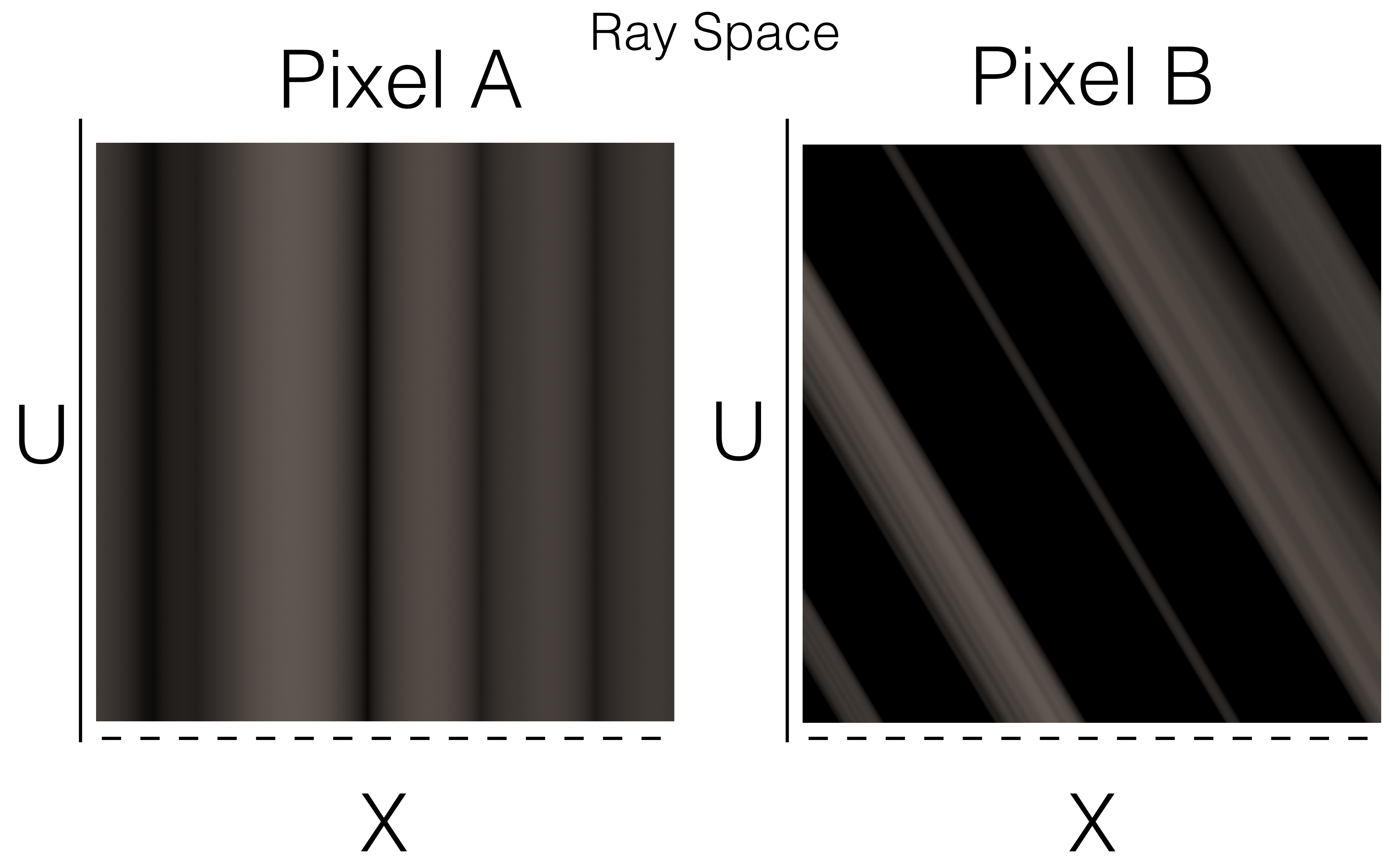


Depth of Field Analysis

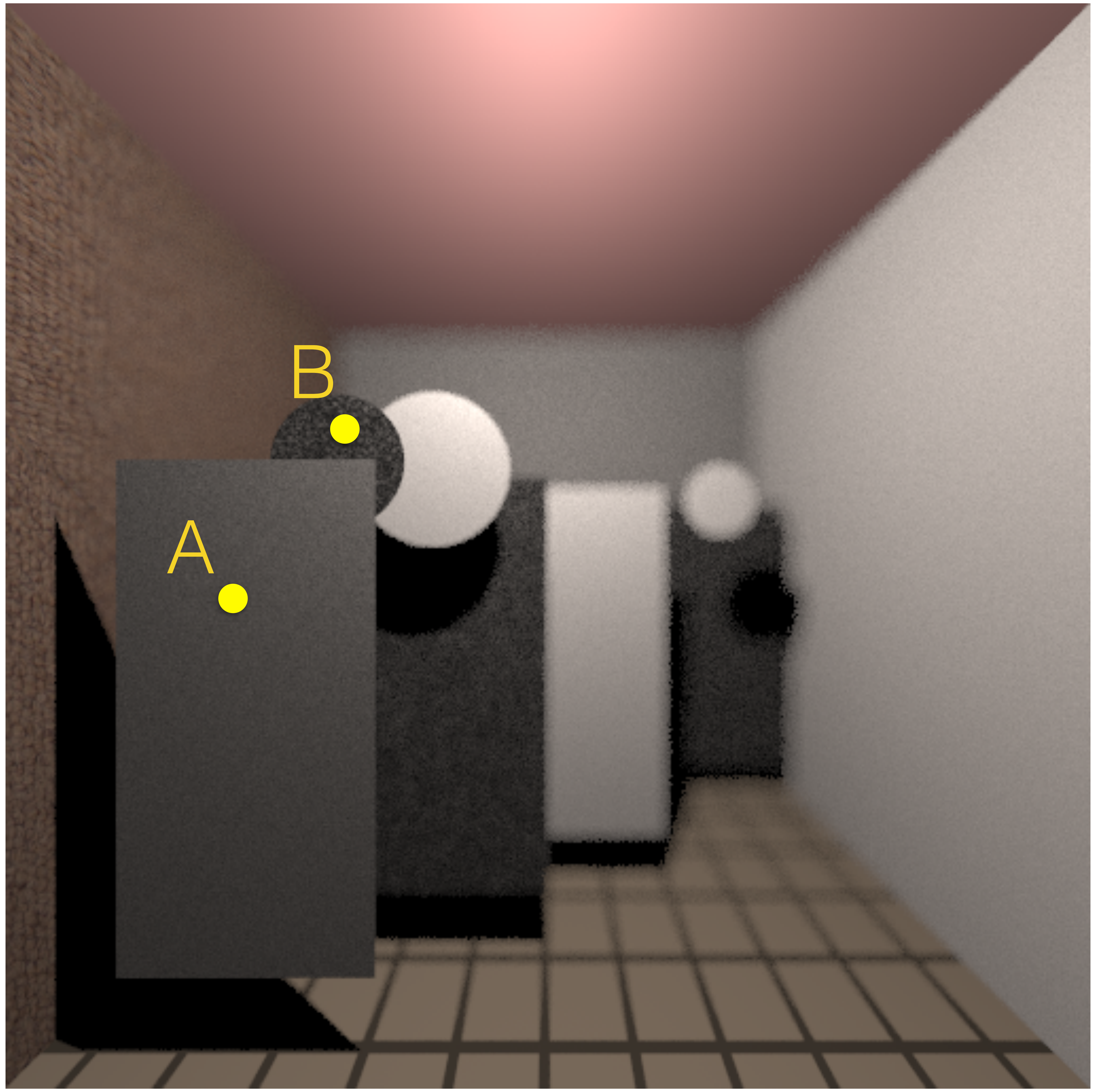




Depth of Field Analysis

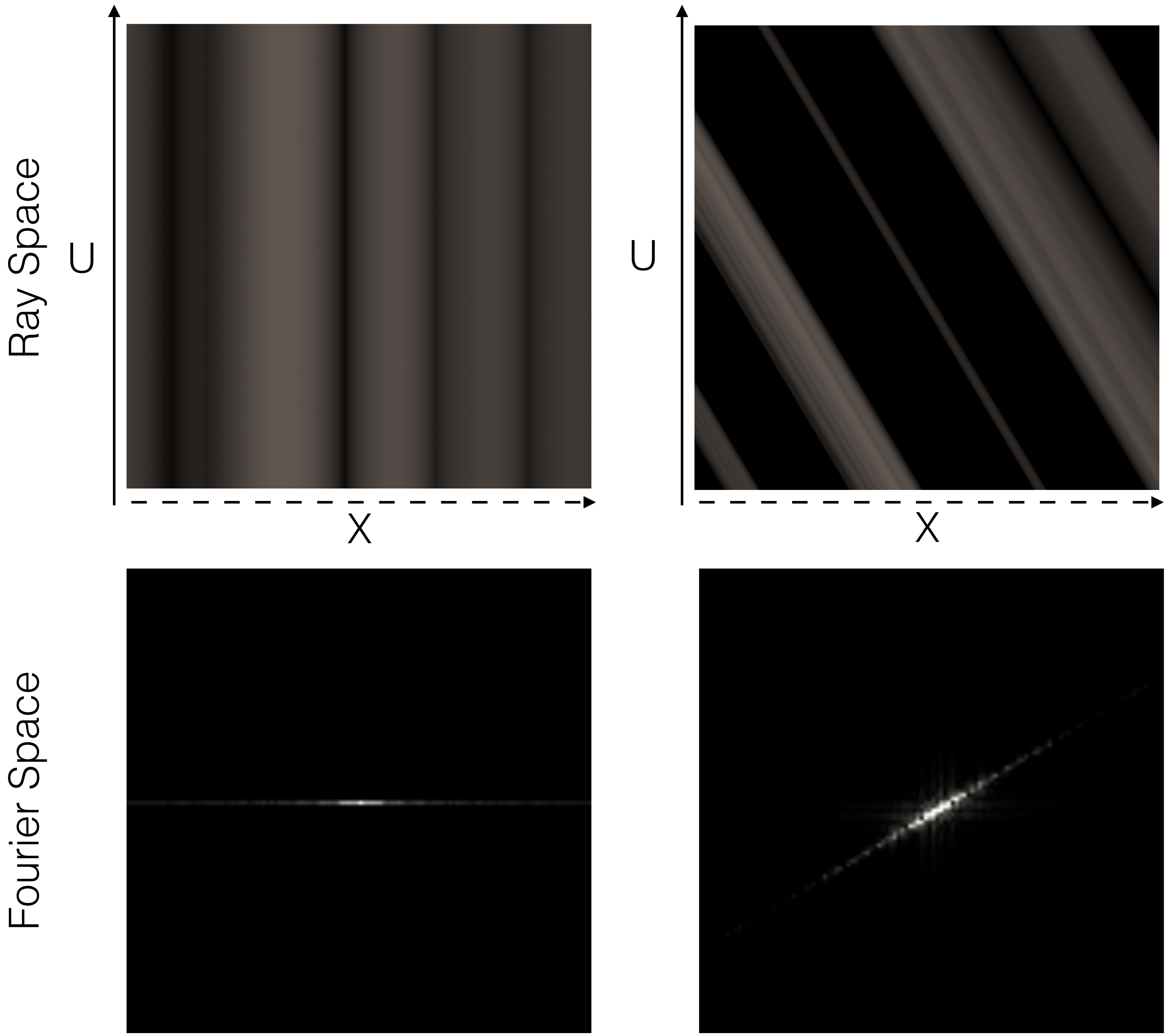


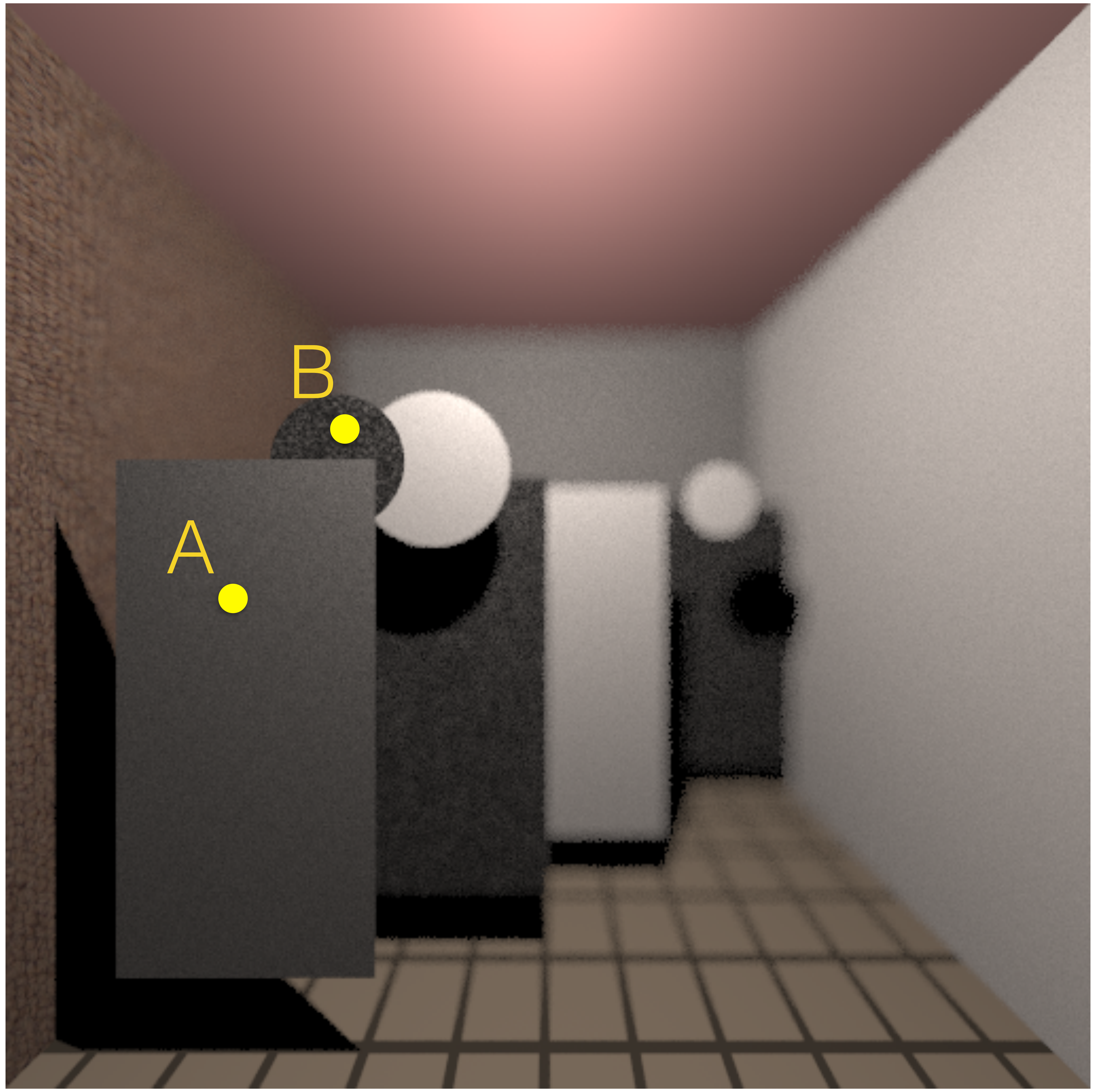
XU Slices



Durand et al. [2005]

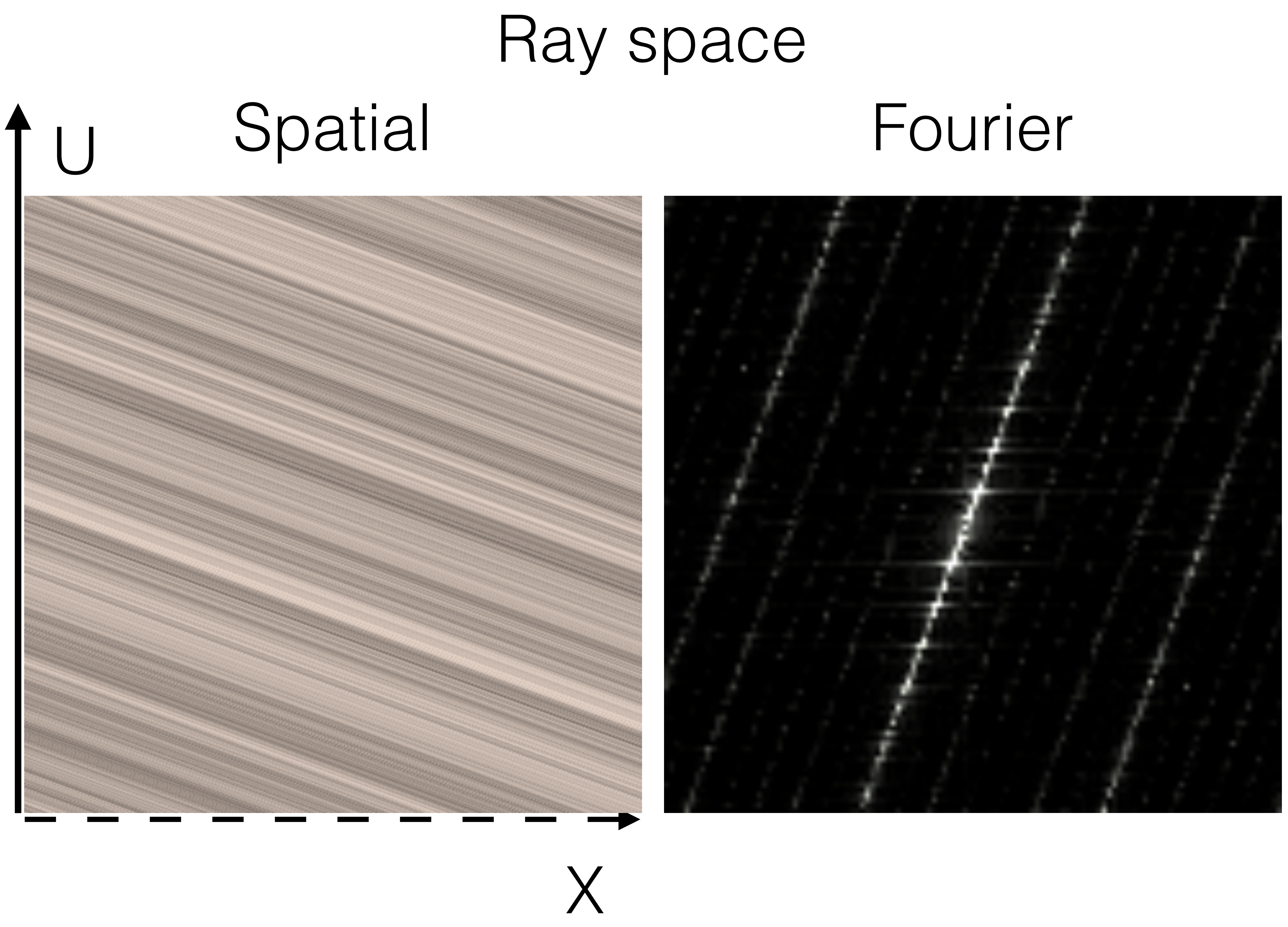
Depth of Field Analysis



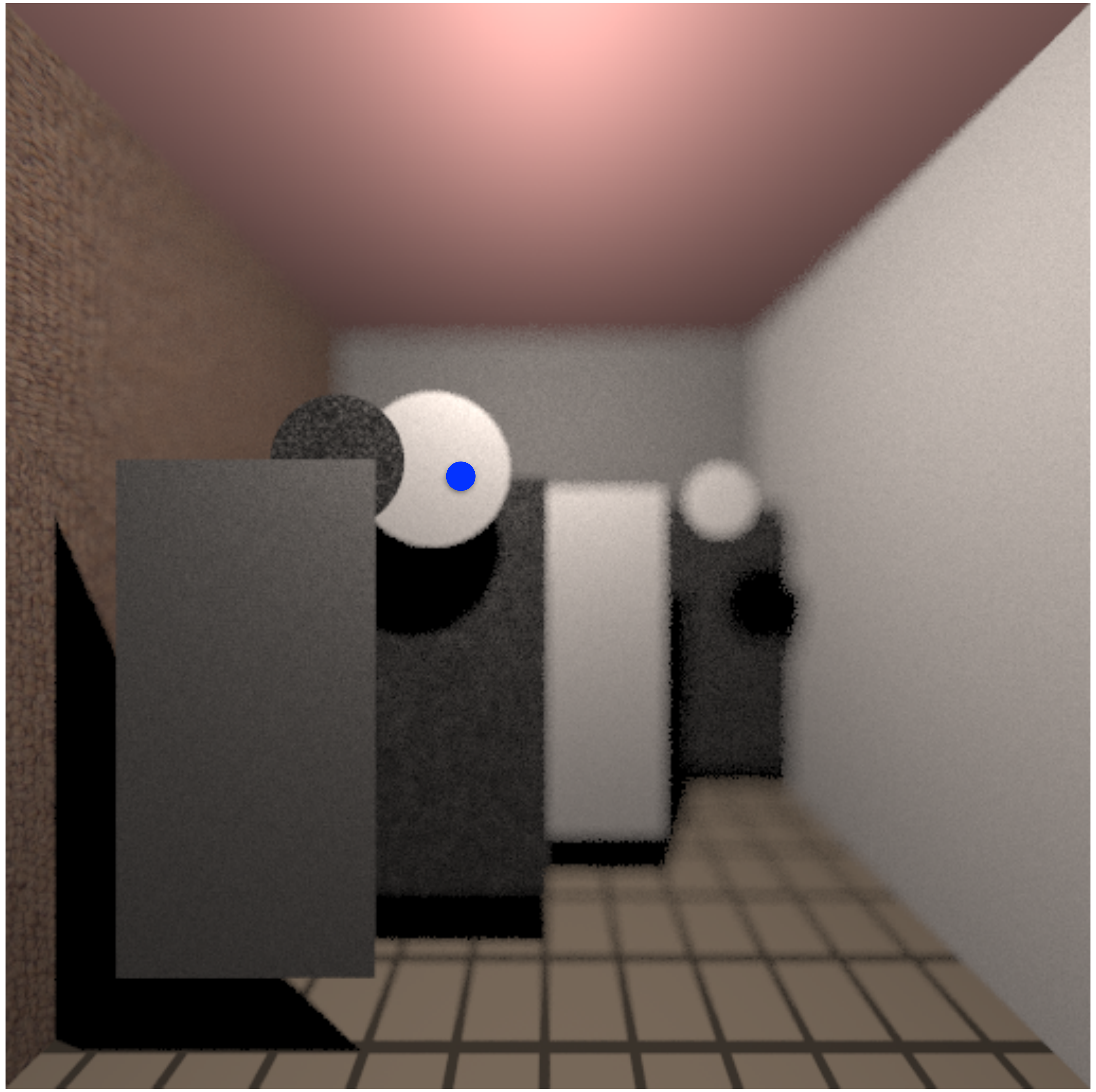


Durand et al. [2005]

Depth of Field Analysis



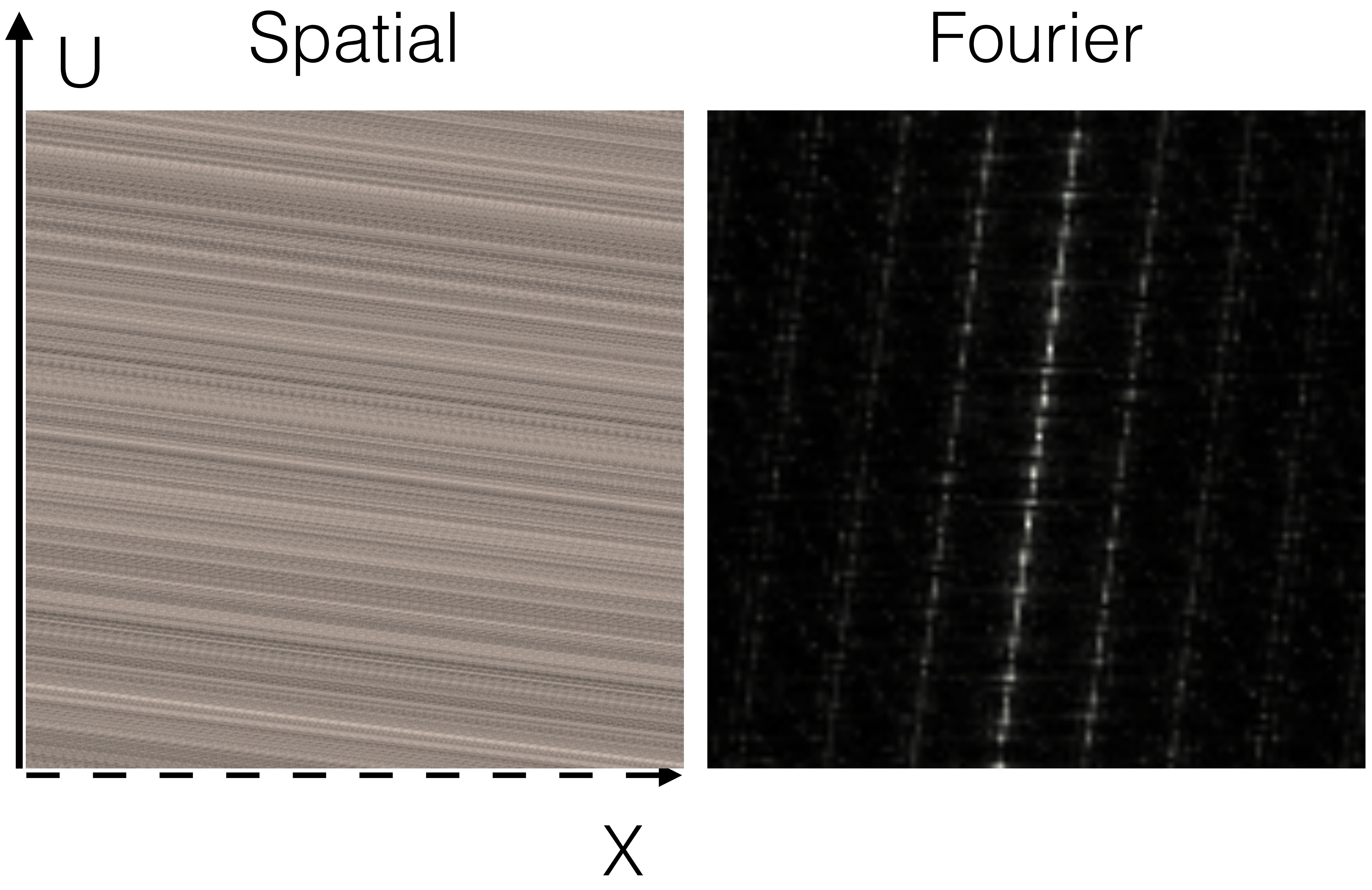
XU Slices



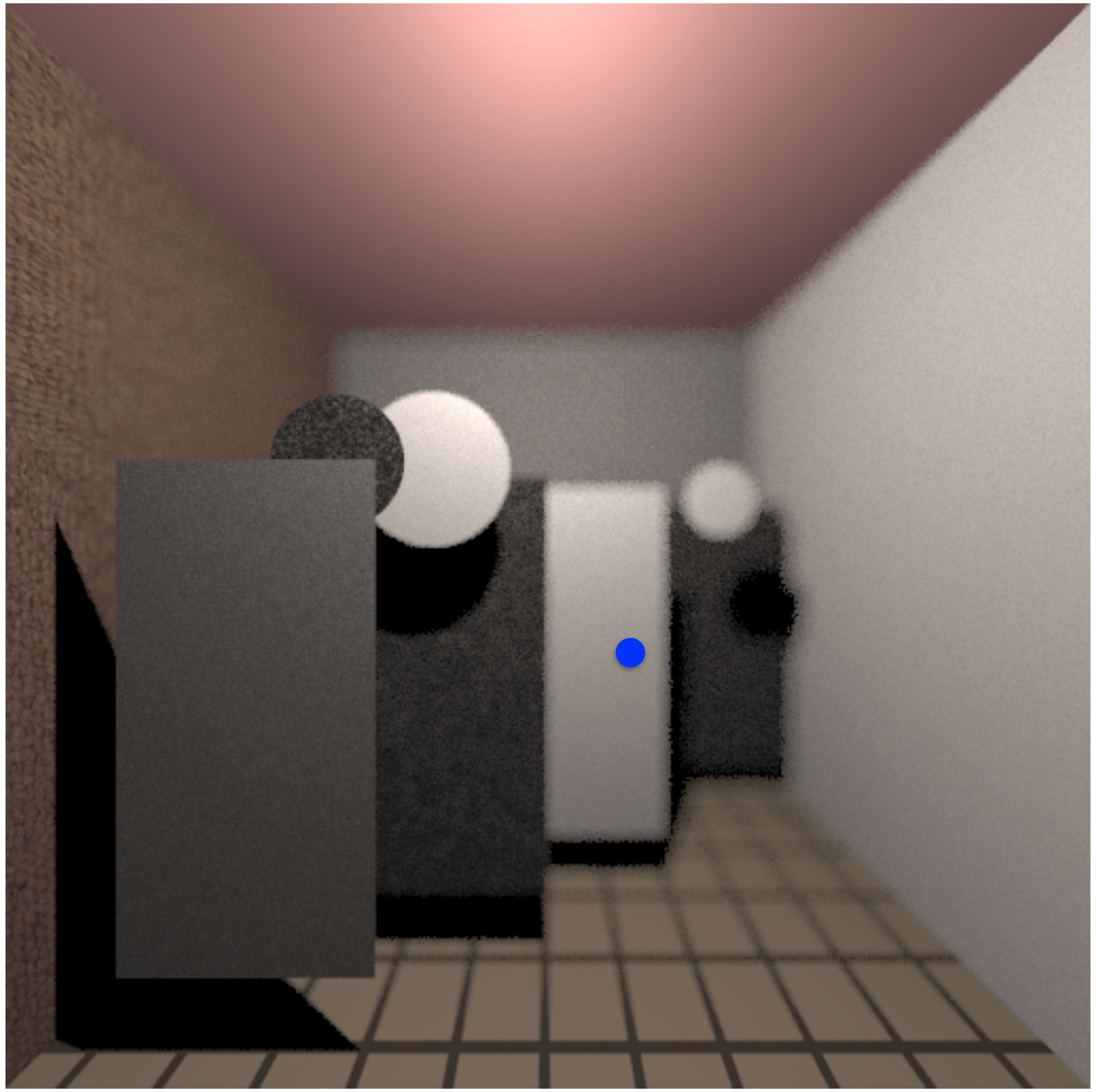
Durand et al. [2005]

Depth of Field Analysis

Ray space



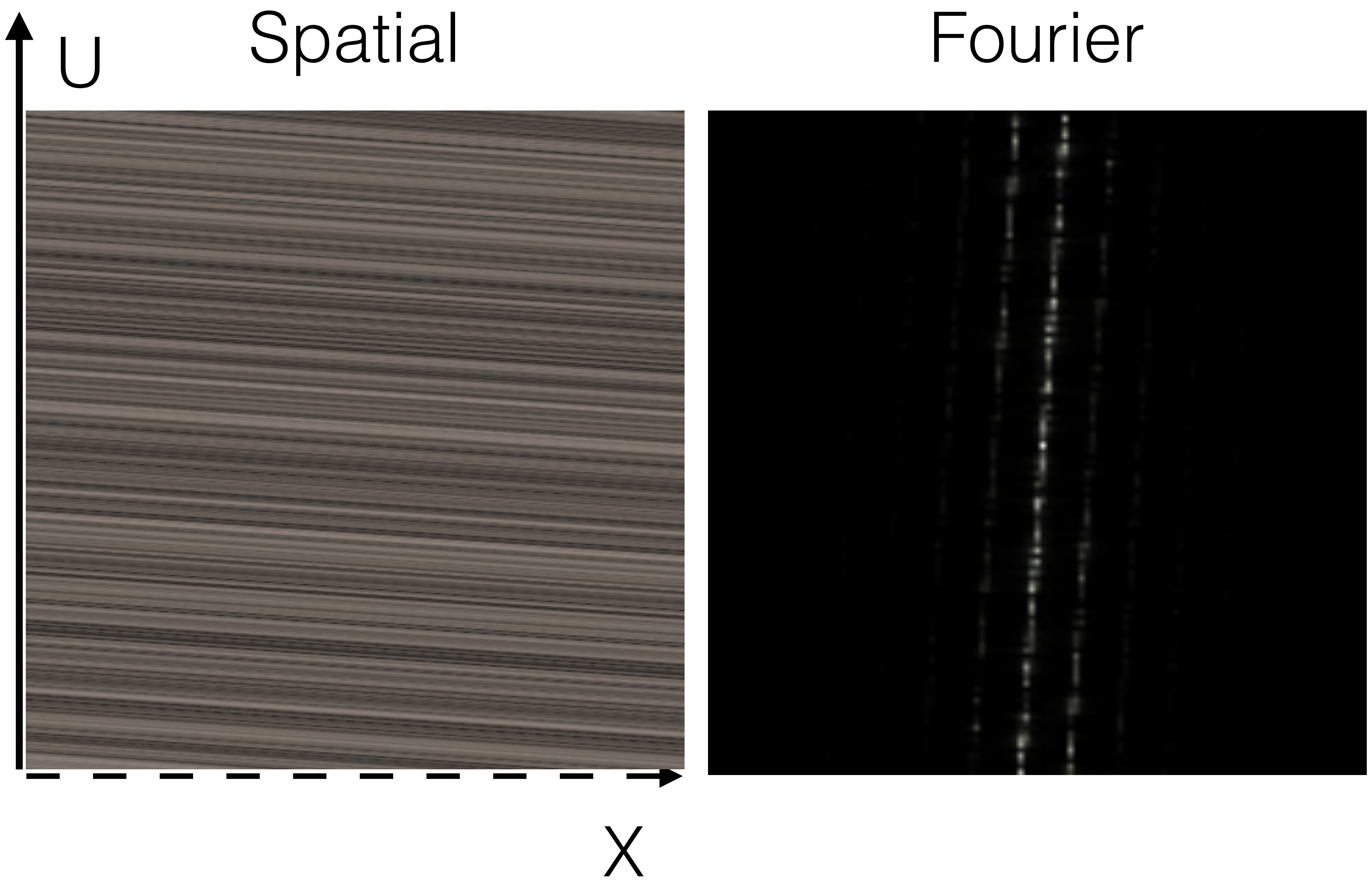
XU Slices



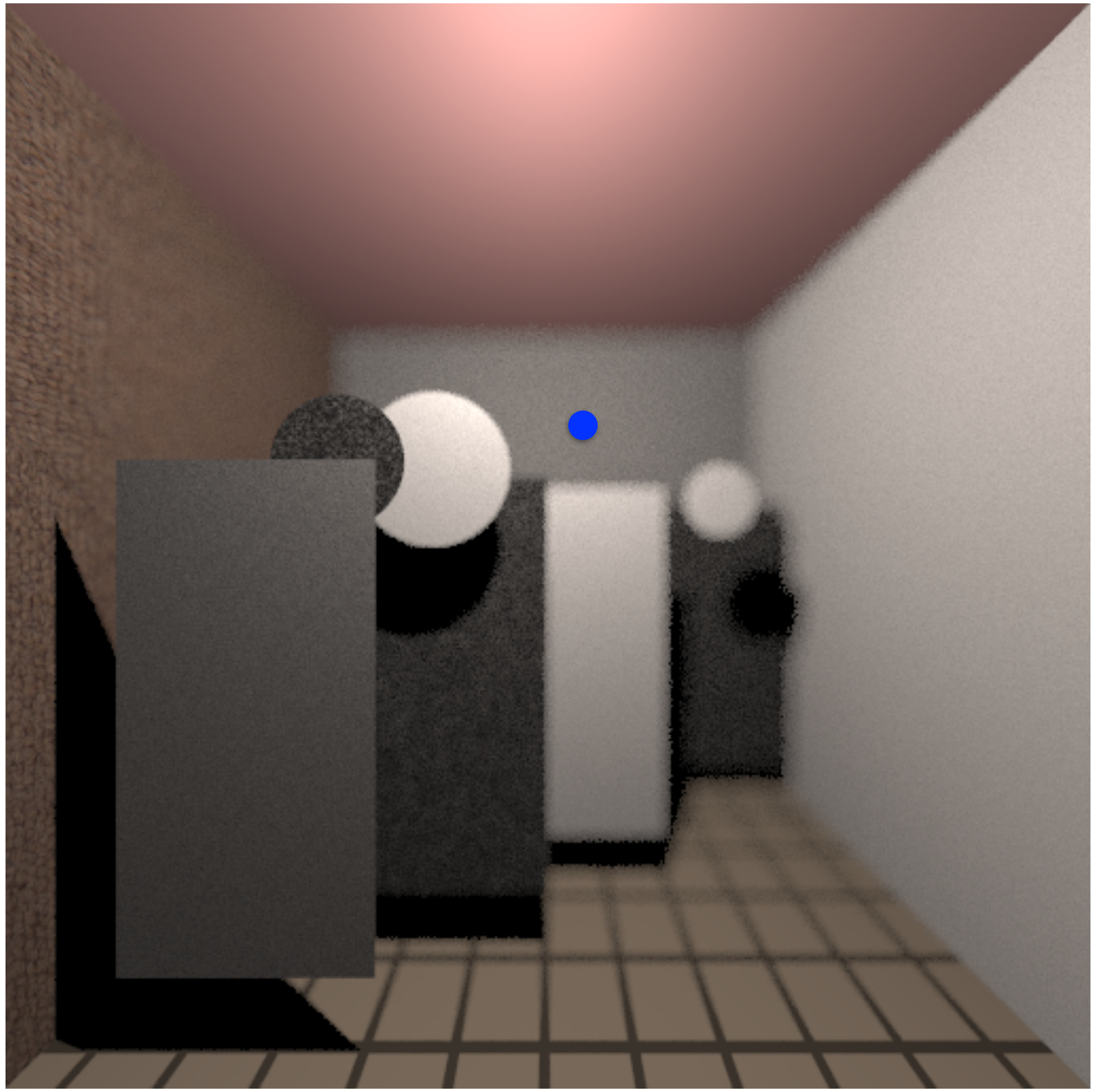
Durand et al. [2005]

Depth of Field Analysis

Ray space



XU Slices



Durand et al. [2005]

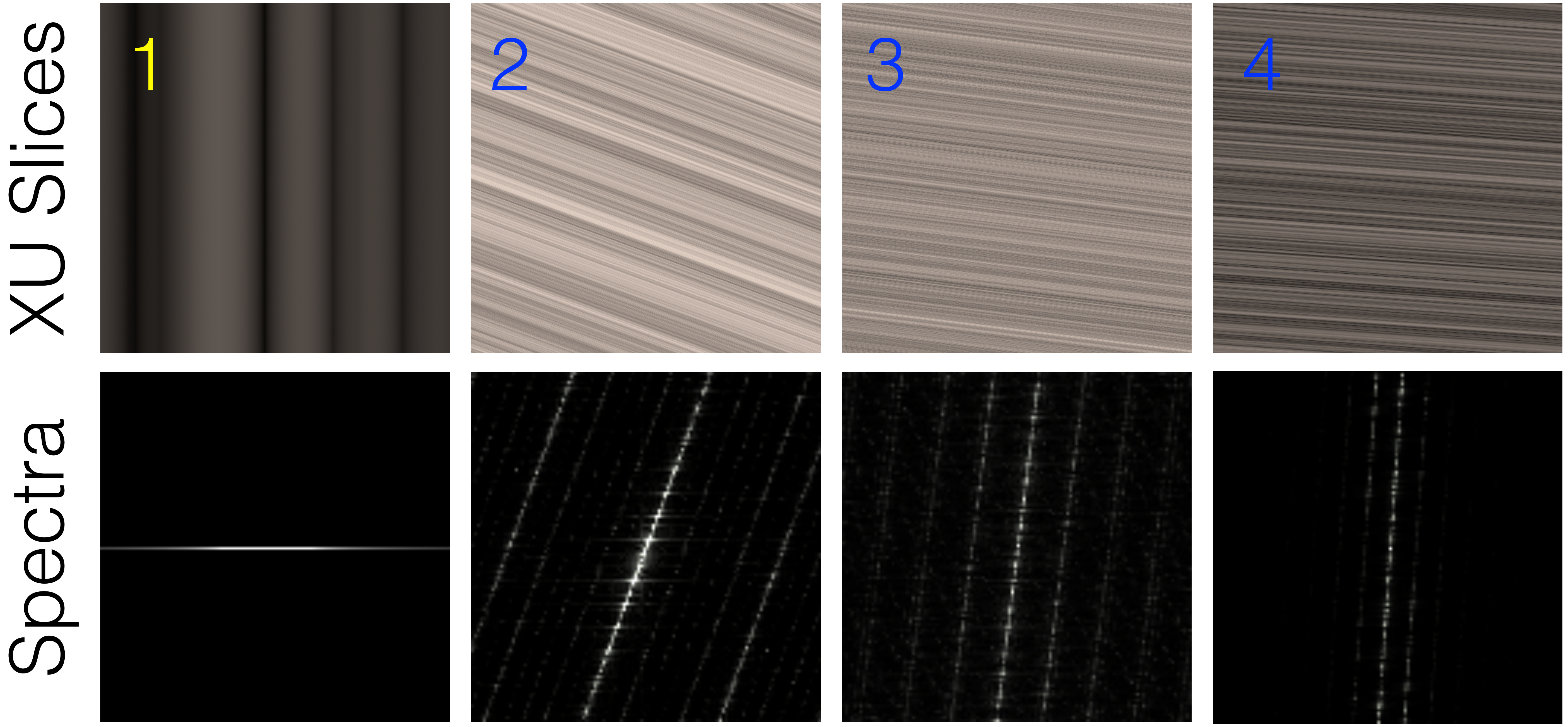
Light Field gets Sheared

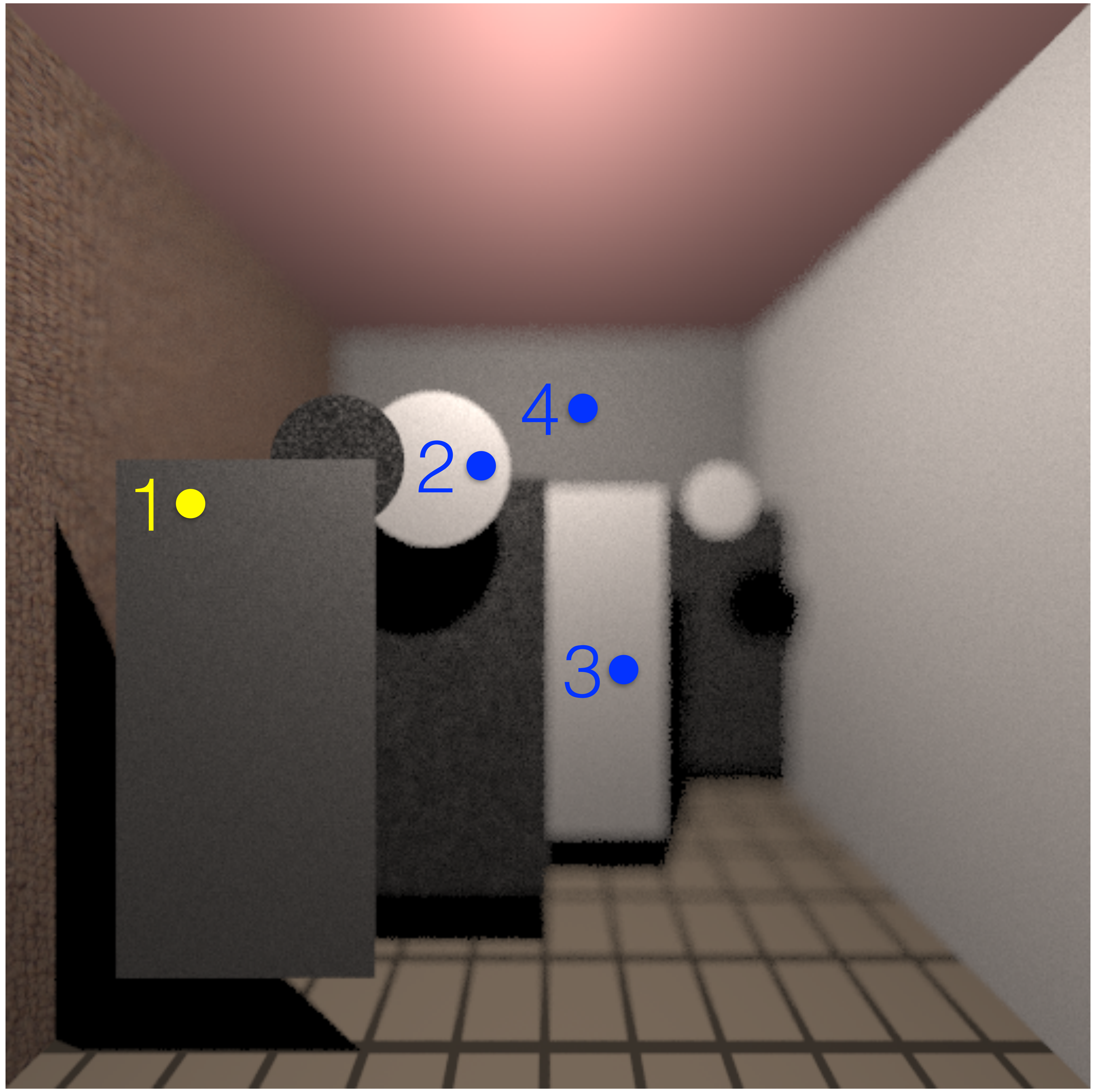
$$x = x + u\frac{F-d}{d},$$

$F:$ focal distance

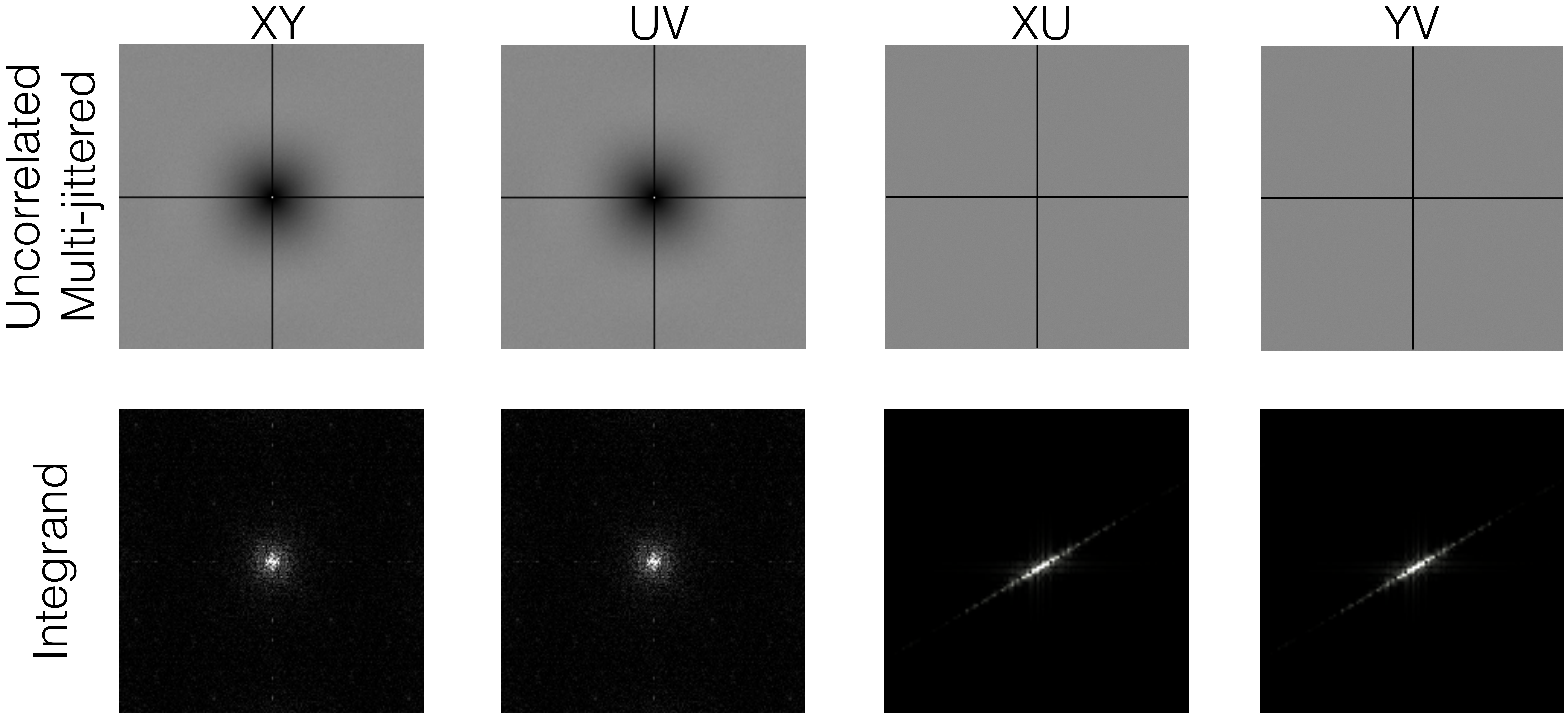
$d:$ depth of the hit object

Shear increases with object depth





Spectra along Different Projections

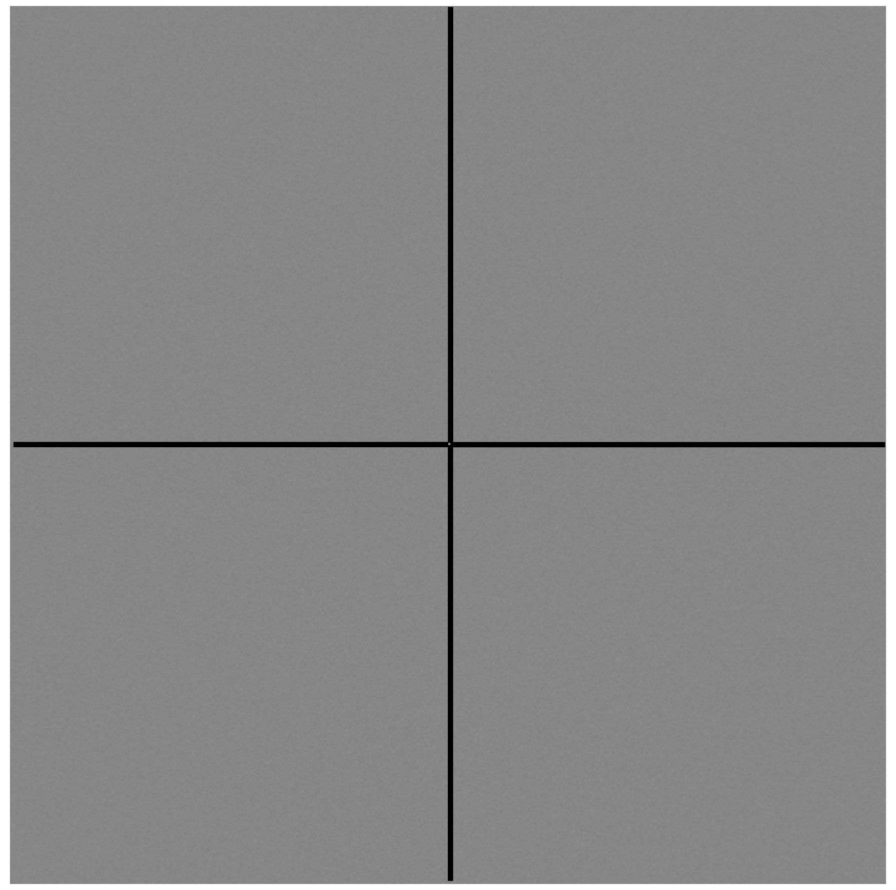


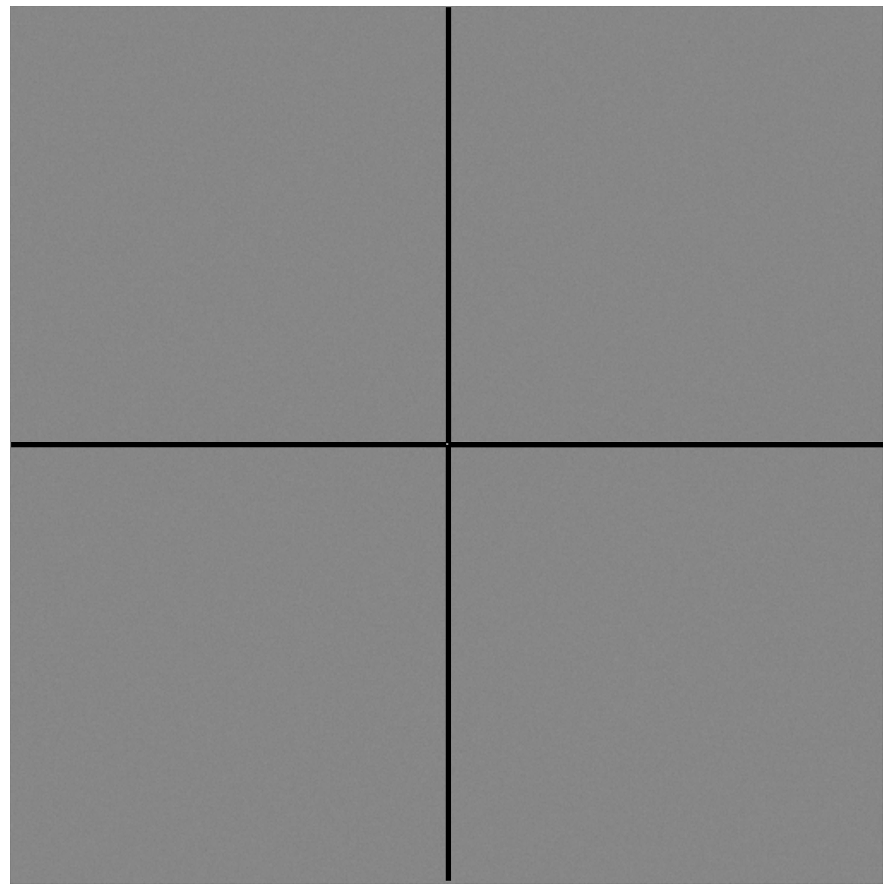
Spectra along Different Projections

XU

YV

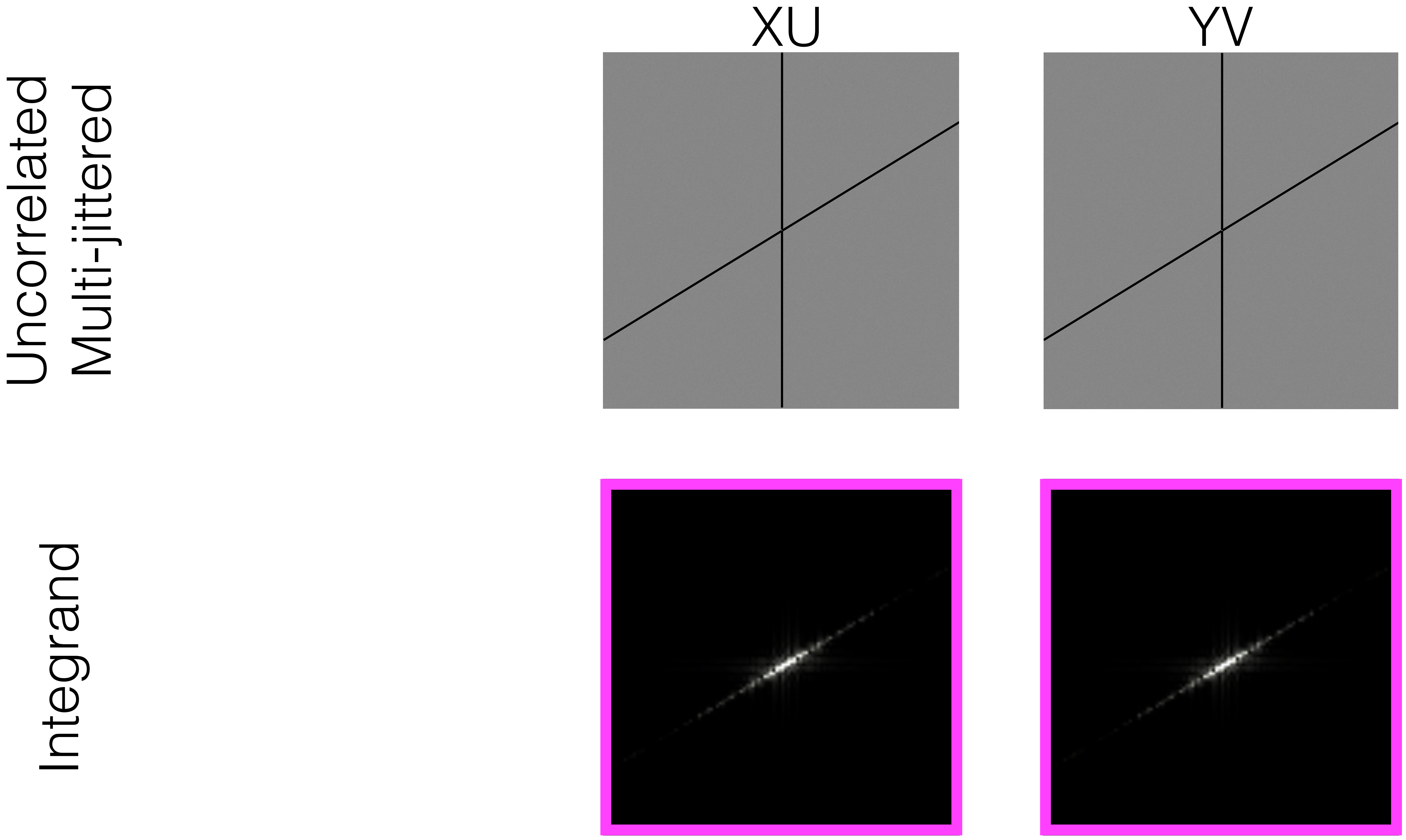
Uncorrelated Multi-jittered





Integrand

Spectra along Different Projections



- Error Formulation in the Spatial Domain
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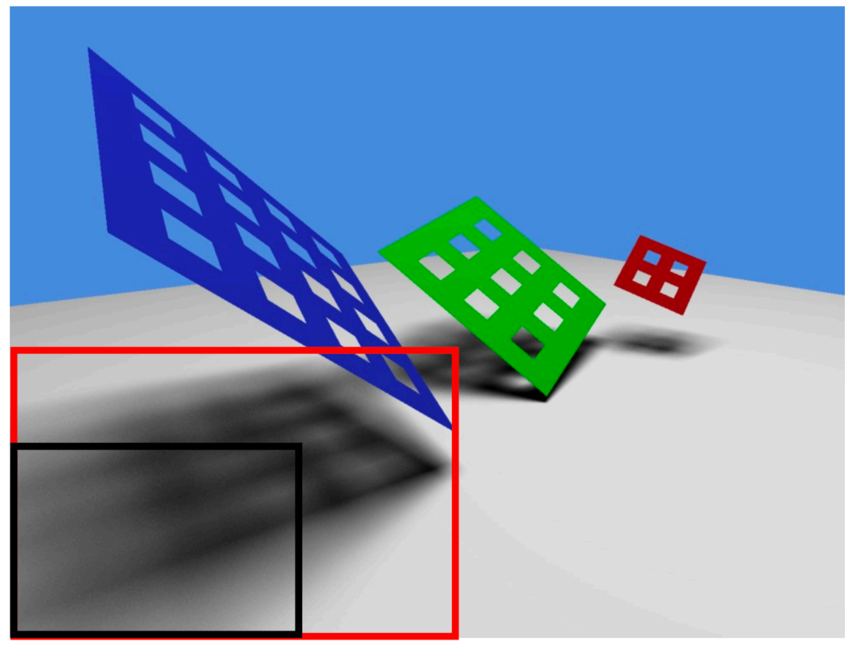
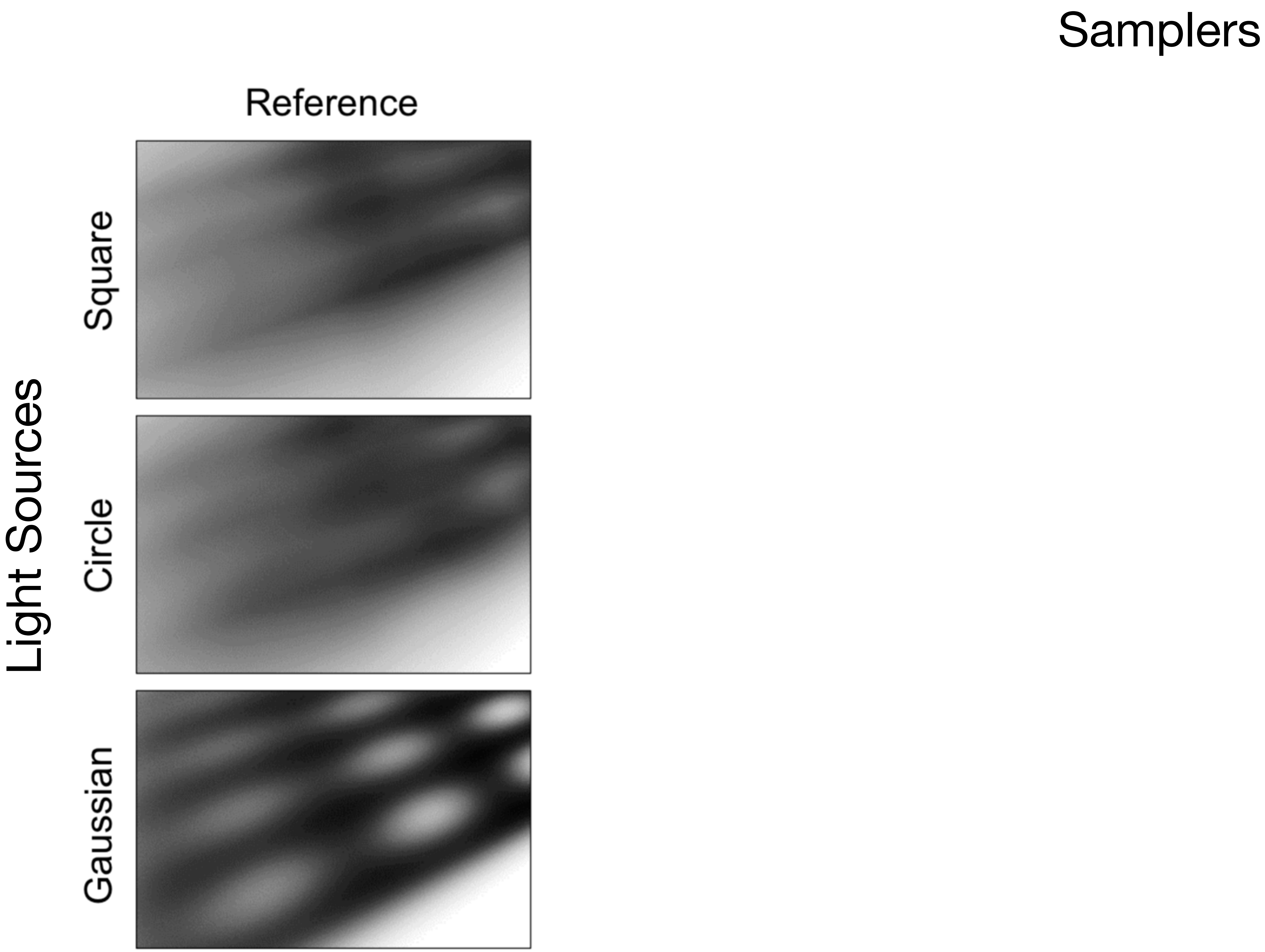
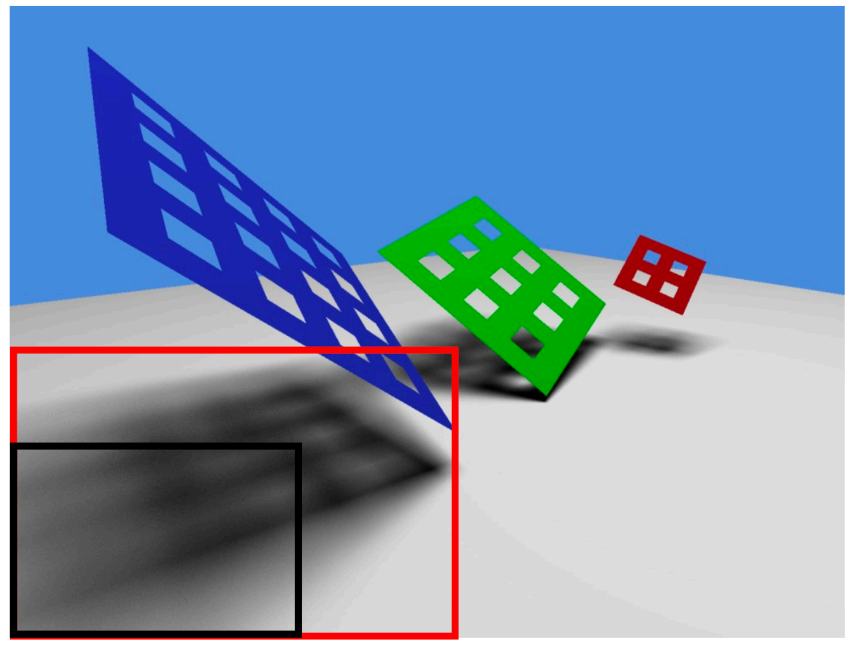
Variance Analysis of Jittered Strategies

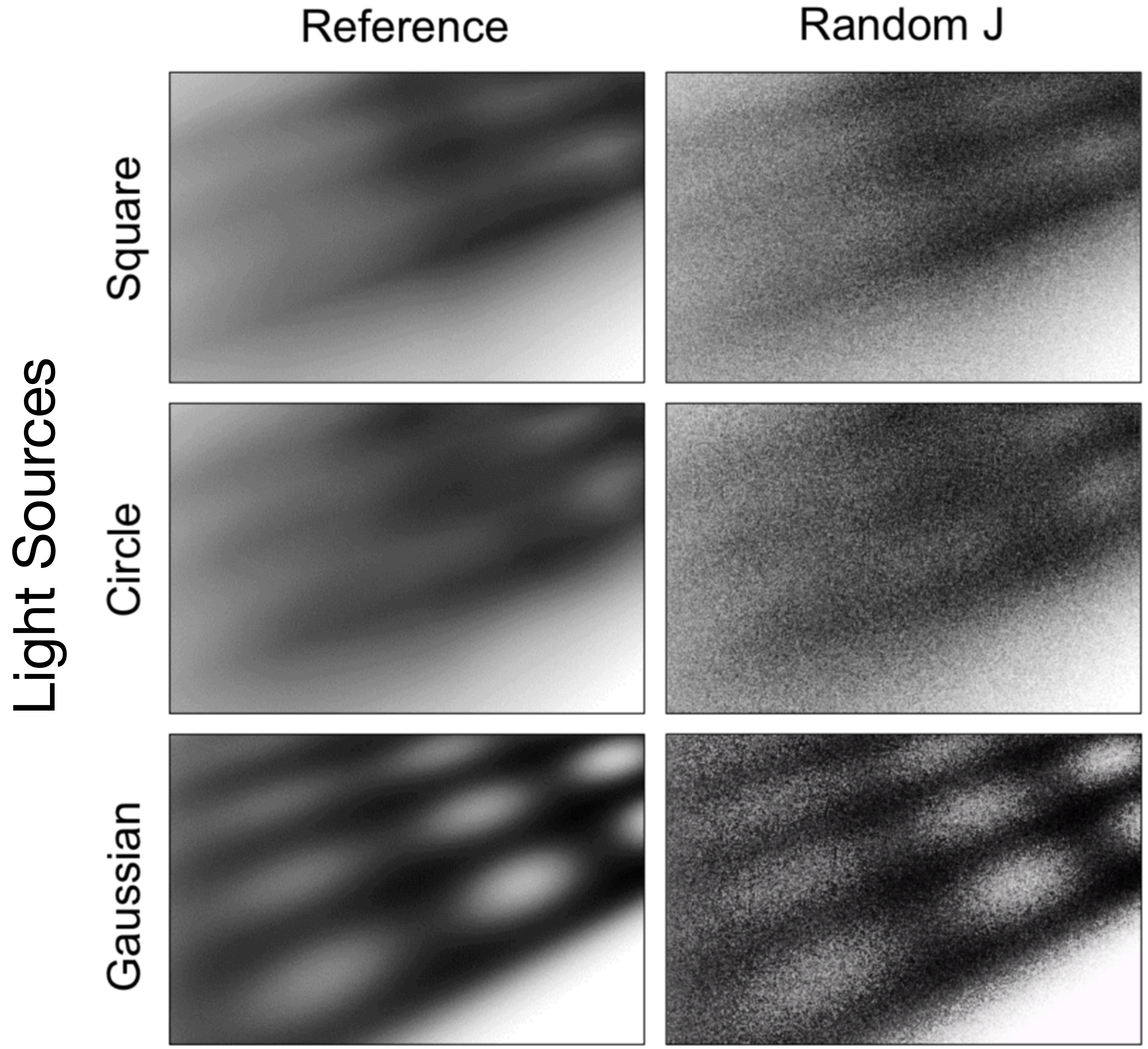




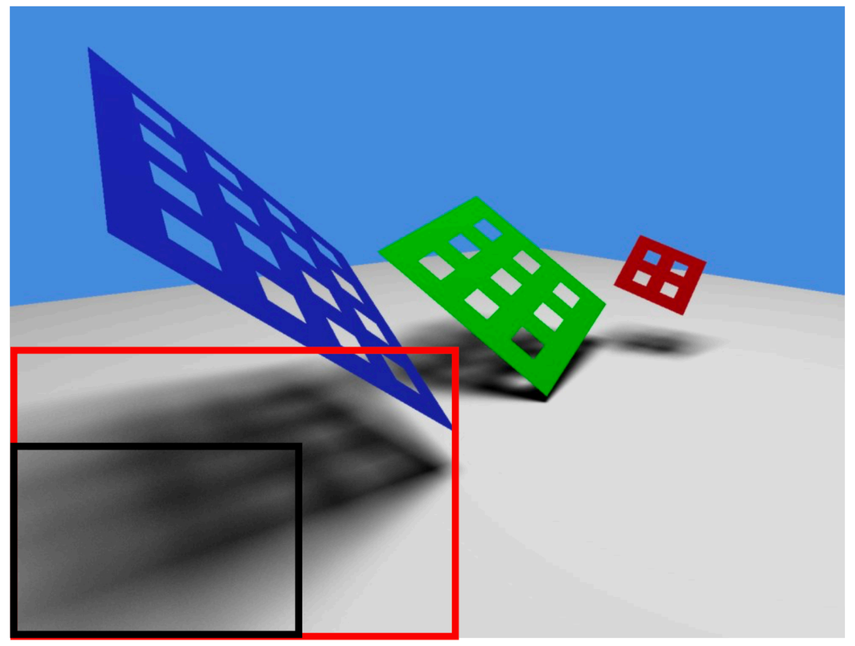
Variance Analysis of Jittered Strategies

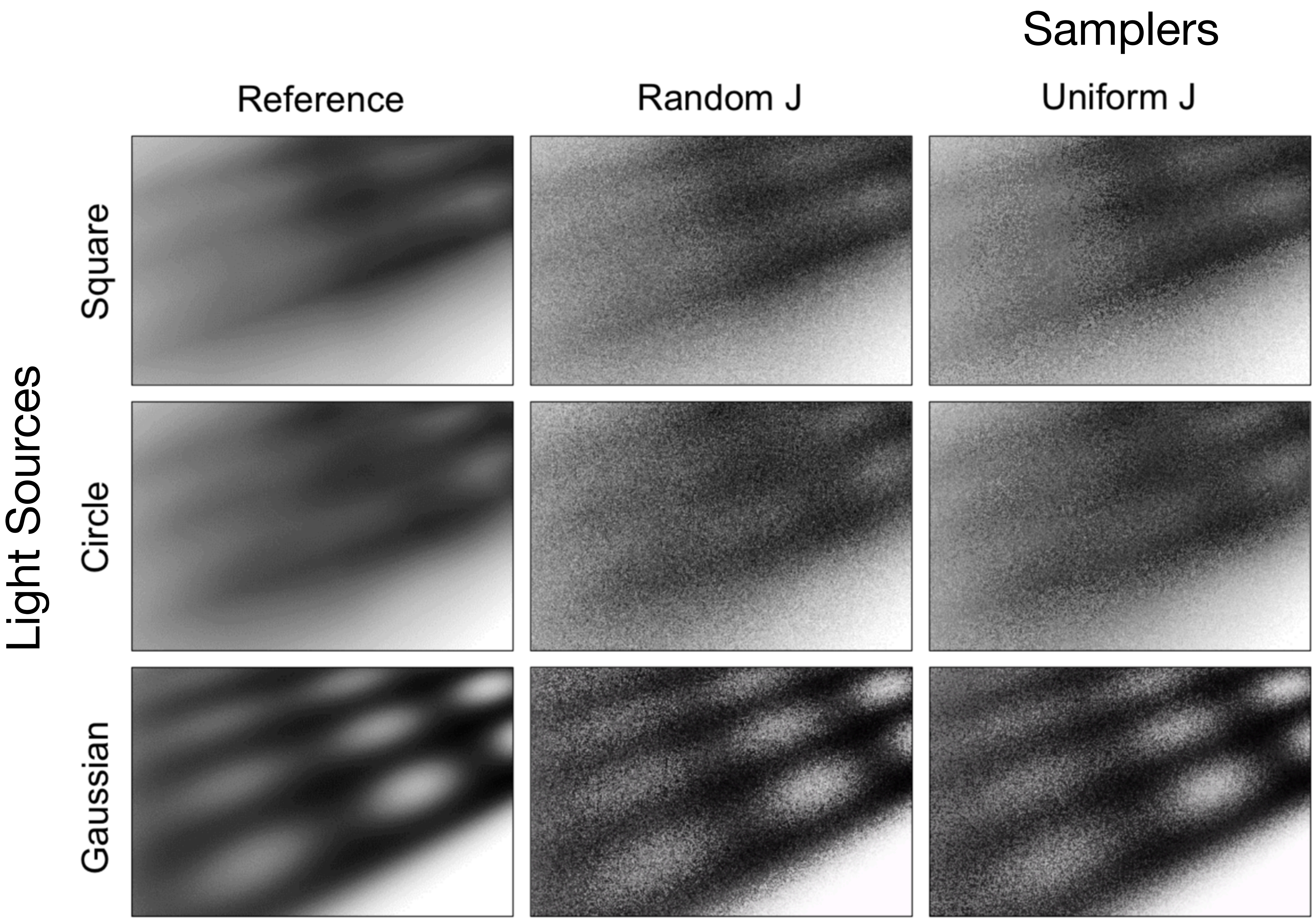


Samplers

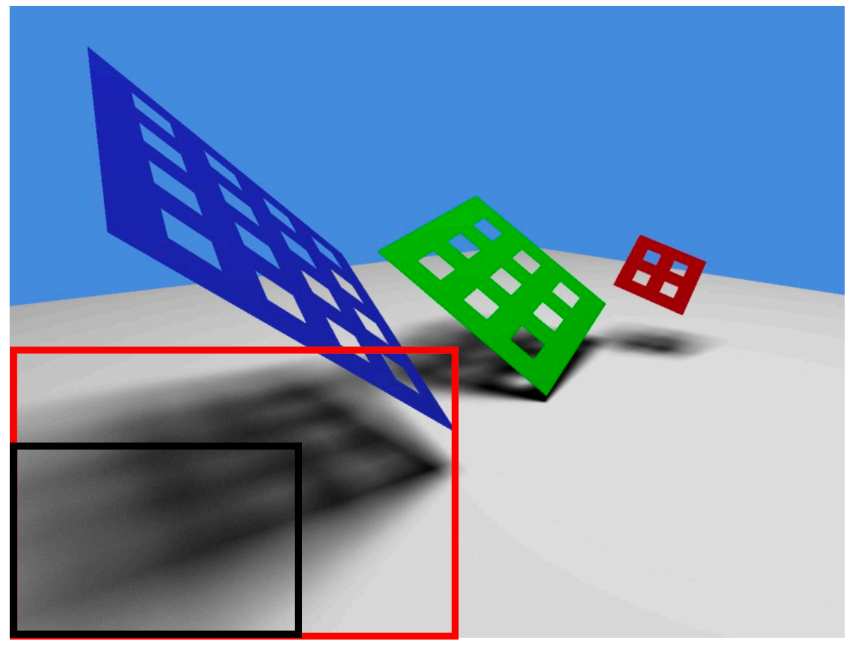


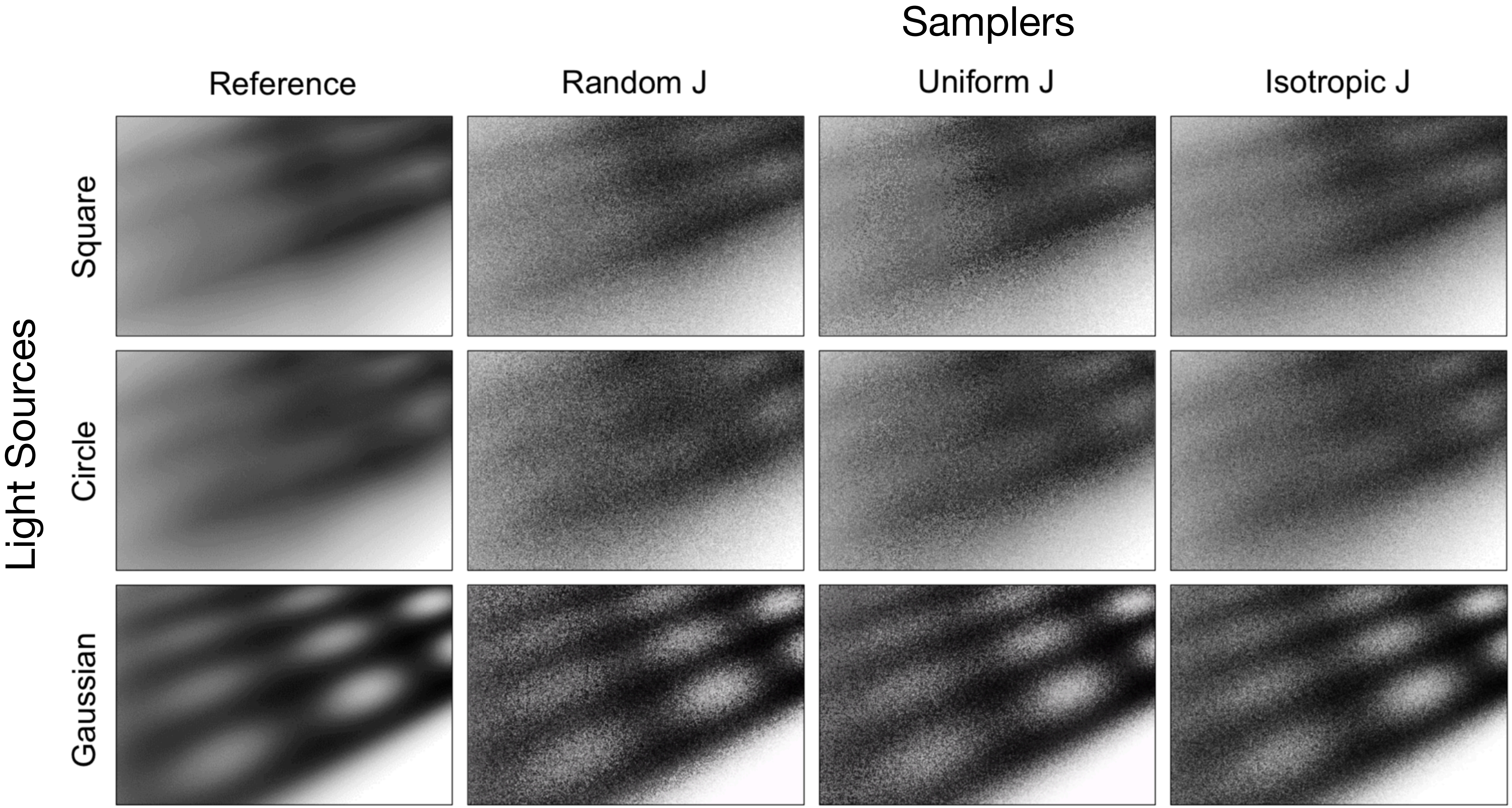
Variance Analysis of Jittered Strategies



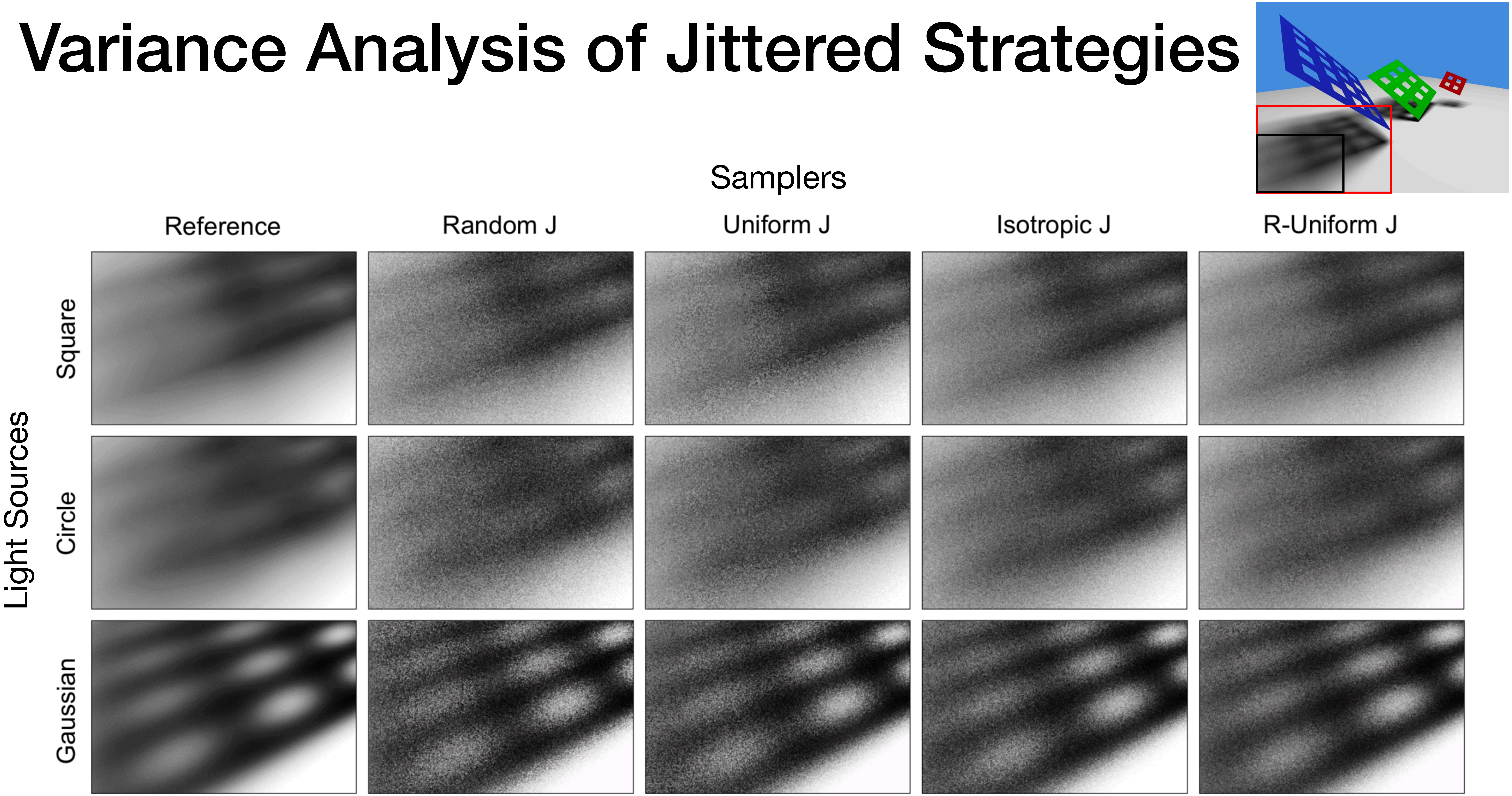


Variance Analysis of Jittered Strategies

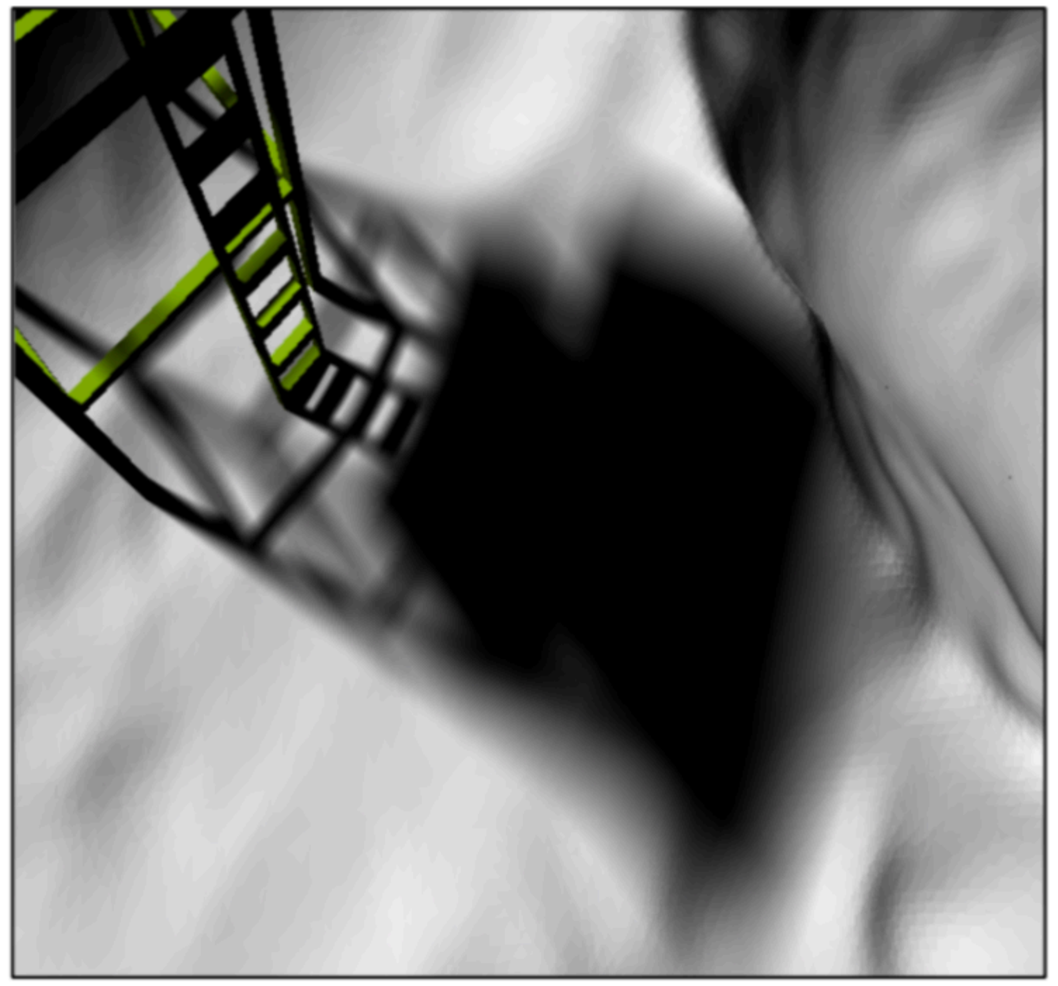


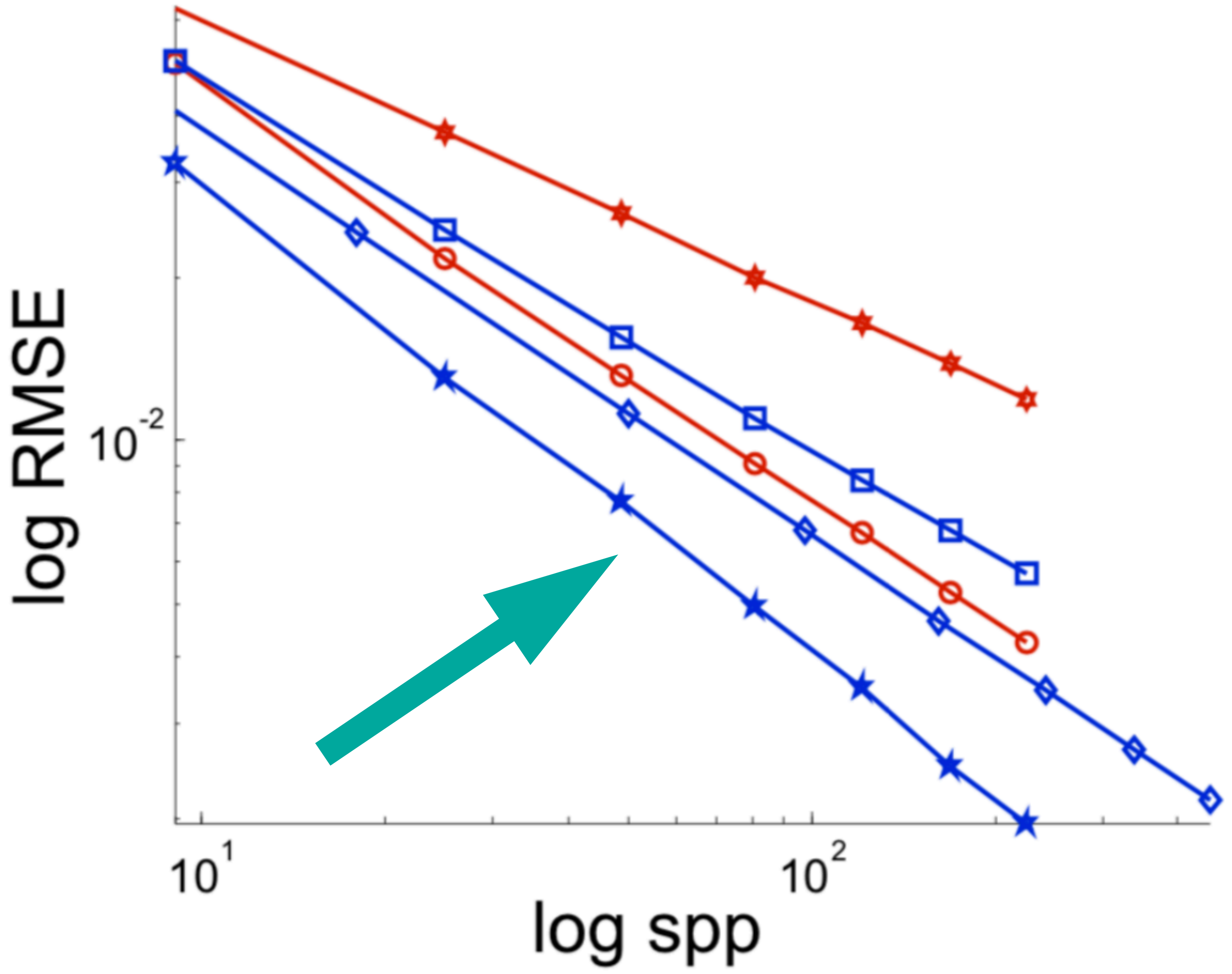


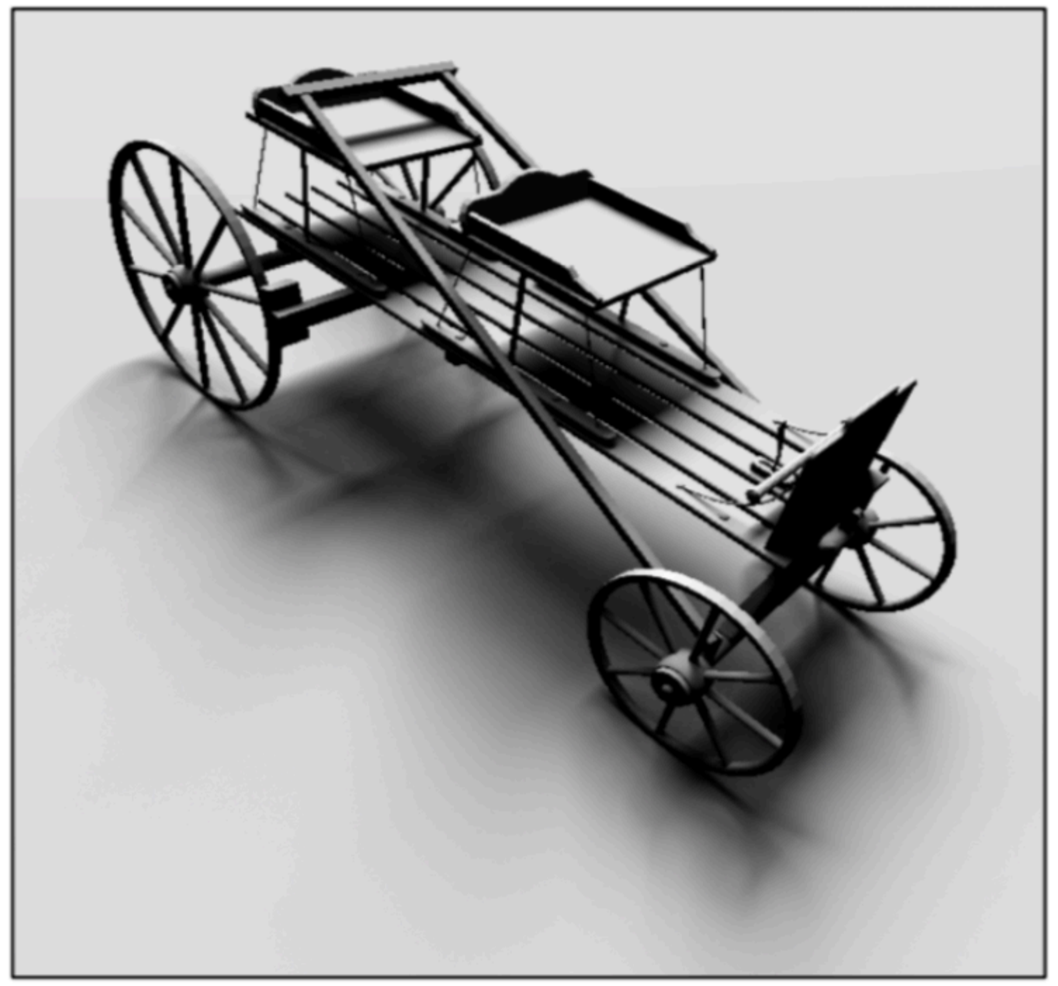
Variance Analysis of Jittered Strategies

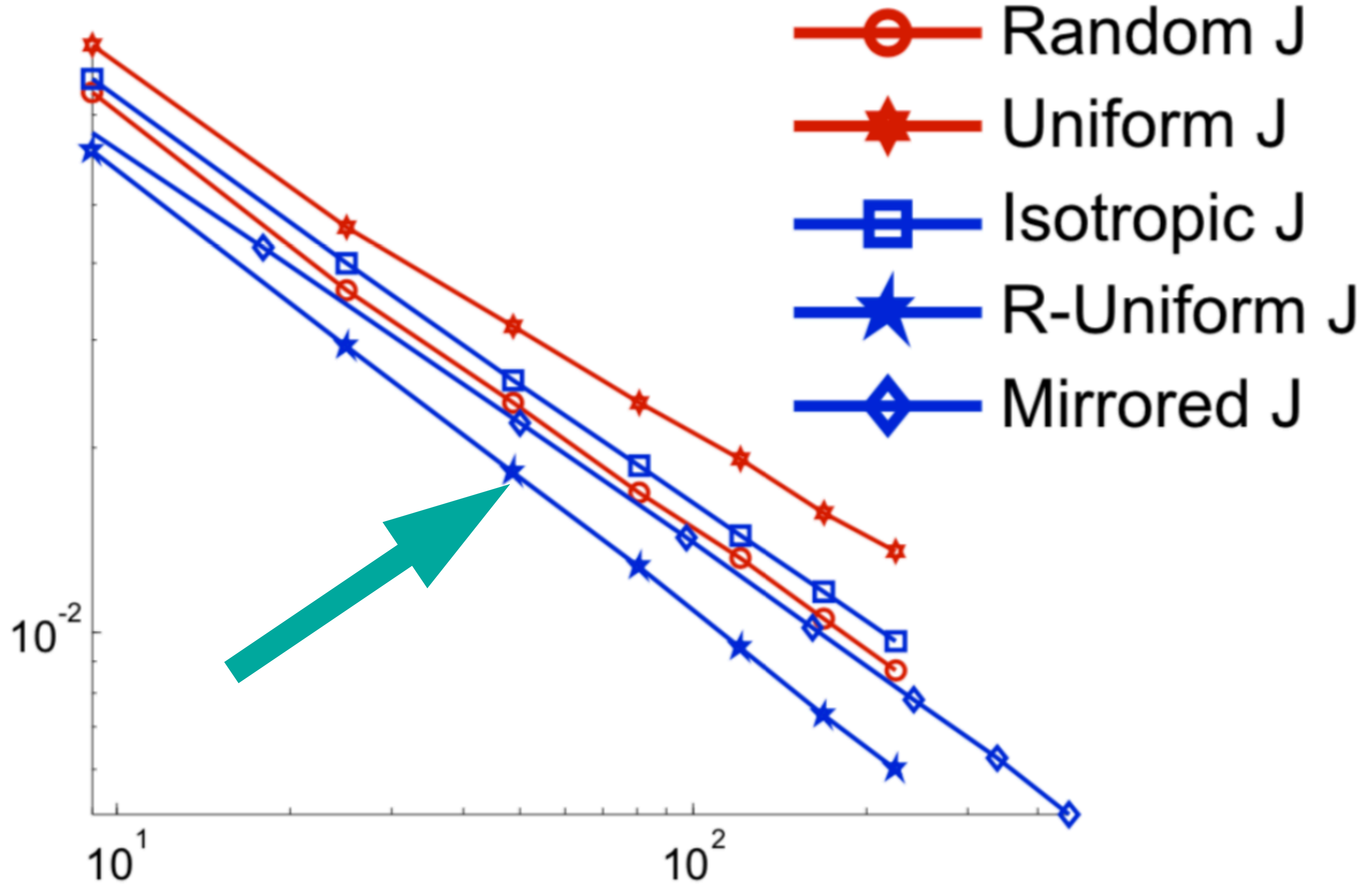


Convergence Analysis of Jittered Strategies



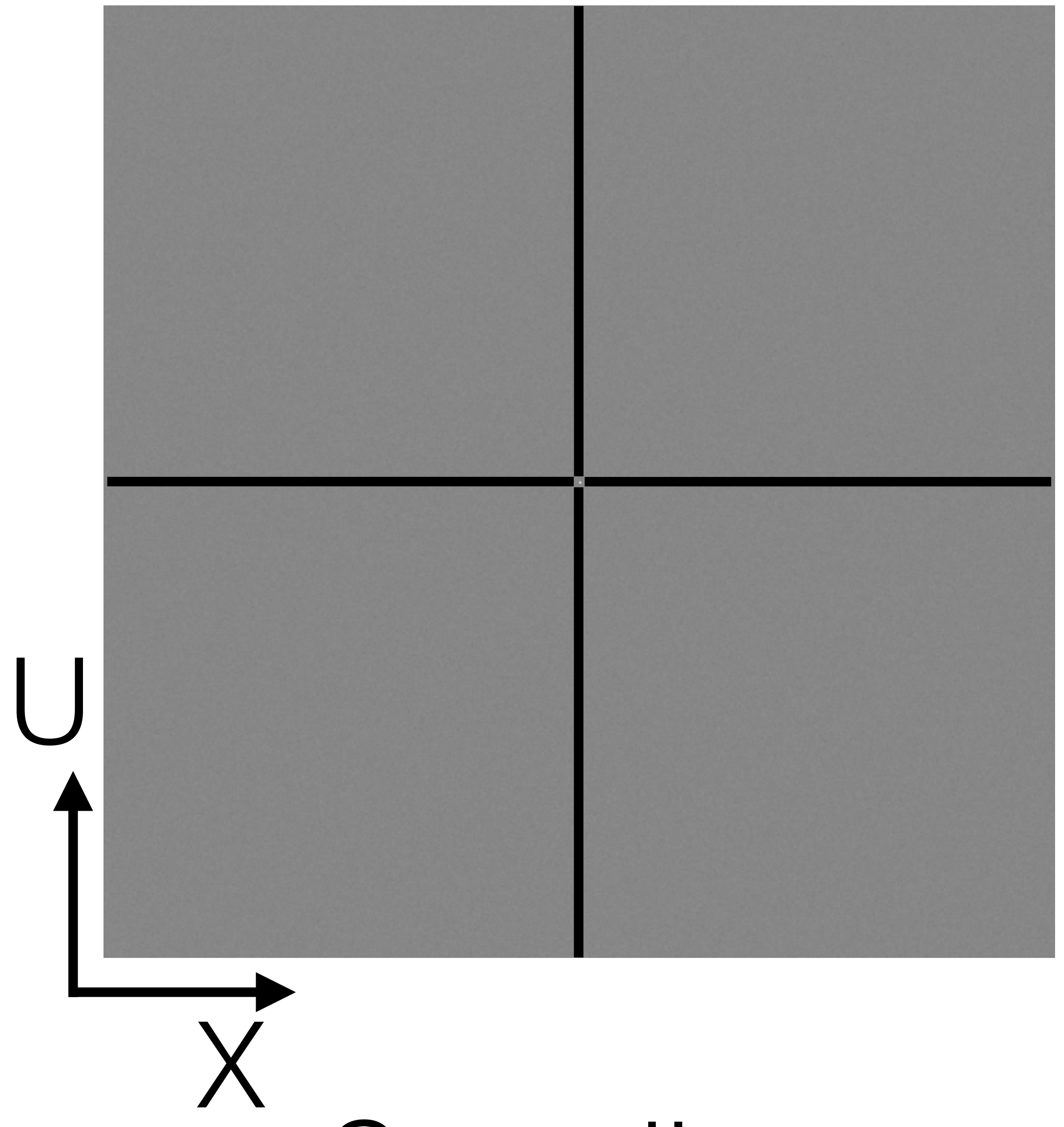




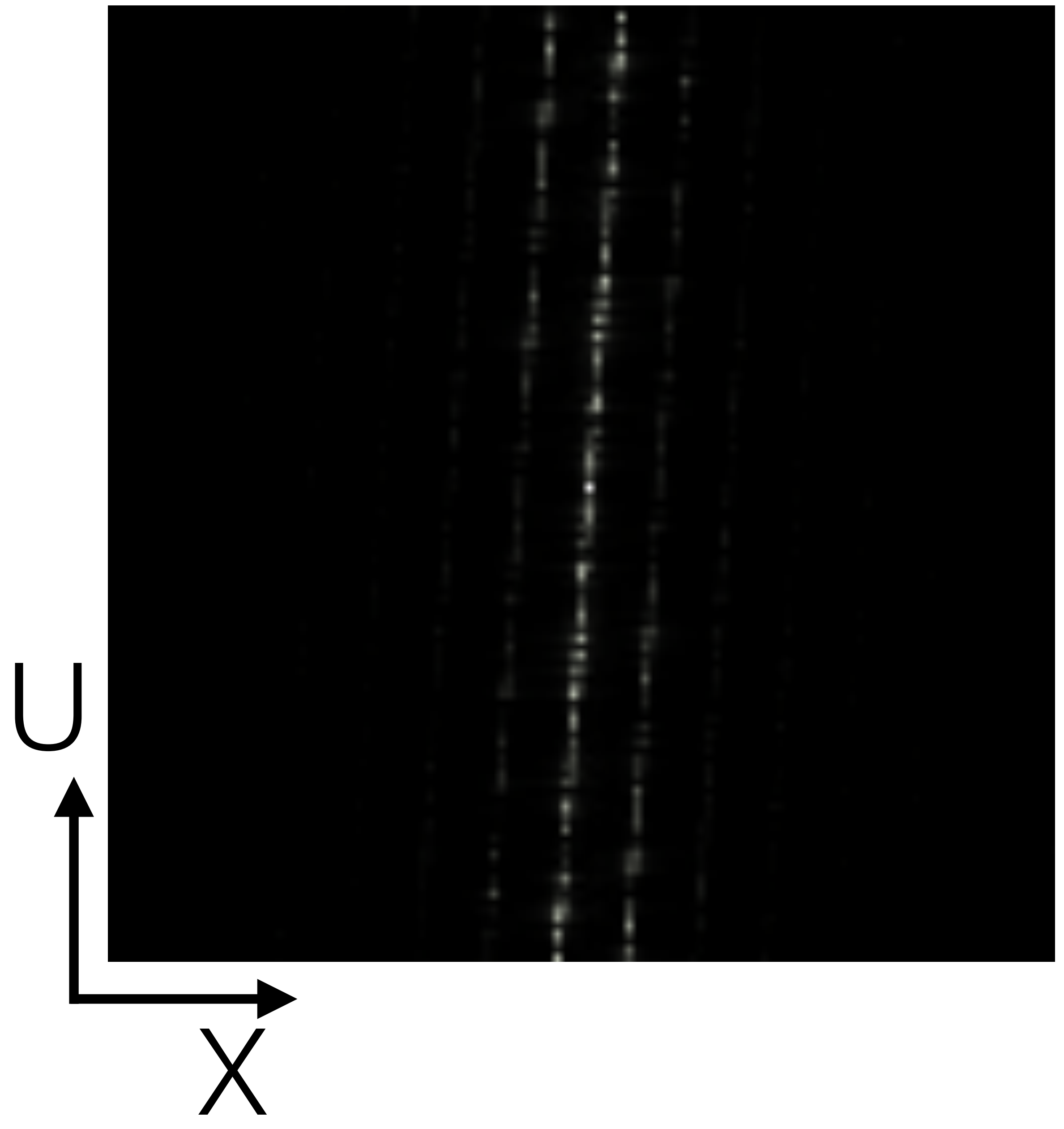


Original Uncorrelated-MultiJittered Samples

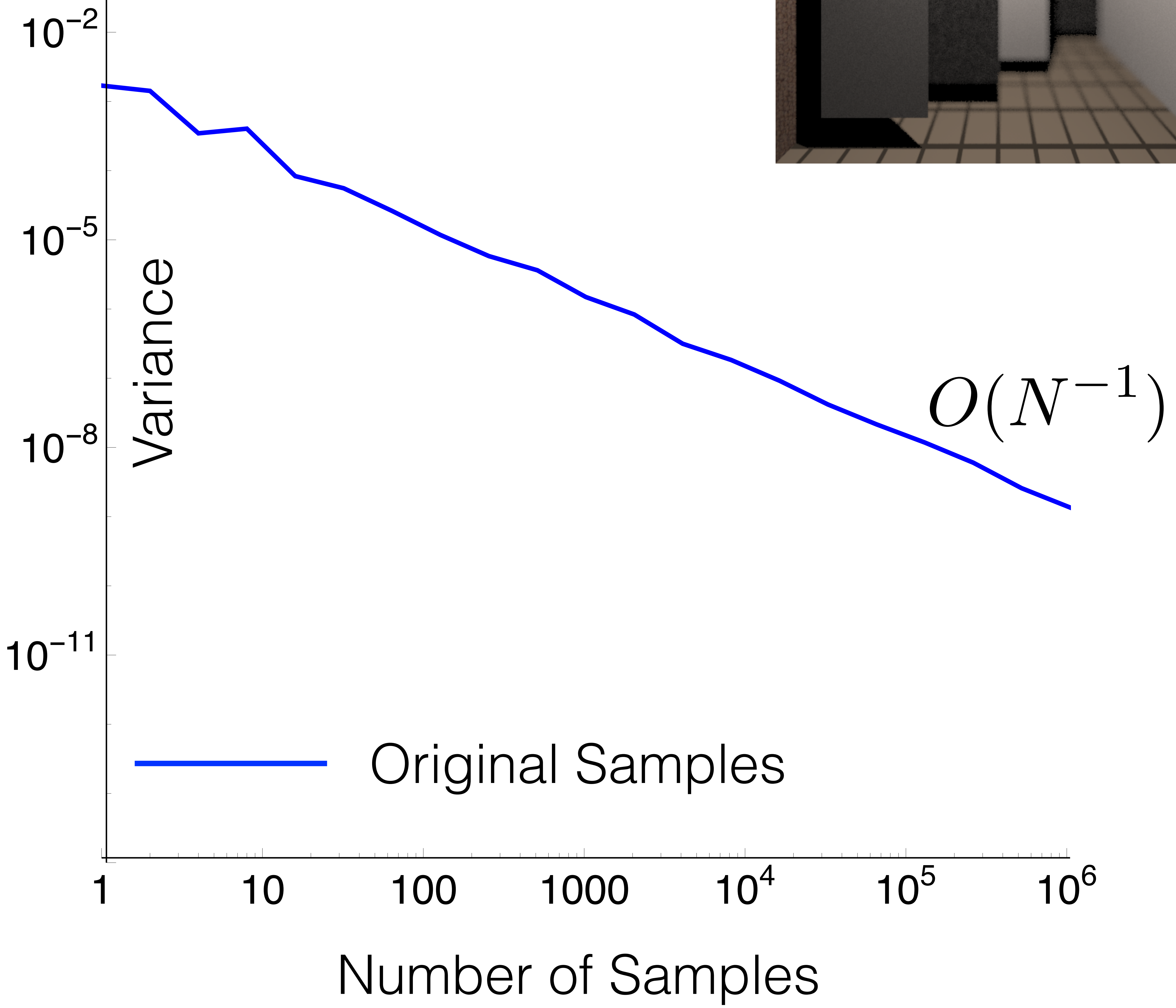
XU Projection

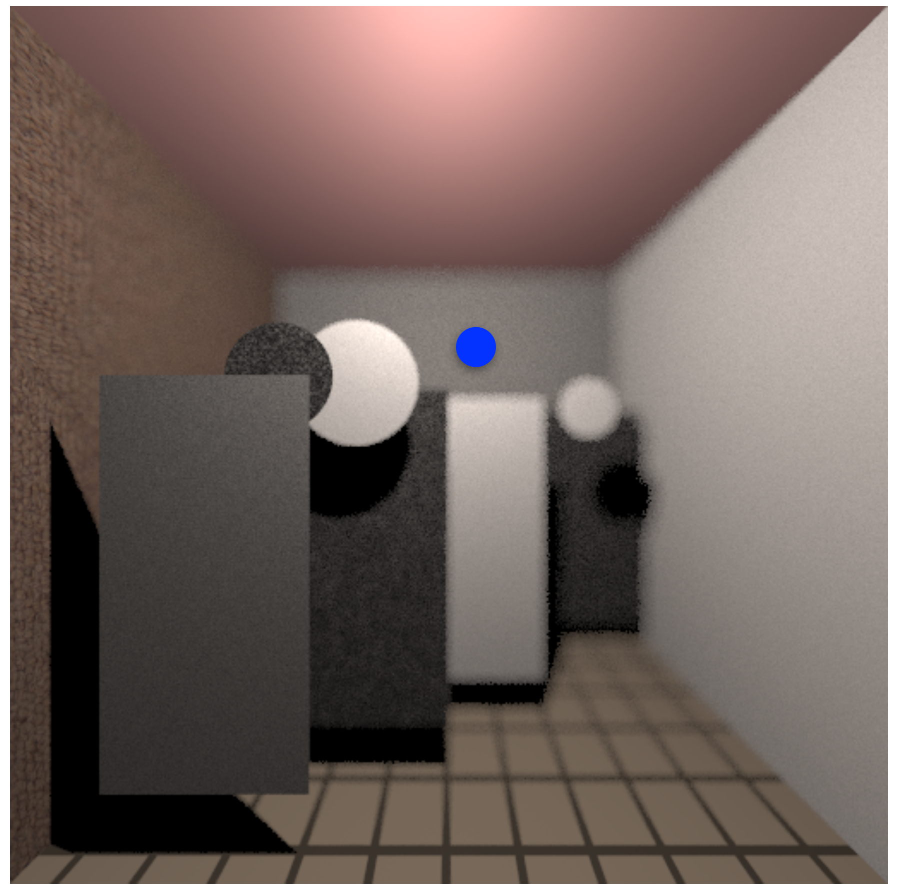


Sampling Spectrum

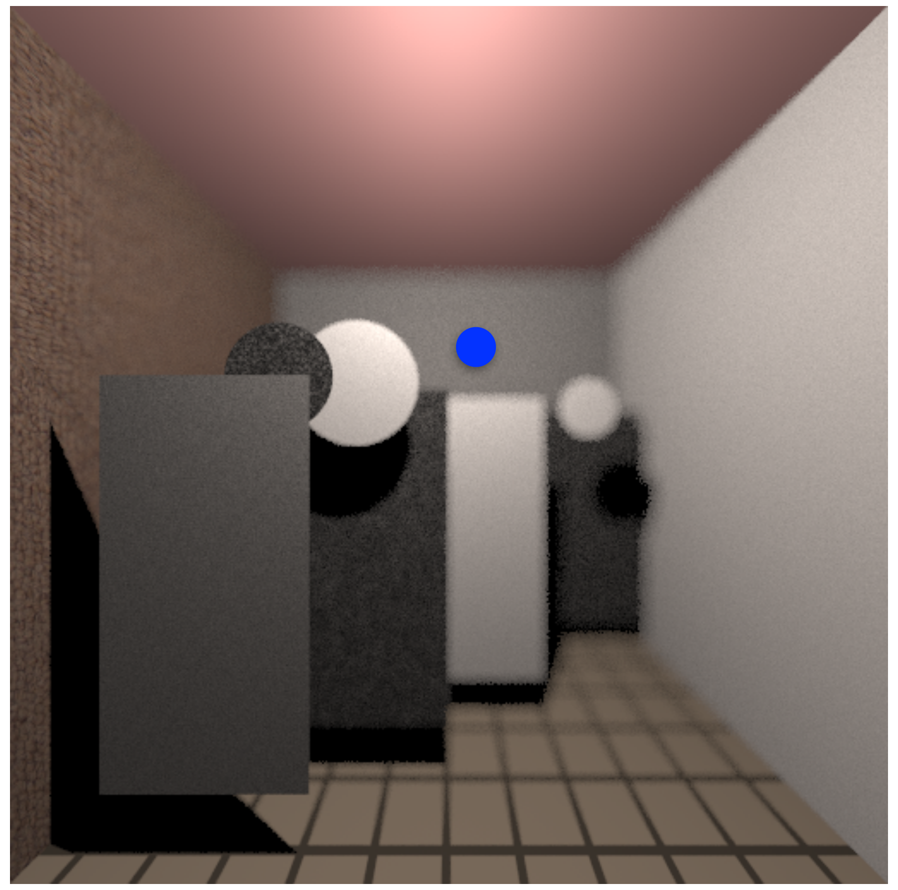


Integrand Spectrum

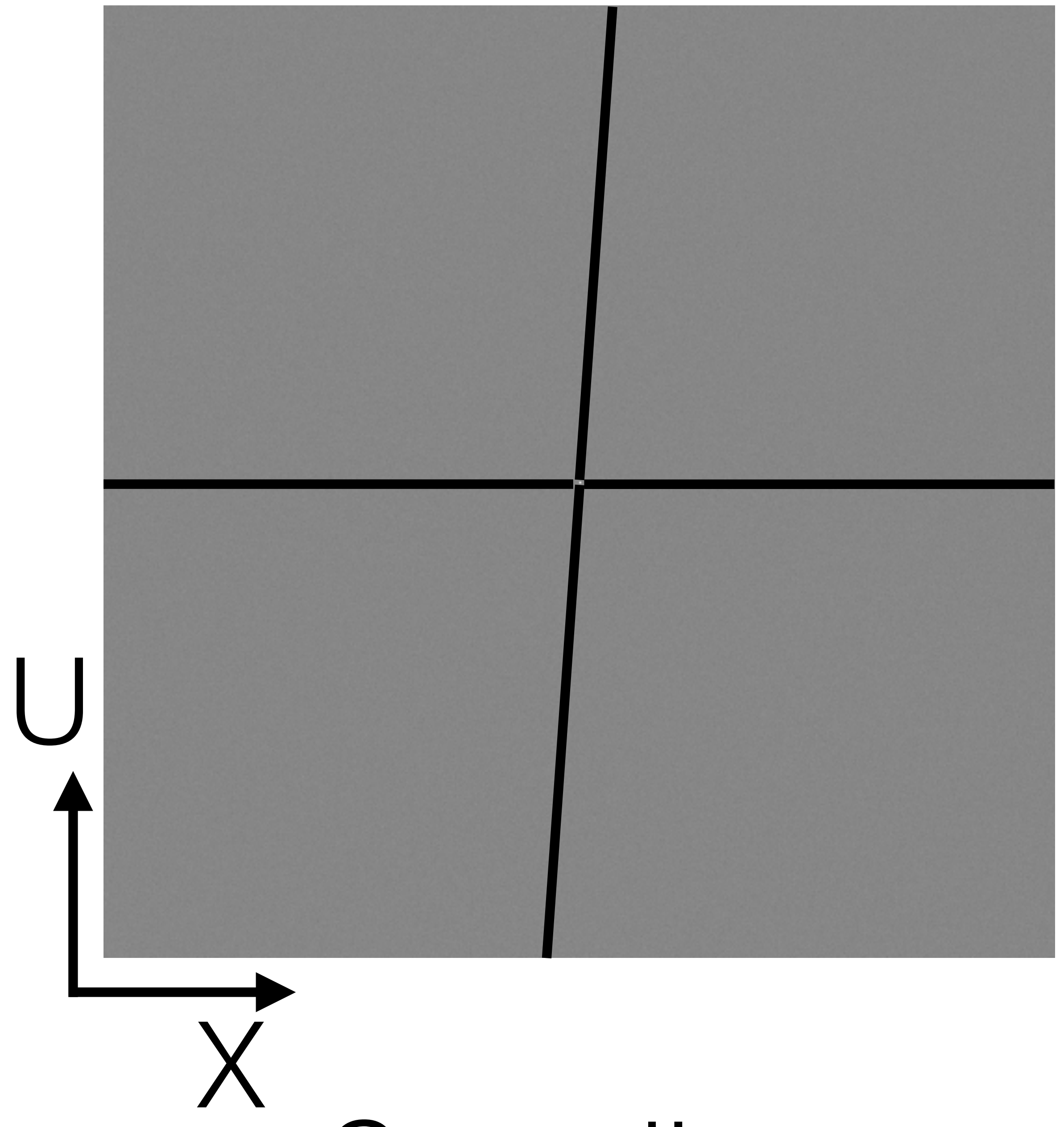




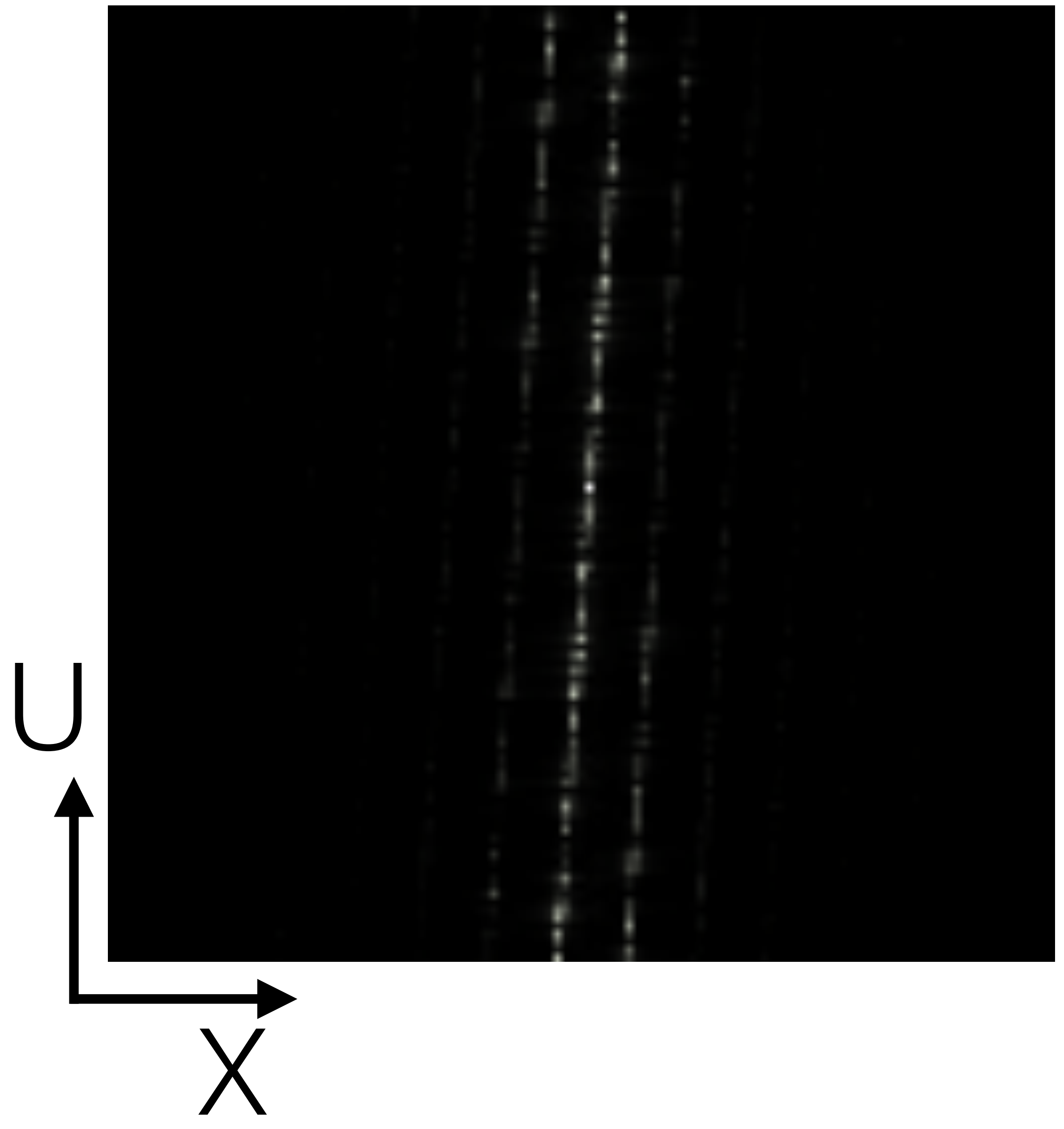
Convergence improvement after Shearing



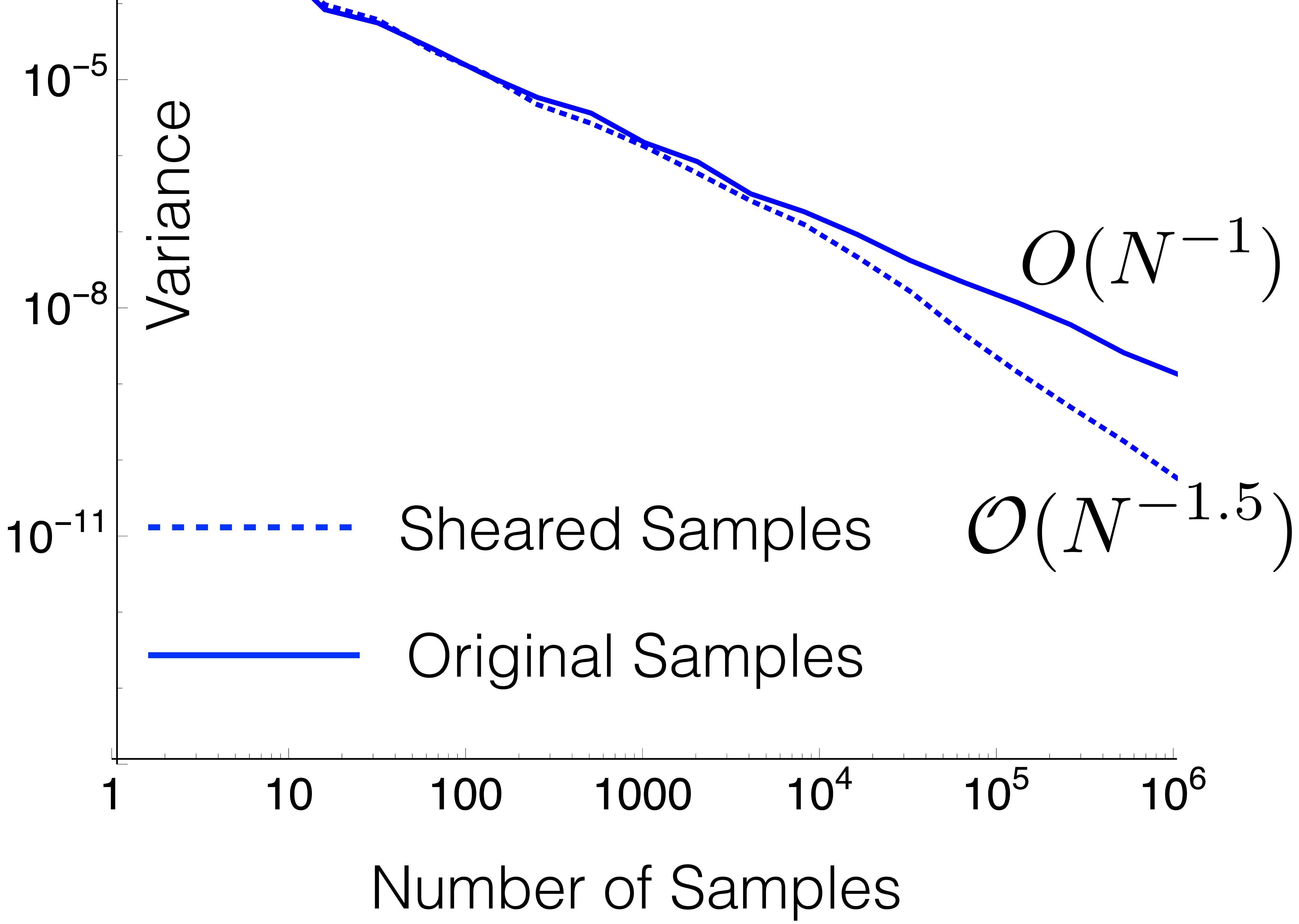
XU Subspace



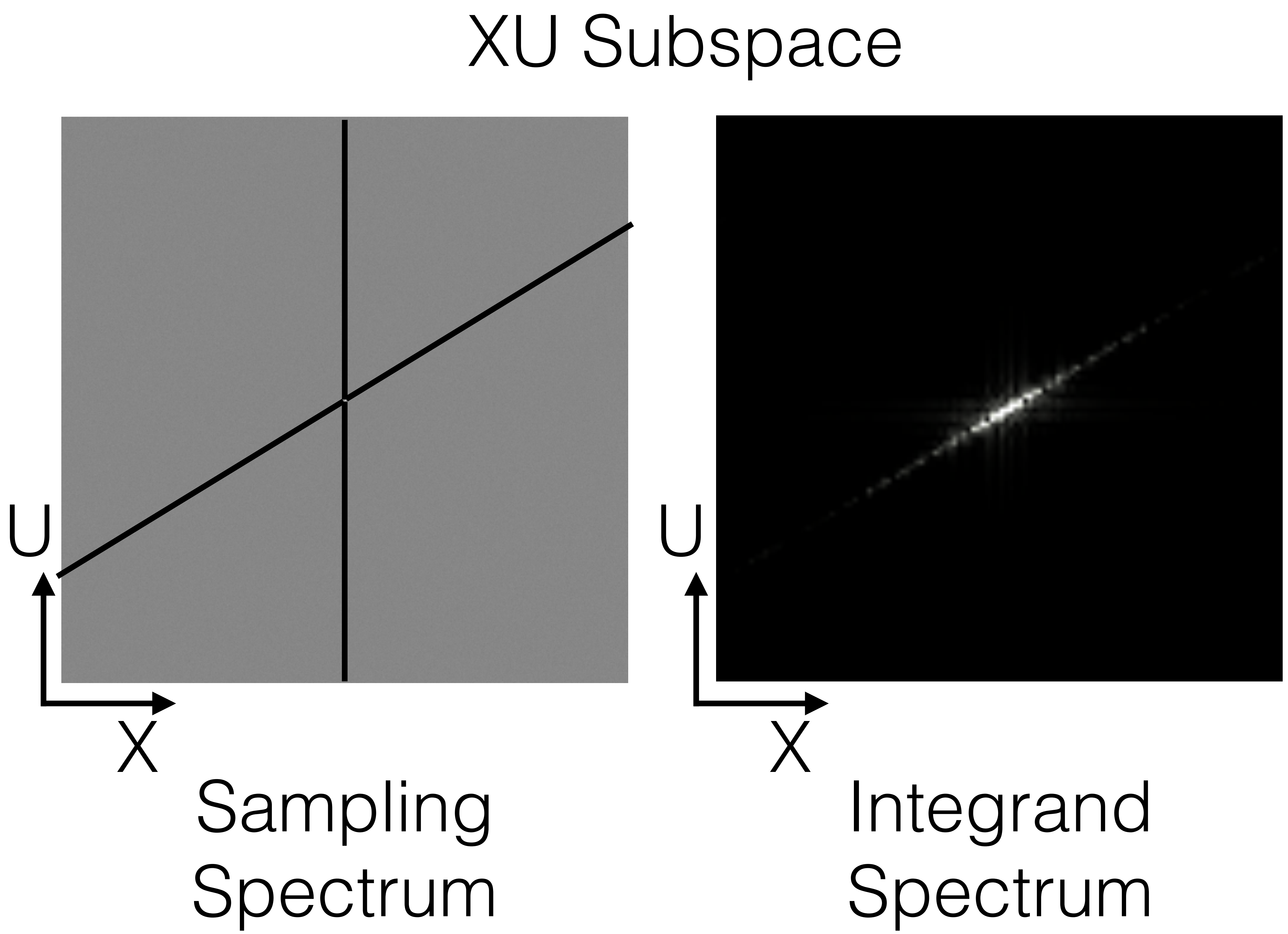
Sampling Spectrum

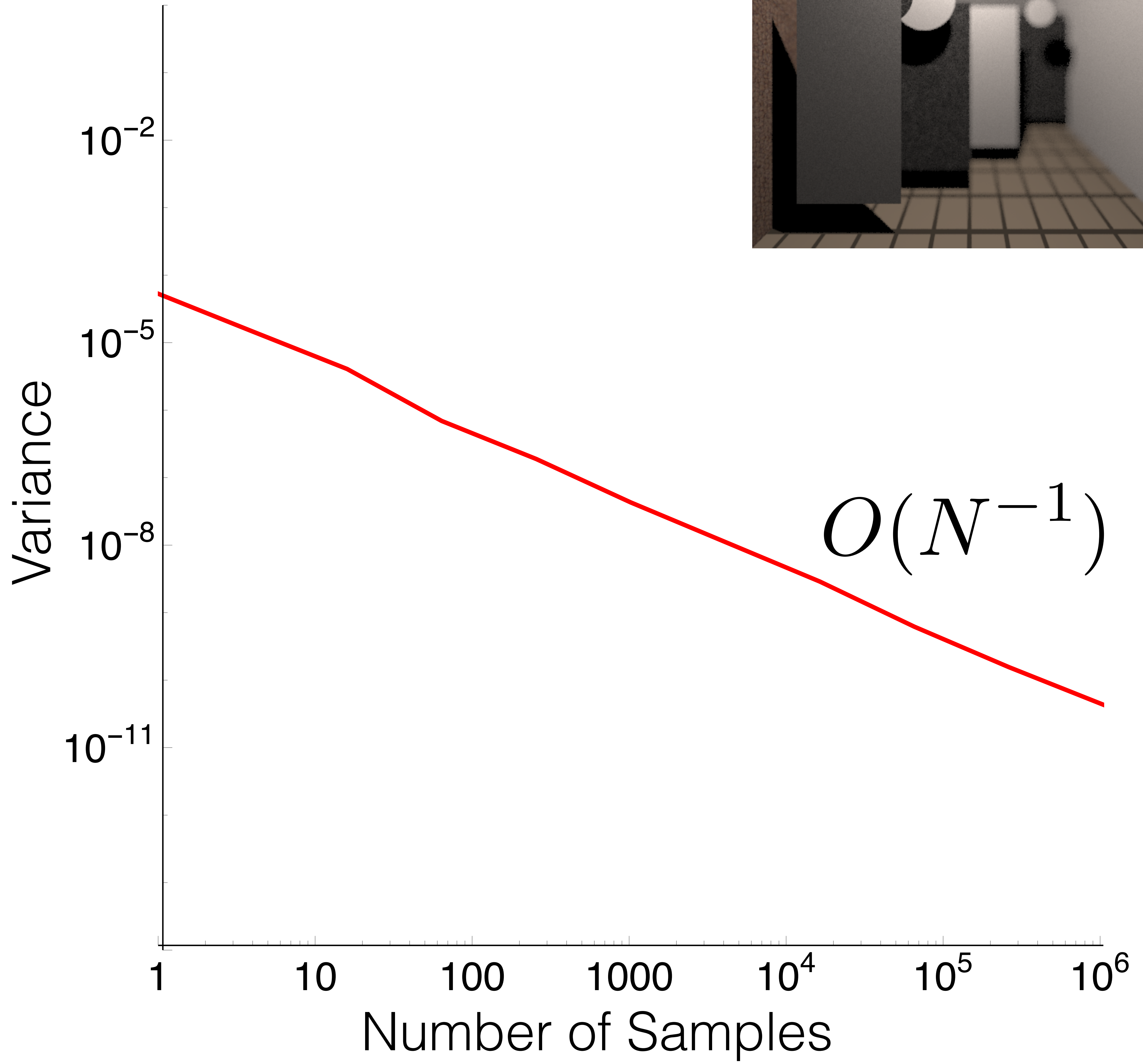


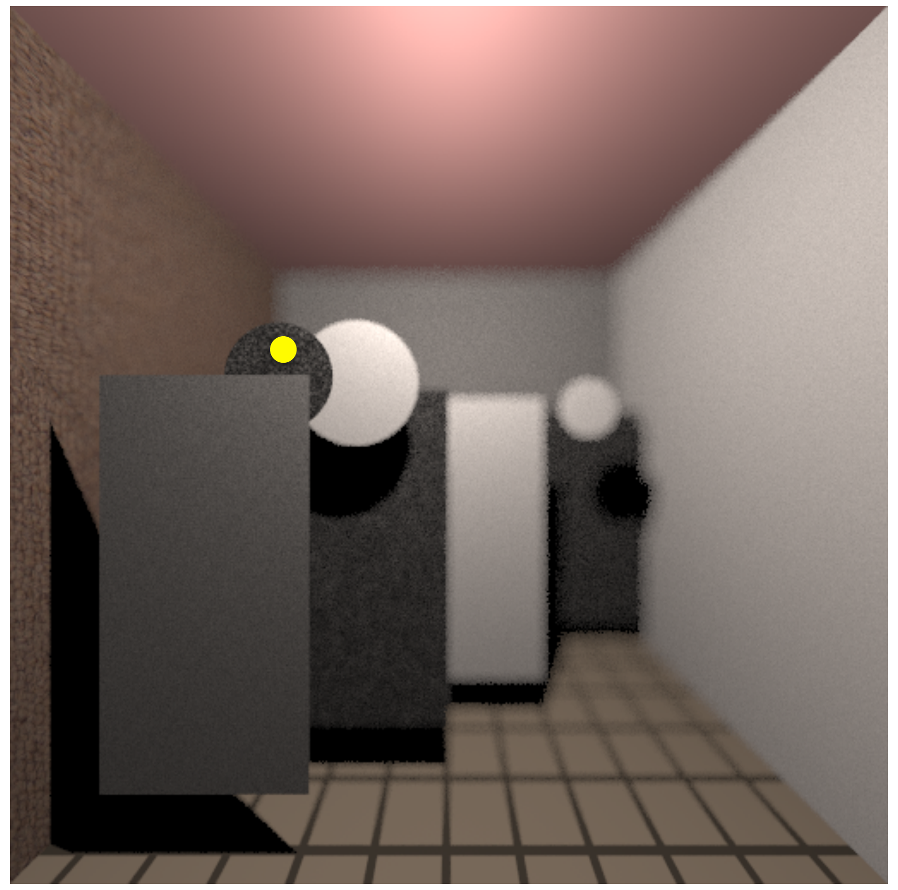
Integrand Spectrum



Original Uncorrelated Multi-jittered Samples

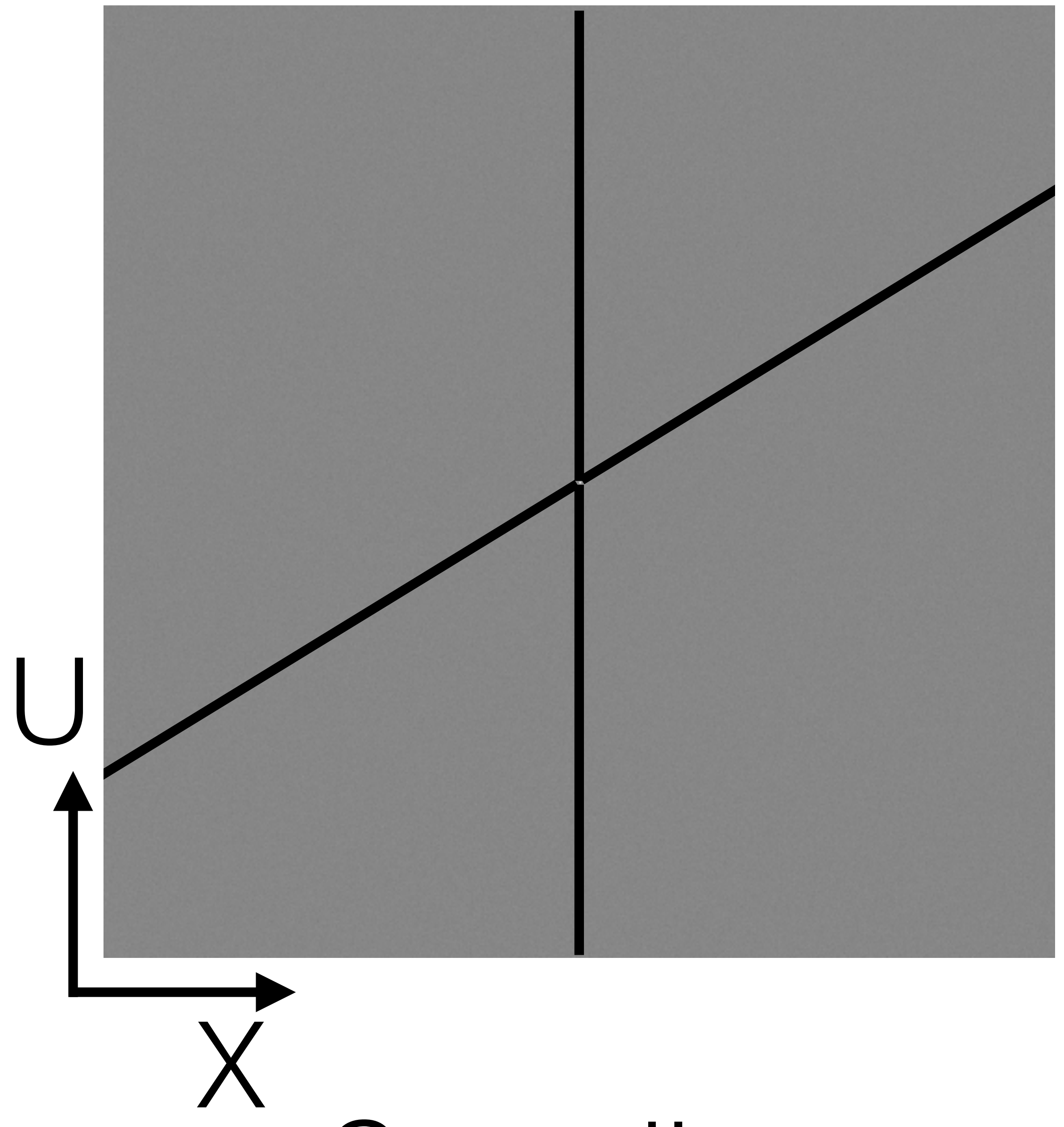




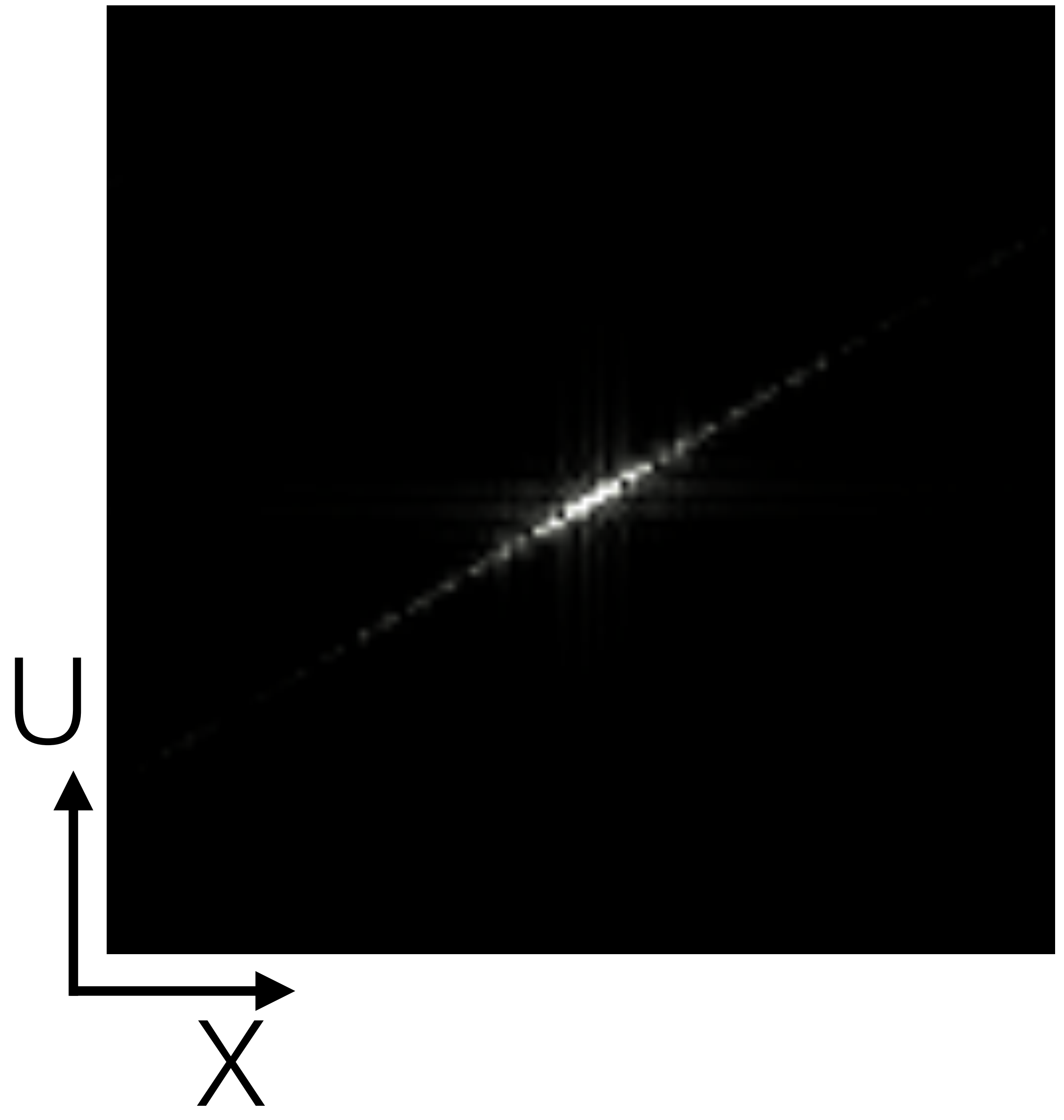


Sheared Uncorrelated Multi-jittered Samples

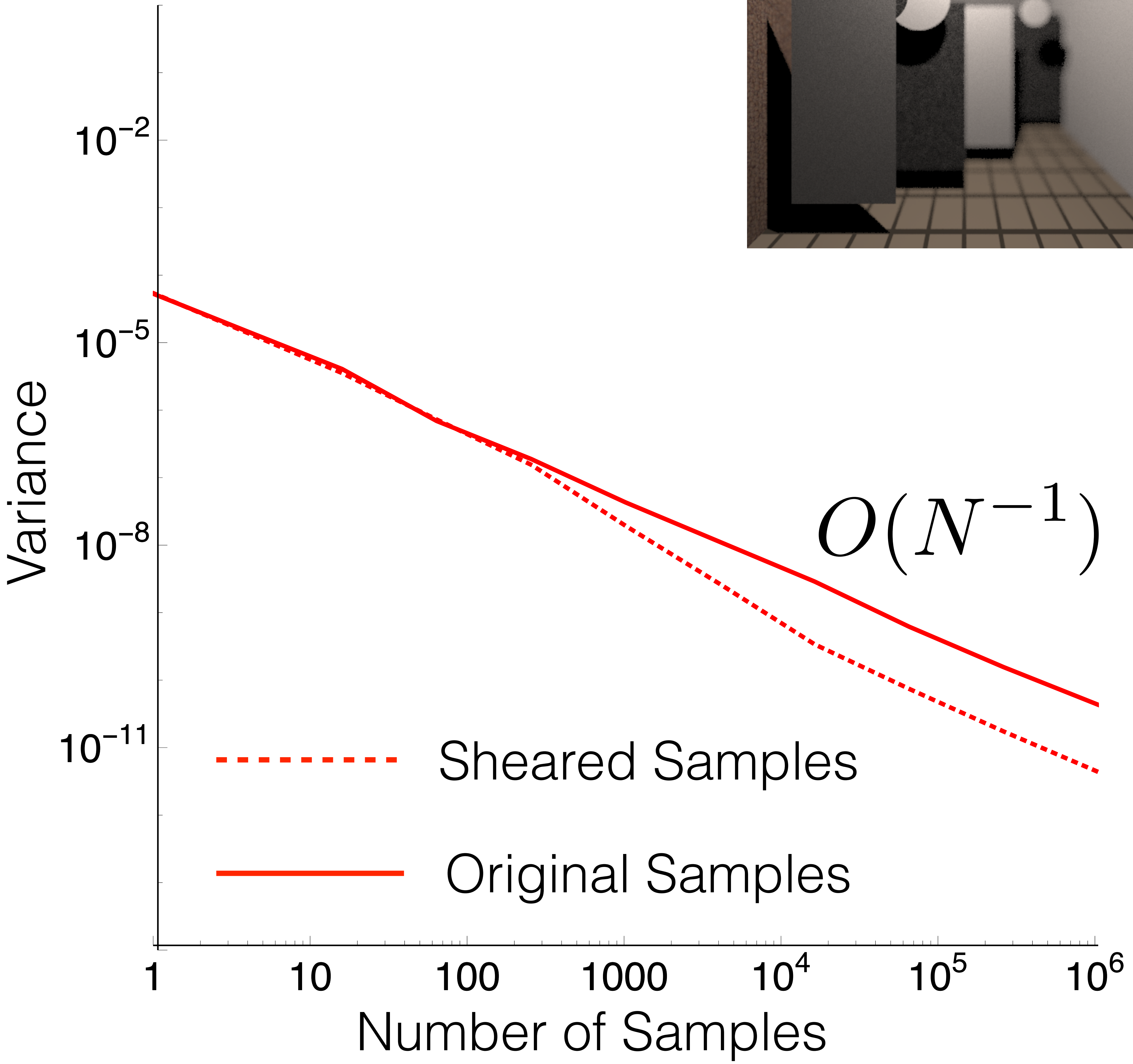
XU Subspace

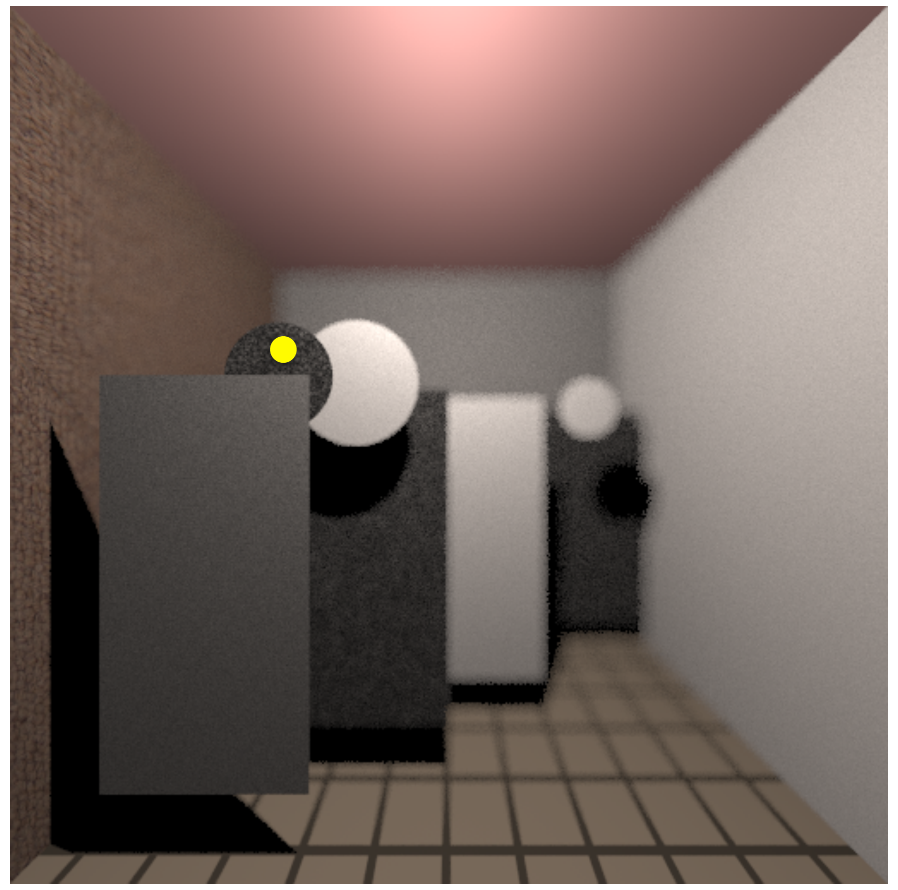


Sampling Spectrum



Integrand Spectrum

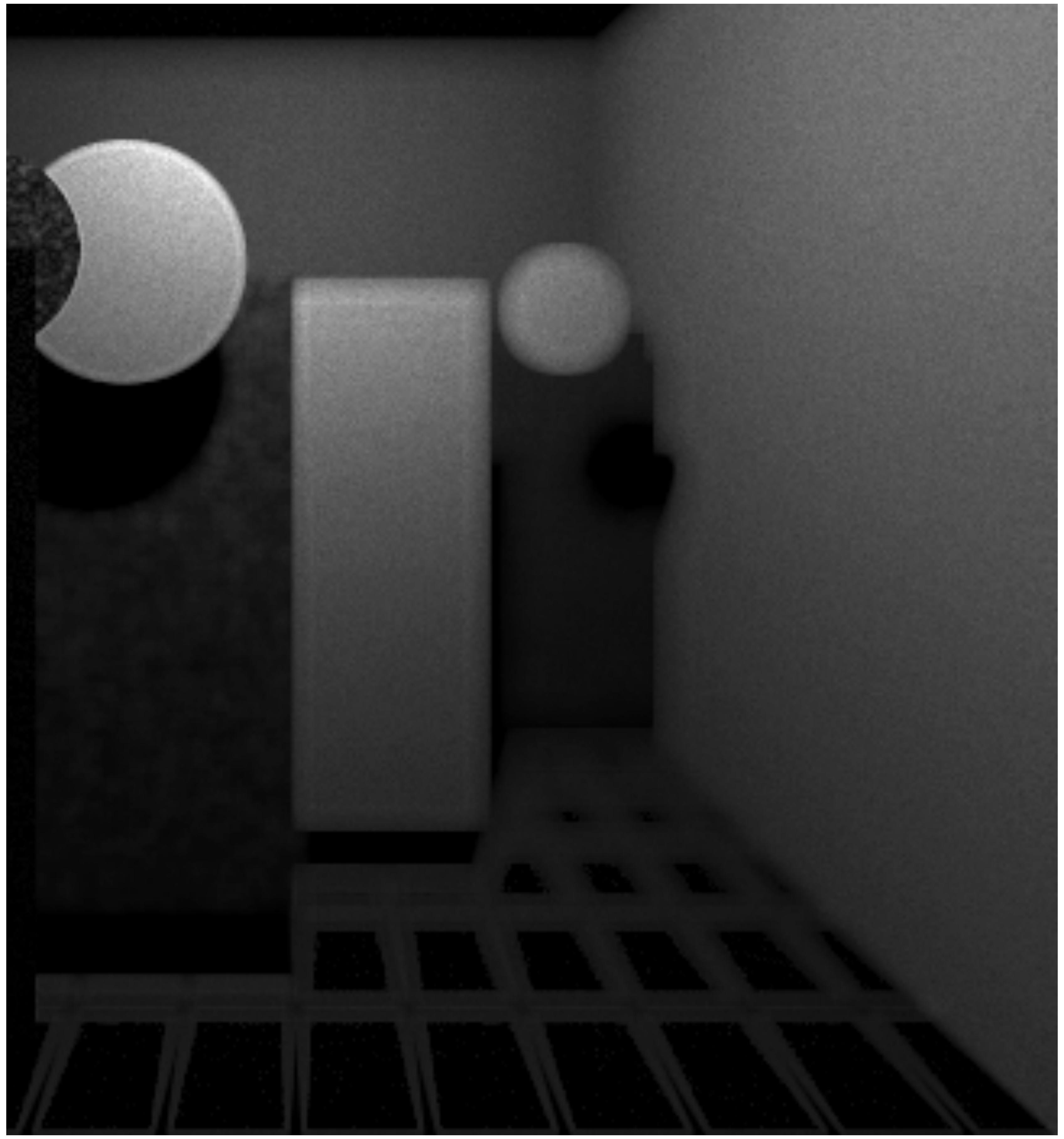




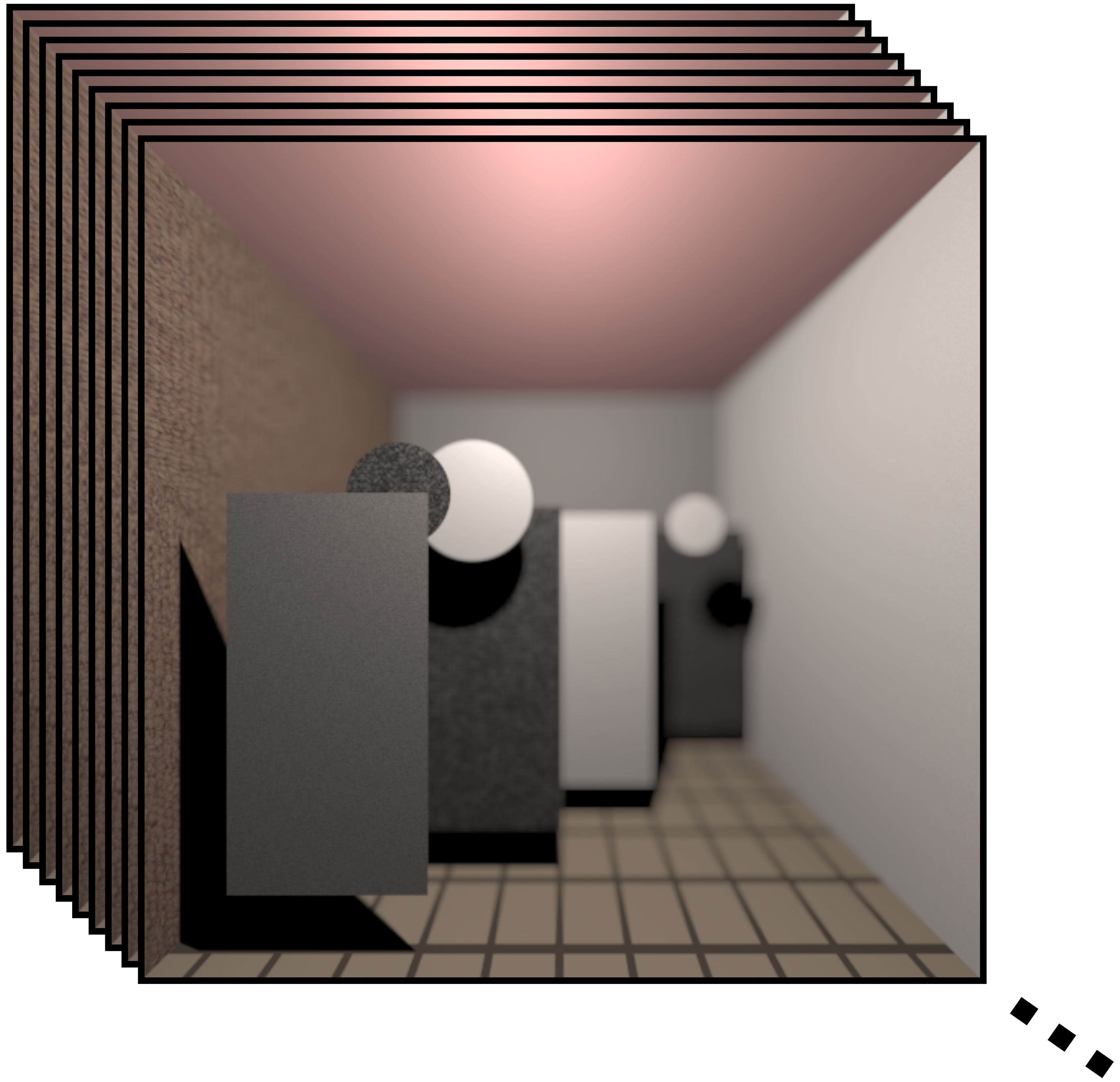
Variance Heatmap

With Original Samples

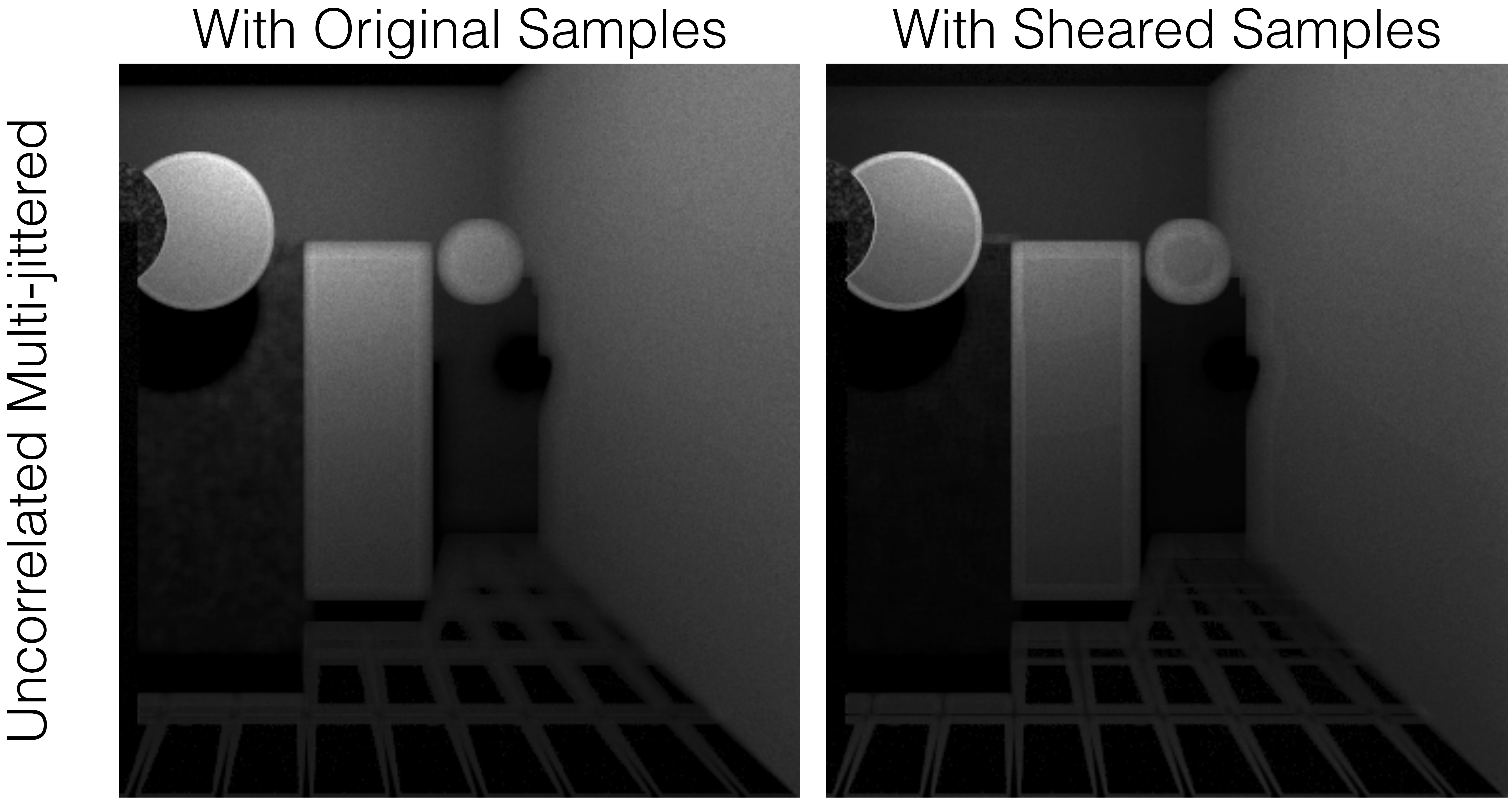
Uncorrelated Multi-jittered



Multiple images



Variance Heatmap



- Error Formulation in the Spatial Domain
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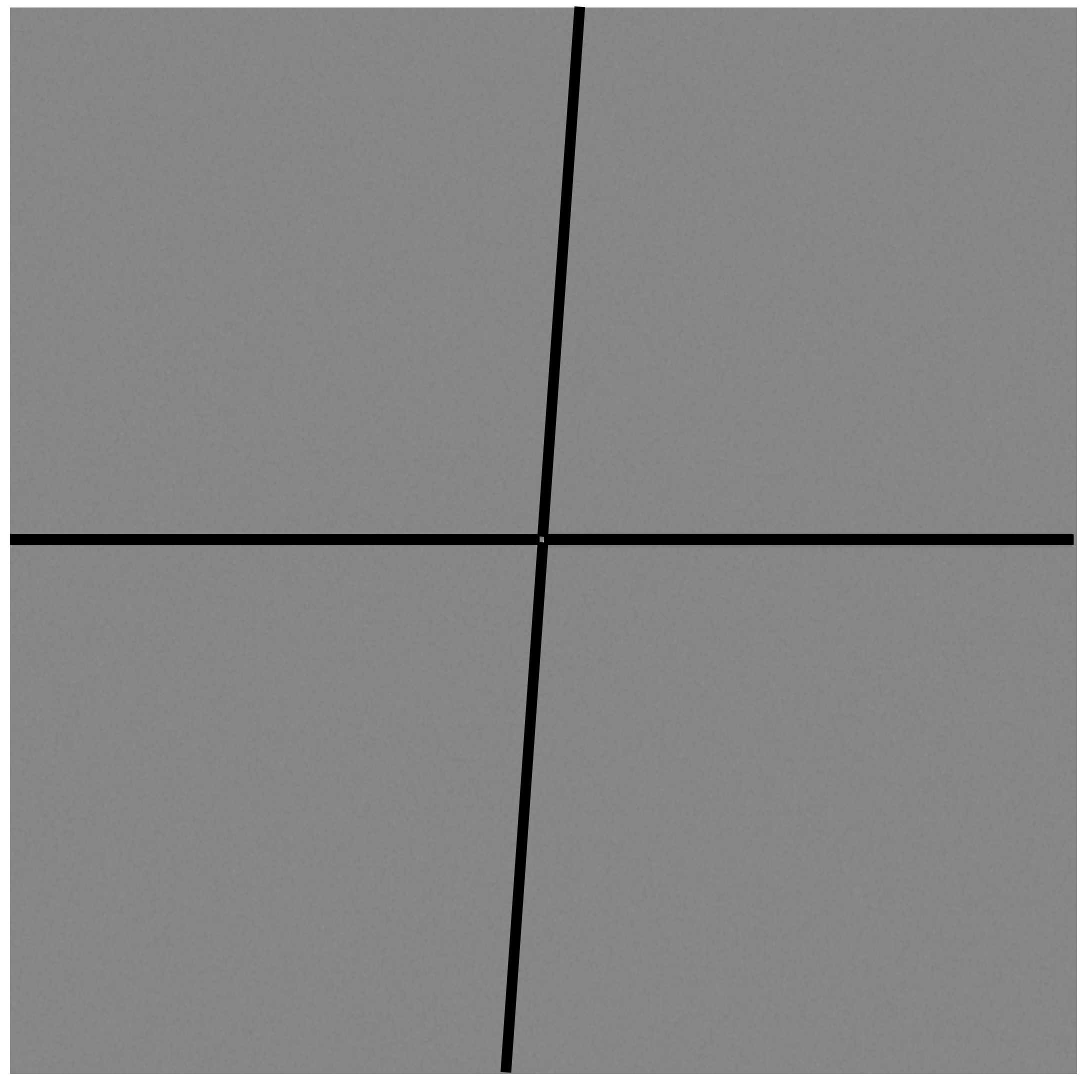
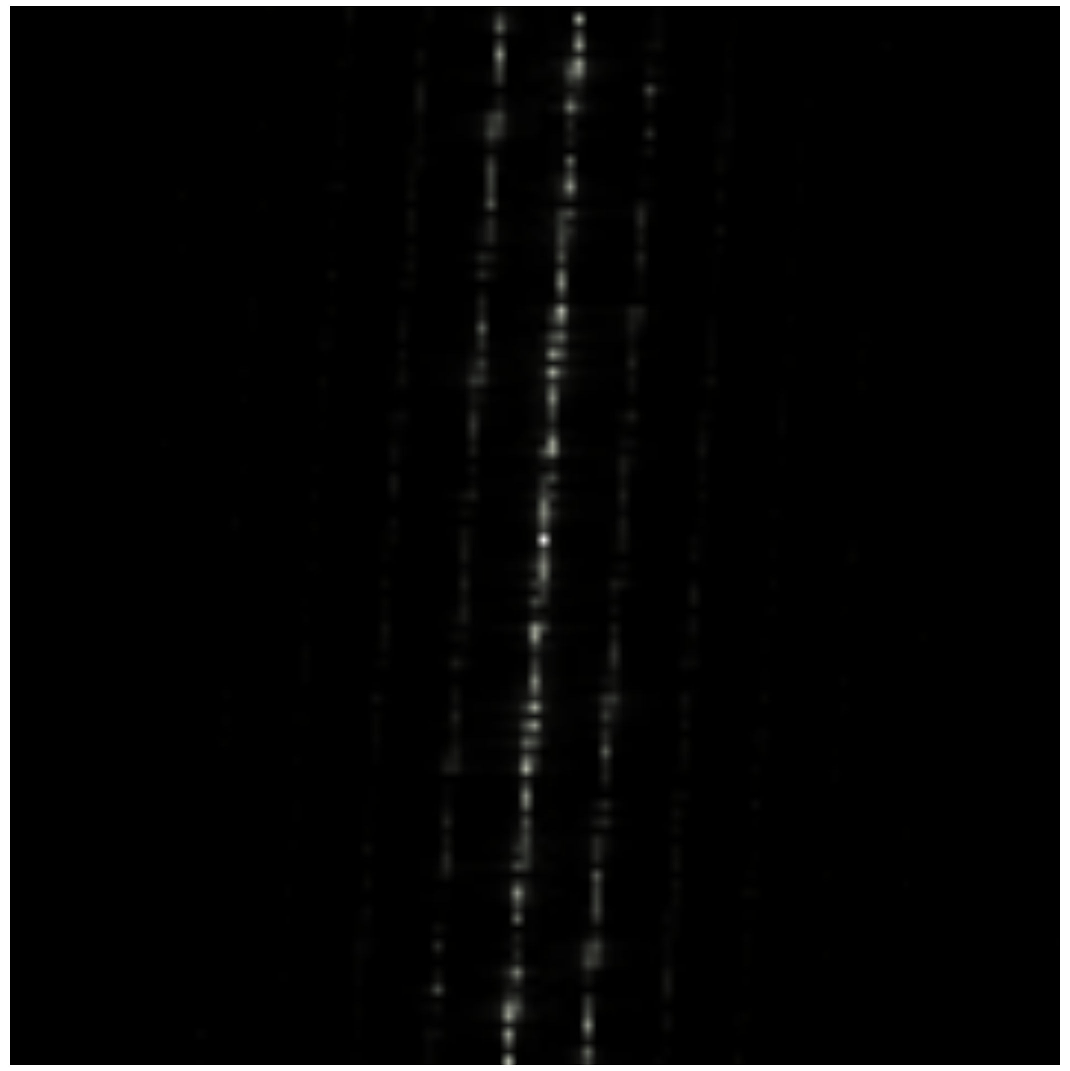
Challenging Cases: XU & YV Projections

Hairline Anisotropy

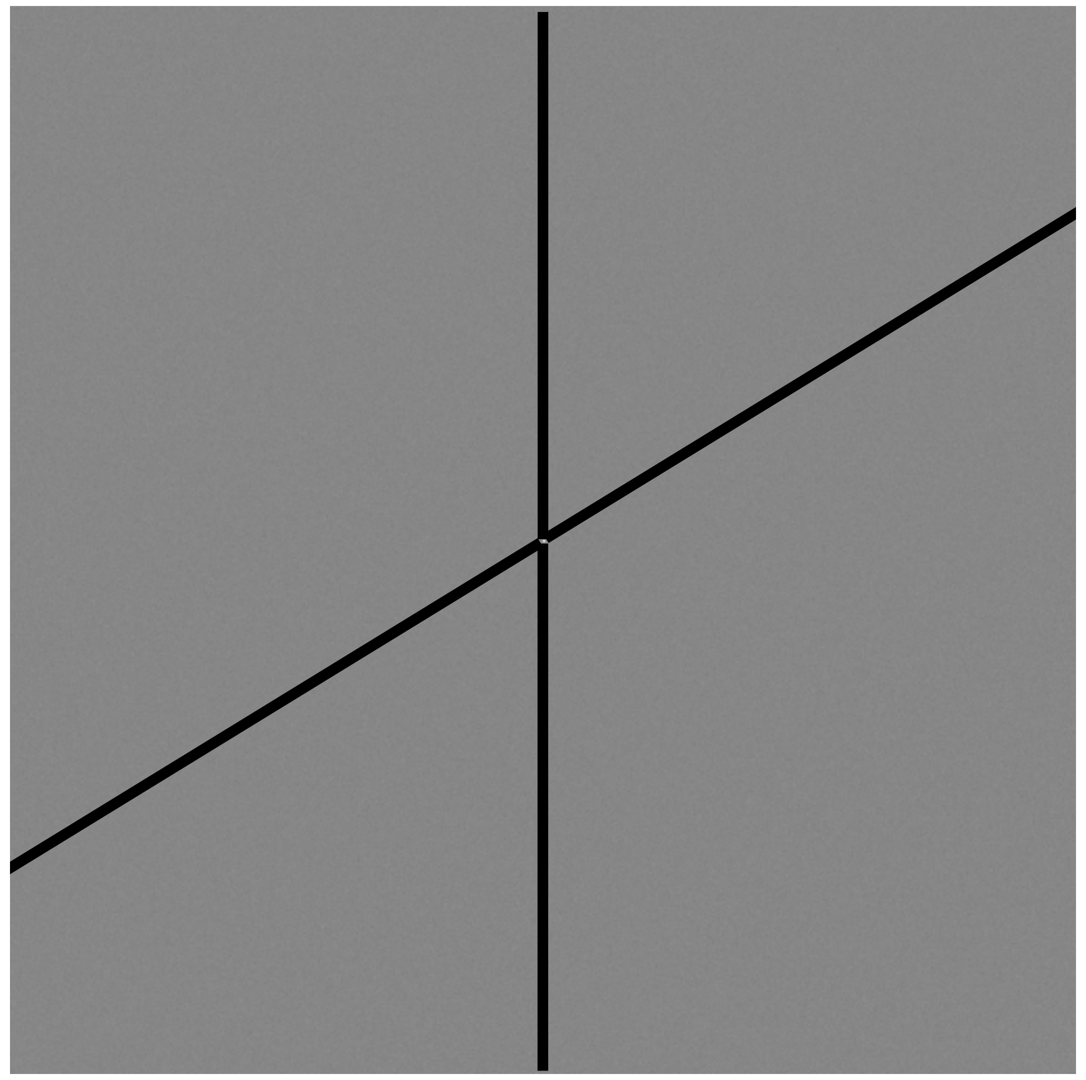


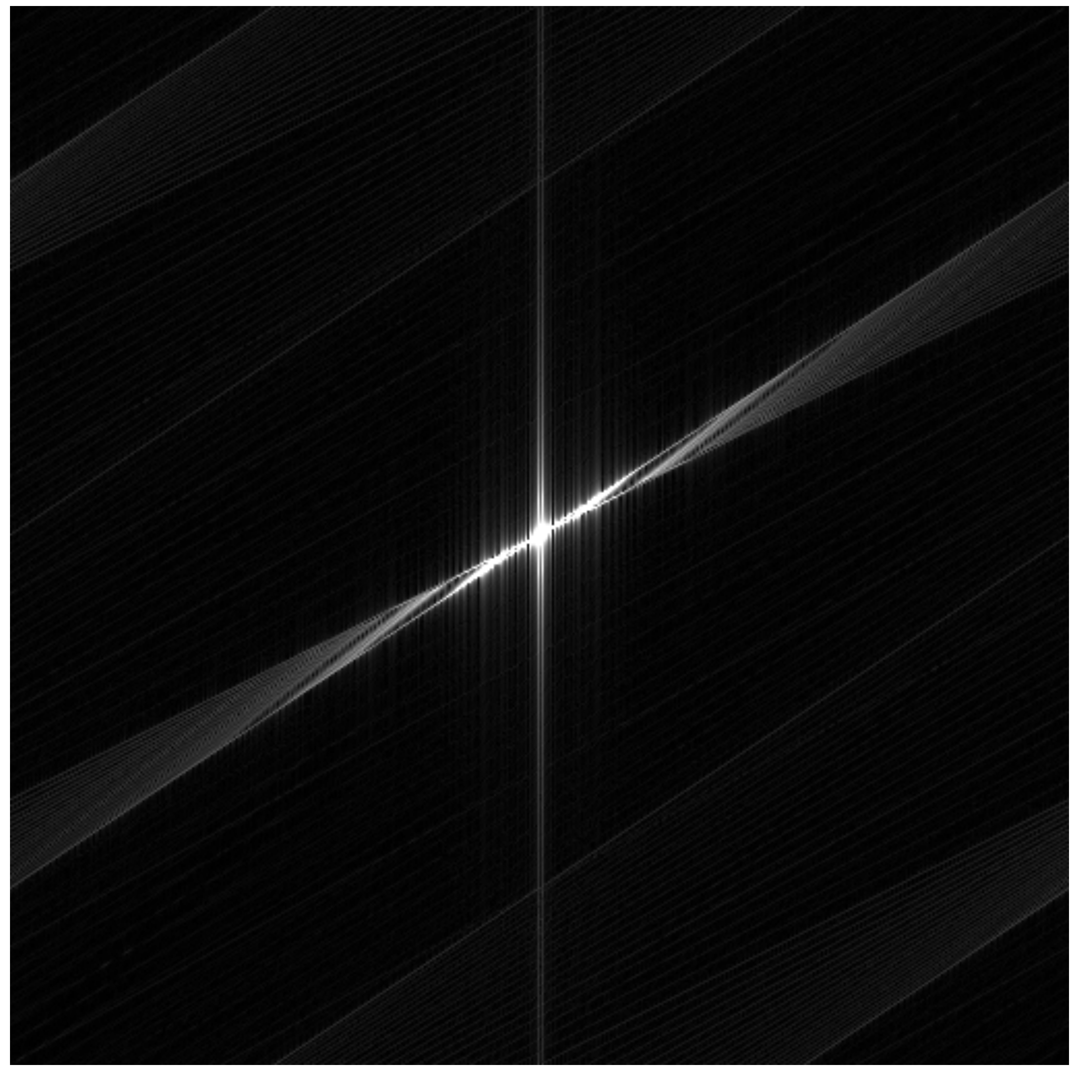


Sampling
XU Spectrum

Pixel A
XU Spectrum

Double-wedge Anisotropy





Sampling
XU Spectrum

Pixel B
XU Spectrum

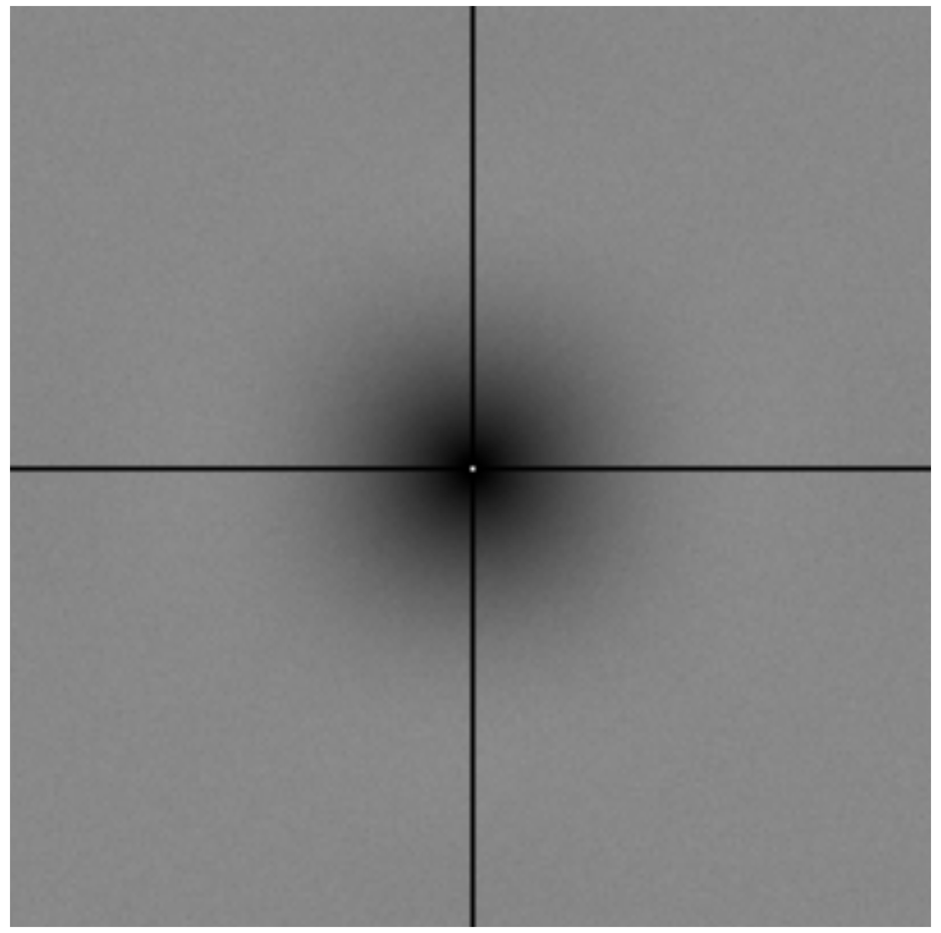
Oracle Accuracy

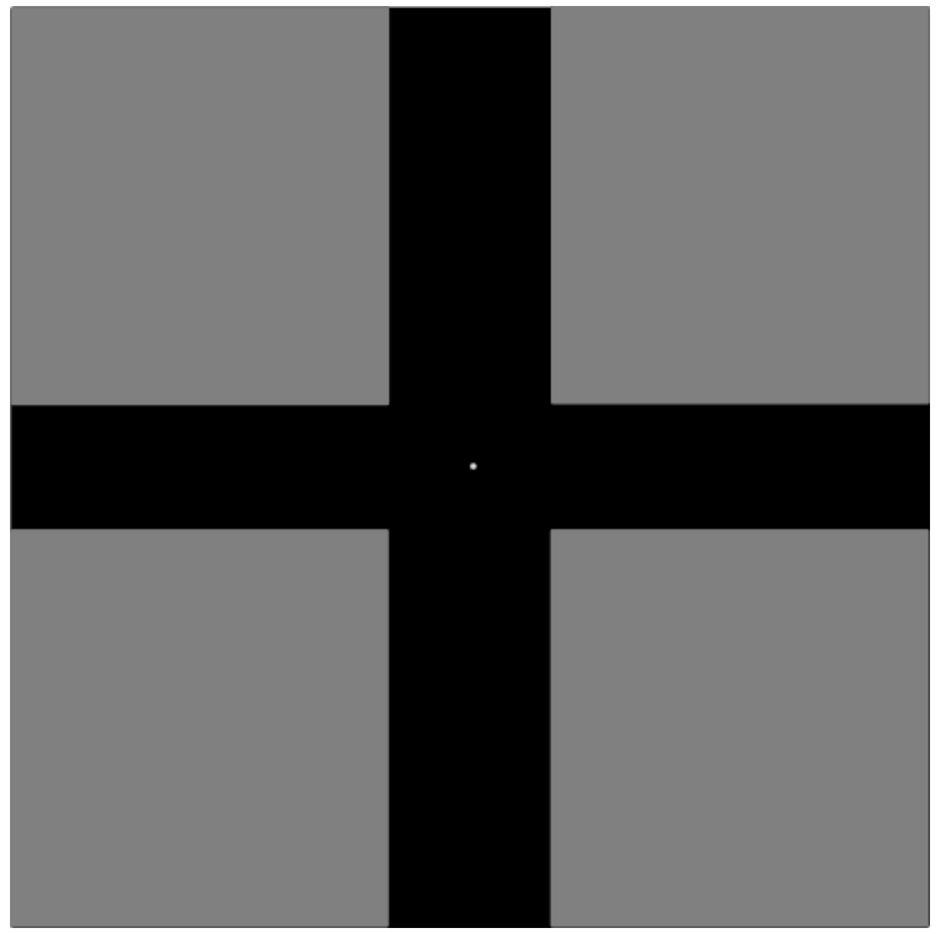
Double-wedge Spectrum

Design Principles for New Sampling Patterns

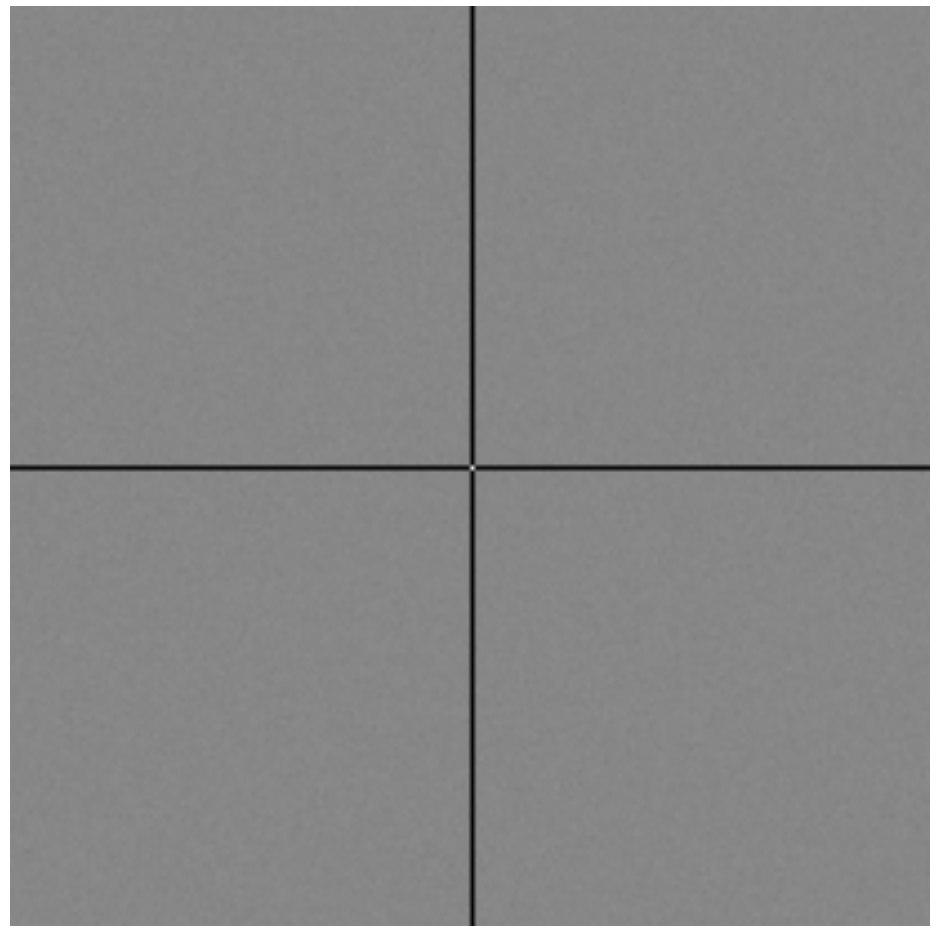
Multi-Jittered Spectra | Desired Sampling Spectra

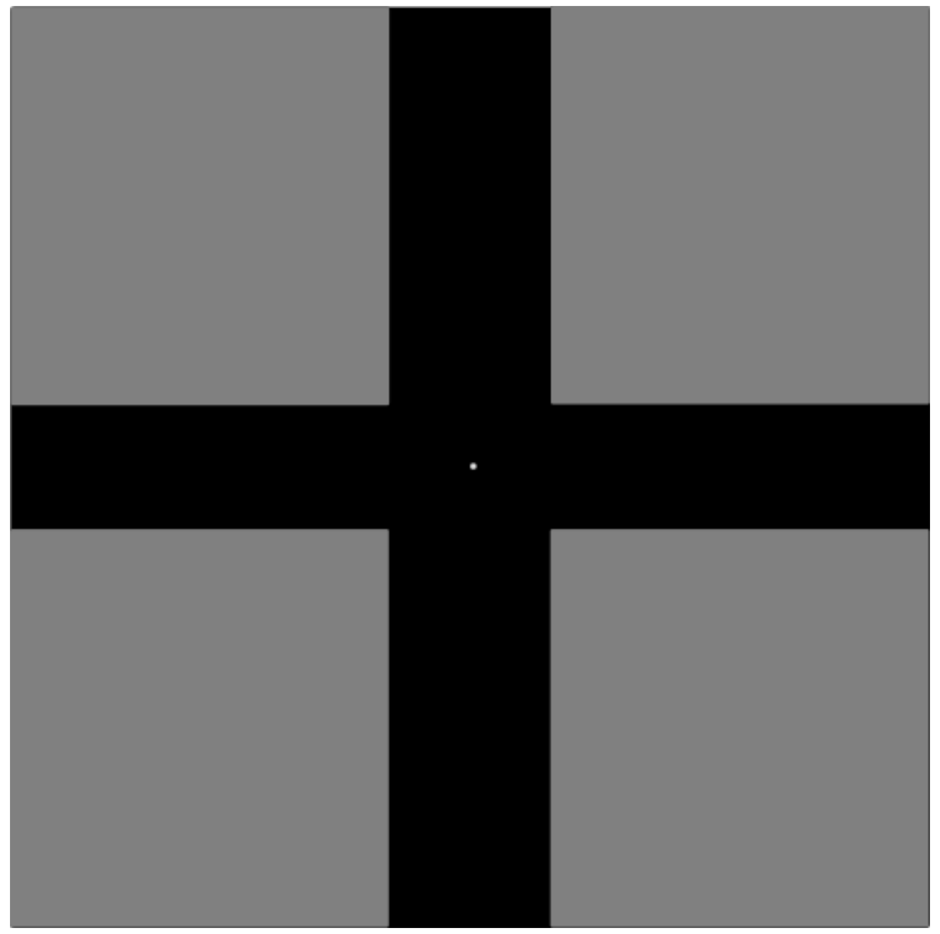
XY





XU





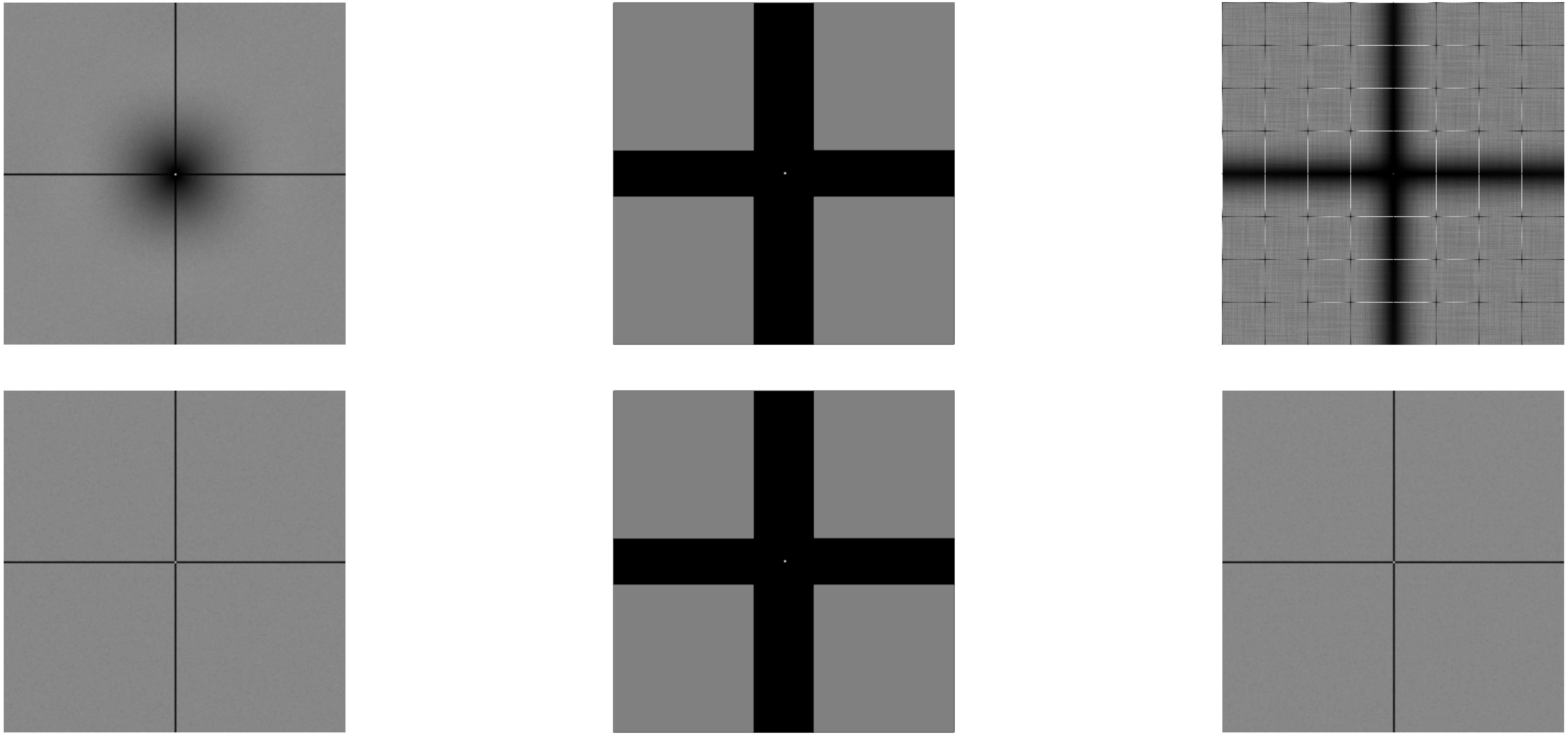
Singh and Jarosz [2017]

Design Principles for New Sampling Patterns

Multi-Jittered Spectra | Desired Sampling Spectra | Correlated Multi-Jitter

XY

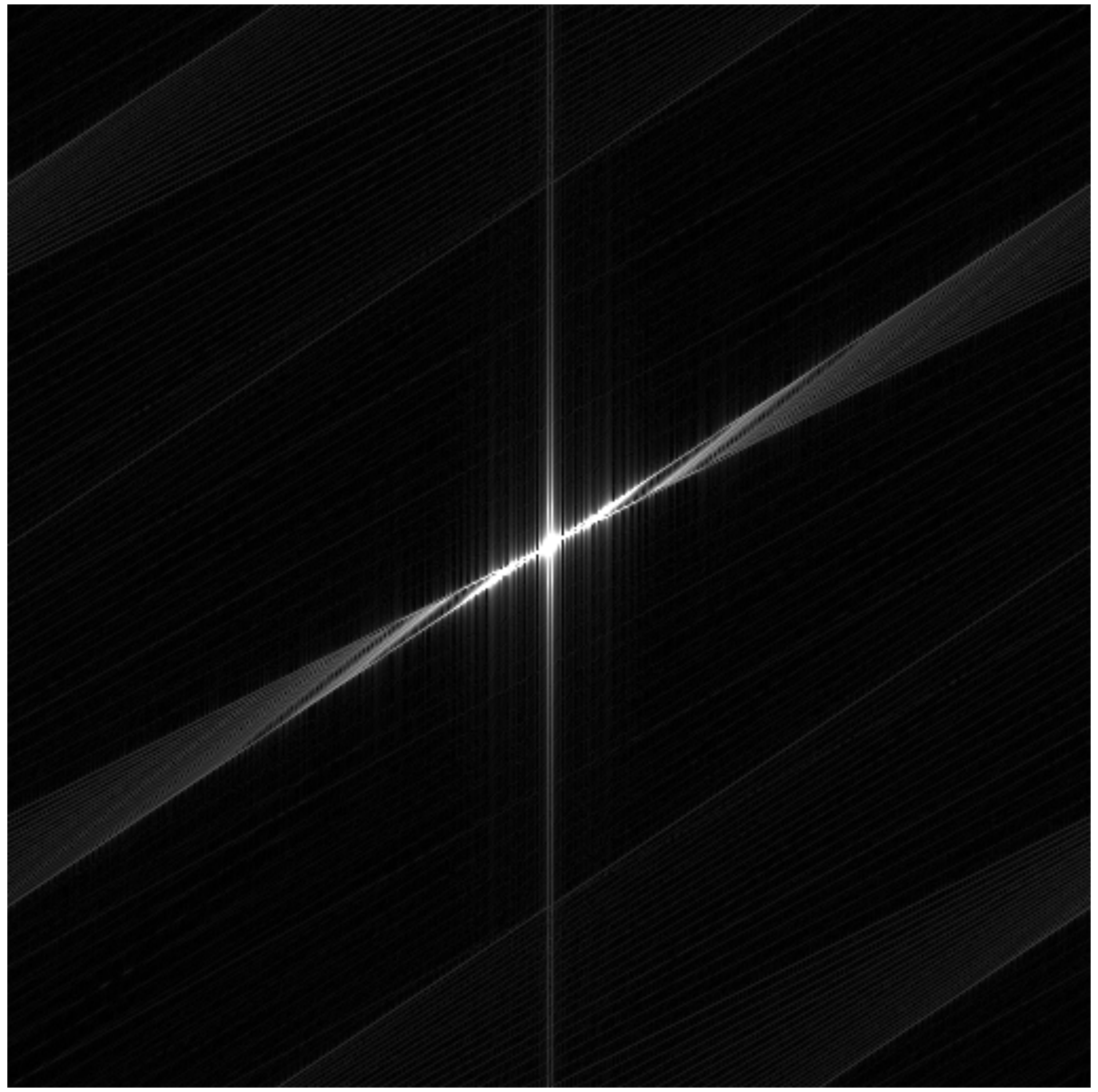
XU



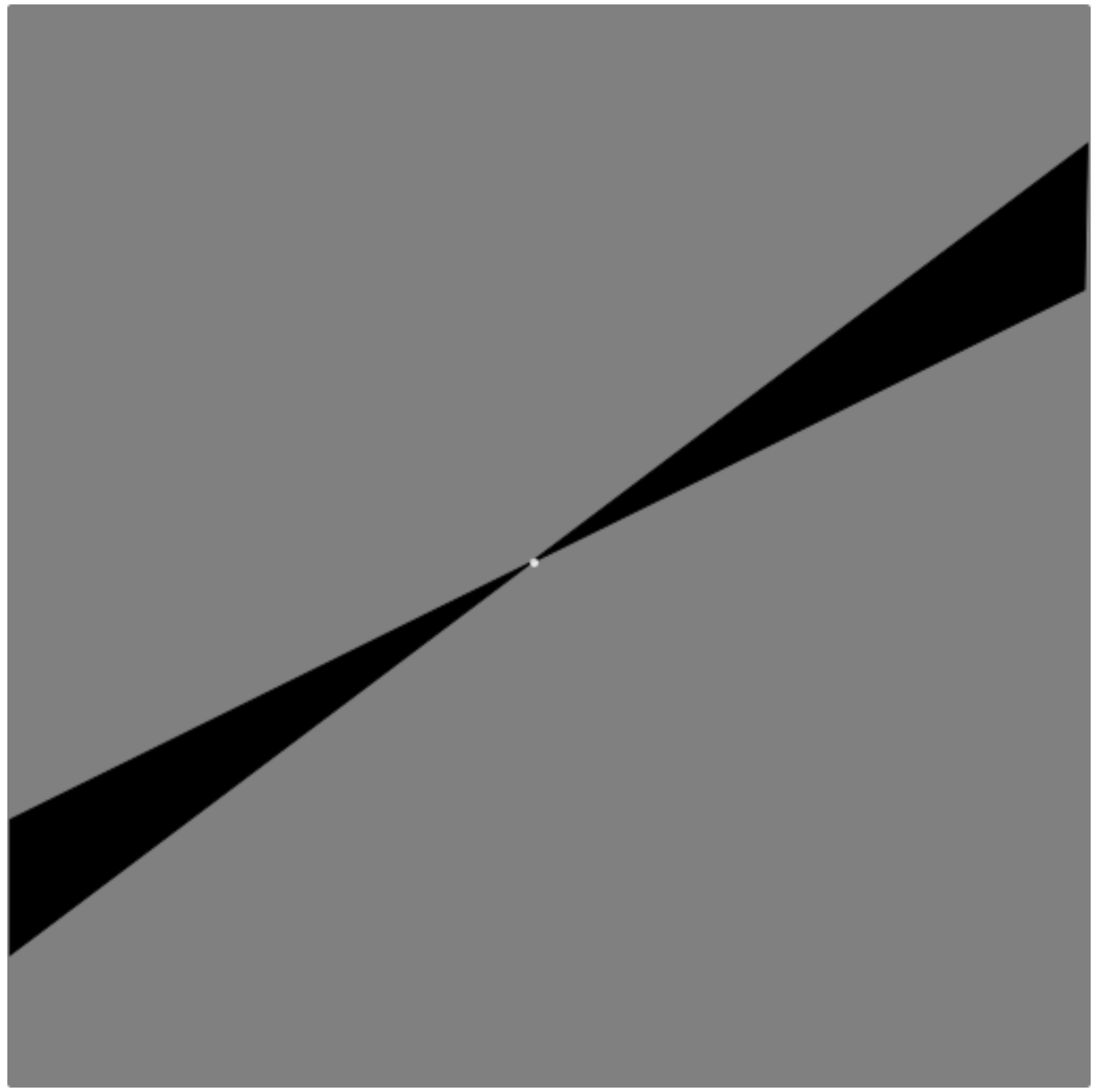
Kensler [2013]

Design Principles for New Sampling Patterns

Integrand Spectrum



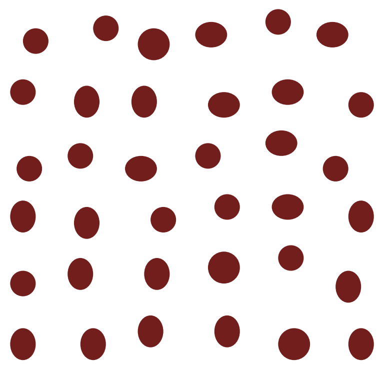
Desired Sampling Spectra



In both XU and YV Projections

Singh and Jarosz [2017]

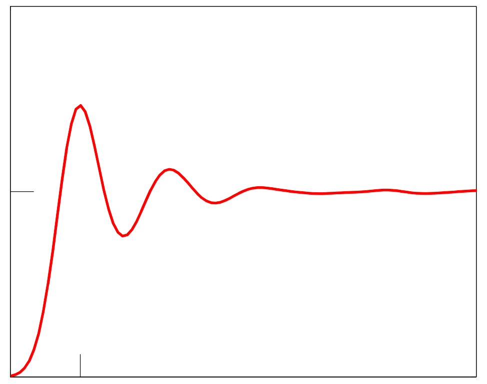
Summary



Point processes to understand error in integration

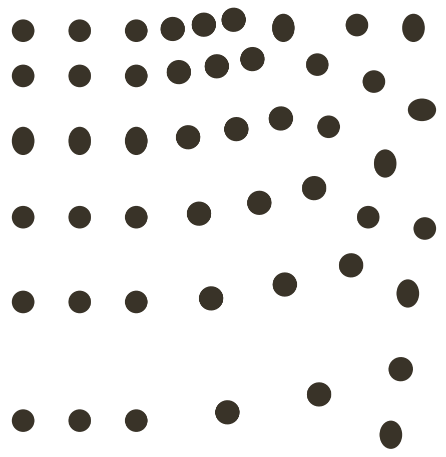
$$\frac{1}{n^2}\sum_{i=1}^{n} n \int s_i^2(\mathbf{x})d\mathbf{x}$$

Closed-form formulas amenable to **analysis**



Only **1st & 2nd** order statistics needed

Future Directions



Sampling patterns with
adaptive density & correlations

General **domains &**
local scene analysis

Anti-aliasing & reconstruction