Why walking the dog takes time: Fréchet distance has no strongly subquadratic algorithms unless SETH fails

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Abstract

The Fréchet distance is a well-studied and very popular measure of similarity of two curves. Many variants and extensions have been studied since Alt and Godau introduced this measure to computational geometry in 1991. Their original algorithm to compute the Fréchet distance of two polygonal curves with n vertices has a runtime of $\mathcal{O}(n^2 \log n)$. More than 20 years later, the state of the art algorithms for most variants still take time more than $\mathcal{O}(n^2/\log n)$, but no matching lower bounds are known, not even under reasonable complexity theoretic assumptions.

To obtain a conditional lower bound, in this paper we assume the Strong Exponential Time Hypothesis or, more precisely, that there is no $\mathcal{O}^*((2-\delta)^N)$ algorithm for CNF-SAT for any $\delta > 0$. Under this assumption we show that the Fréchet distance cannot be computed in strongly subquadratic time, i.e., in time $\mathcal{O}(n^{2-\delta})$ for any $\delta > 0$. This means that finding faster algorithms for the Fréchet distance is as hard as finding faster CNF-SAT algorithms, and the existence of a strongly subquadratic algorithm can be considered unlikely.

Our result holds for both the continuous and the discrete Fréchet distance. We extend the main result in various directions. Based on the same assumption we (1) show non-existence of a strongly subquadratic 1.001-approximation, (2) present tight lower bounds in case the numbers of vertices of the two curves are imbalanced, and (3) examine realistic input assumptions (c-packed curves).

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1 Introduction

Intuitively, the (continuous) Fréchet distance of two curves P, Q is the minimal length of a leash required to connect a dog to its owner, as they walk along P or Q, respectively, without backtracking. The Fréchet distance is a very popular measure of similarity of two given curves. In contrast to distance notions such as the Hausdorff distance, it takes into account the order of the points along the curve, and thus better captures the similarity as perceived by human observers [3].

Alt and Godau introduced the Fréchet distance to computational geometry in 1991 [5, 24]. For polygonal curves P and Q with n and m vertices[1], respectively, they presented an $\mathcal{O}(nm\log(nm))$ algorithm. Since Alt and Godau's seminal paper, Fréchet distance has become a rich field of research, with various directions such as generalizations to surfaces (see, e.g., [4]), approximation algorithms for realistic input curves ([6, 7, 21]), the geodesic and homotopic Fréchet distance (see, e.g., [15, 17]), and many more variants (see, e.g., [11, 20, 28, 30]). Being a natural measure for curve similarity, the Fréchet distance has found applications in various areas such as signature verification (see, e.g., [31]), map-matching tracking data (see, e.g., [9]), and moving objects analysis (see, e.g., [12]).

A particular variant that we will also discuss in this paper is the *discrete* Fréchet distance. Here, intuitively the dog and its owner are replaced by two frogs, and in each time step each frog can jump to the next vertex along its curve or stay at its current vertex. Defined in [22], the original algorithm for the discrete Fréchet distance has runtime $\mathcal{O}(nm)$.

Recently, improved algorithms have been found for some variants. Agarwal et al. [2] showed how to compute the discrete Fréchet distance in (mildly) subquadratic time $\mathcal{O}\big(nm\frac{\log\log n}{\log n}\big)$. Buchin et al. [13] gave algorithms for the continuous Fréchet distance in time $\mathcal{O}(n^2\sqrt{\log n}(\log\log n)^{3/2})$ on the Real RAM and $\mathcal{O}(n^2(\log\log n)^2)$ on the Word RAM. However, the problem remains open whether there is a *strongly subquadratic*[2] algorithm for the Fréchet distance, i.e., an algorithm in time $\mathcal{O}(n^{2-\delta})$ for any $\delta > 0$. For a particular variant, the discrete Fréchet distance with shortcuts, strongly subquadratic algorithms have been found recently [8], however, this seems to have no implications for the classical continuous or discrete Fréchet distance.

The only known lower bound shows that the Fréchet distance takes time $\Omega(n\log n)$ (in the algebraic decision tree model) [10]. The typical way of proving (conditional) quadratic lower bounds for geometric problems is 3SUM-hardness [23]. Alt conjectured that the Fréchet distance is 3SUM-hard, but this remains an open problem. Buchin et al. [13] recently argued that the Fréchet distance is unlikely to be 3SUM-hard.

Strong Exponential Time Hypothesis The Exponential Time Hypothesis (ETH) and the Strong Exponential Time Hypothesis (SETH), both introduced by Impagliazzo, Paturi, and Zane [26, 27], provide alternative ways of proving conditional lower bounds. ETH asserts that 3-SAT has no $2^{o(N)}$ algorithm, where N is the number of variables, and can be used to prove matching lower bounds for a wealth of problems, see [29] for a survey. However, since this hypothesis does not specify the exact exponent, it is not suited for proving polynomial time lower bounds, where the exponent is important.

The stronger hypothesis SETH asserts that there is no $\delta > 0$ such that k-SAT has an $\mathcal{O}((2-\delta)^N)$ algorithm for all k. In this paper, we will use the following weaker variant, which has also been used in [32, 33].

[1] We always assume that $m \leqslant n$.

[2] We use the term *strongly subquadratic* to differentiate between this runtime and the *(mildly) subquadratic* $\mathcal{O}(n^2\log\log n/\log n)$ algorithm from [2].

Hypothesis SETH': *There is no $\mathcal{O}^*((2-\delta)^N)$ algorithm for $\mathsf{CNF\text{-}SAT}$ for any $\delta > 0$. Here, $\mathcal{O}^*$ hides polynomial factors in the number of variables N and the number of clauses M.*

While SETH deals with formulas of width k, SETH' deals with $\mathsf{CNF\text{-}SAT}$, i.e., unbounded width clauses. Thus, it is a weaker assumption and more likely to be true. Note that exhaustive search takes time $\mathcal{O}^*(2^N)$, and the fastest known algorithms for $\mathsf{CNF\text{-}SAT}$ are only slighly faster than that, namely of the form $\mathcal{O}^*(2^{N(1-C/\log(M/N))})$ for some positive constant C [14, 19]. Thus, SETH' is a reasonable assumption that can be considered unlikely to fail. It has been observed that one can use SETH and SETH' to prove lower bounds for polynomial time problems such as k-Dominating Set and others [32], the diameter of sparse graphs [33], and dynamic connectivity problems [1]. However, it seems to be applicable only for few problems, e.g., it seems to be a wide open problem to prove that 3SUM has no strongly subquadratic algorithms unless SETH fails, similarly for matching, maximum flow, edit distance, and other classic problems.

Main result Our main theorem gives strong evidence that the Fréchet distance may have no strongly subquadratic algorithms by relating it to the Strong Exponential Time Hypothesis.

Theorem 1.1. *There is no $\mathcal{O}(n^{2-\delta})$ algorithm for the (continuous or discrete) Fréchet distance for any $\delta > 0$, unless SETH' fails.*

Since SETH and its weaker variant SETH' are reasonable hypotheses, by this theorem one can consider it unlikely that the Fréchet distance has strongly subquadratic algorithms. In particular, any strongly subquadratic algorithm for the Fréchet distance would not only give improved algorithms for $\mathsf{CNF\text{-}SAT}$ that are much faster than exhaustive search, but also for various other problems such as Hitting Set, Set Splitting, and NAE-SAT via the reductions in [18]. Alternatively, in the spirit of [32], one can view the above theorem as a possible attack on $\mathsf{CNF\text{-}SAT}$, as algorithms for the Fréchet distance now could provide a route to faster $\mathsf{CNF\text{-}SAT}$ algorithms. In any case, anyone trying to find strongly subquadratic algorithms for the Fréchet distance should be aware that this is as hard as finding improved $\mathsf{CNF\text{-}SAT}$ algorithms, which might be impossible.

We remark that all our lower bounds (unless stated otherwise) hold in the Euclidean plane, and thus also in $\mathbb{R}^d$ for any $d \geqslant 2$.

Extensions We extend our main result in two important directions: We show approximation hardness and we prove that the lower bound still holds for restricted classes of curves.

First, it would be desirable to have good approximation algorithms in strongly subquadratic time, say a near-linear time approximation scheme. We exclude such algorithms by proving that there is no 1.001-approximation for the Fréchet distance in strongly subquadratic time unless SETH' fails. Hence, within $n^{o(1)}$-factors any 1.001-approximation takes as much time as an exact algorithm. We did not try to optimize the constant 1.001, but only to find the asymptotically largest possible approximation ratio, which seems to be a constant. We leave it as an open problem whether there is a strongly subquadratic $\mathcal{O}(1)$-approximation. The literature so far contains no strongly subquadratic approximation algorithms for general curves at all.

Second, it might be conceivable that if one curve has much fewer vertices than the other, i.e., $m \ll n$, then after some polynomial preprocessing on the smaller curve we can compute the Fréchet distance of the two curves quickly, e.g., in total time $\mathcal{O}((n + m^3)\log n)$. Note that such a runtime is not ruled out by the trivial argument that any algorithm needs time $\Omega(n + m)$ for reading the input, and is also not ruled out by Theorem 1.1, since the runtime is not subquadratic for $n = m$. We rule out such runtimes by proving that there is no $\mathcal{O}((nm)^{1-\delta})$ algorithm "for any m", unless

SETH' fails. More precisely, we prove this lower bound for the "special case" $m \approx n^\gamma$ for any constant $0 \leqslant \gamma \leqslant 1$. To make this formal, for any input parameter α and constants $\gamma_0 < \gamma_1$ in $\mathbb{R} \cup \{-\infty, \infty\}$, we say that a statement holds *for any polynomial restriction of* $n^{\gamma_0} \leqslant \alpha \leqslant n^{\gamma_1}$ if it holds restricted to instances with $n^{\gamma-\delta} \leqslant \alpha \leqslant n^{\gamma+\delta}$ for any constants $\delta > 0$ and $\gamma_0 + \delta \leqslant \gamma \leqslant \gamma_1 - \delta$. We obtain the following extension of the main result Theorem 1.1, which yields tight lower bounds for any behaviour of m and any $(1+\varepsilon)$-approximation with $0 \leqslant \varepsilon \leqslant 0.001$.

Theorem 1.2. *There is no* 1.001*-approximation in time* $\mathcal{O}((nm)^{1-\delta})$ *for the (continuous or discrete) Fréchet distance for any* $\delta > 0$*, unless* SETH' *fails. This holds for any polynomial restriction of* $1 \leqslant m \leqslant n$.

Realistic input curves In attempts to capture the properties of realistic input curves, strongly subquadratic algorithms have been devised for restricted classes of inputs such as backbone curves [7], κ-bounded and κ-straight [6], and ϕ-low density curves [21]. The most popular model are *c-packed curves*, which have been used for various generalizations of the Fréchet distance [16, 20, 25]. Driemel et al. [21] introduced this model and presented a $(1+\varepsilon)$-approximation for the continuous Fréchet distance in time $\mathcal{O}(cn/\varepsilon + cn \log n)$, which works in any $\mathbb{R}^d$, $d \geqslant 2$.

While the algorithm of [21] is near-linear for small c and $1/\varepsilon$, is is not clear whether its dependence on c and $1/\varepsilon$ is optimal for c and $1/\varepsilon$ that grow with n. We give strong evidence that the algorithm of [21] has optimal dependence on c for any constant $0 < \varepsilon \leqslant 0.001$.

Theorem 1.3. *There is no* 1.001*-approximation in time* $\mathcal{O}((cn)^{1-\delta})$ *for the (continuous or discrete) Fréchet distance on c-packed curves for any* $\delta > 0$*, unless* SETH' *fails. This holds for any polynomial restriction of* $1 \leqslant c \leqslant n$.

Since we prove this claim for any polynomial restriction $c \approx n^\gamma$, the above result excludes 1.001-approximations in time, say, $\mathcal{O}(c^2 + n)$.

Regarding the dependence on ε, in any dimension $d \geqslant 5$ we can prove a conditional lower bound that matches the dependency on ε of [21] up to a polynomial.

Theorem 1.4. *There is no* $(1+\varepsilon)$*-approximation for the (continuous or discrete) Fréchet distance on c-packed curves in* $\mathbb{R}^d$*,* $d \geqslant 5$*, in time* $\mathcal{O}(\min\{cn/\sqrt{\varepsilon}, n^2\}^{1-\delta})$ *for any* $\delta > 0$*, unless* SETH' *fails. This holds for sufficiently small* $\varepsilon > 0$ *and any polynomial restriction of* $1 \leqslant c \leqslant n$ *and* $\varepsilon \leqslant 1$.

Outline of the main result To prove the main result we present a reduction from $\mathsf{CNF\text{-}SAT}$ to the Fréchet distance. Given a $\mathsf{CNF\text{-}SAT}$ instance φ, we partition its variables into sets V_1, V_2 of equal size. In order to find a satisfying assignment of φ we have to choose (partial) assignments a_1 of V_1 and a_2 of V_2. We will construct curves P_1, P_2 where P_k is responsible for choosing a_k. To this end, P_k consists of *assignment gadgets*, one for each assignment of V_k. Assignment gadgets are build of *clause gadgets*, one for each clause. The assignment gadgets of assignments a_1 of V_1 and a_2 of V_2 are constructed such that they have Fréchet distance at most 1 if and only if (a_1, a_2) forms a satisfying assignment of φ. In P_1 and P_2 we connect these assignment gadgets with some additional curves to implement an OR-gadget, which forces any traversal of (P_1, P_2) to walk along two assignment gadgets in parallel. If φ is not satisfiable, then any pair of assignment gadgets has Fréchet distance larger than 1, so that P_1, P_2 have Fréchet distance larger than 1. If, on the other hand, a satisfying assignment (a_1, a_2) of φ exists, then we ensure that there is a traversal of P_1, P_2 that essentially only traverses the assignment gadgets of a_1 and a_2 in parallel, so that it always stays in distance 1.

To argue about the runtime, since P_k contains an assignment gadget for every assignment of one half of the variables, and every assignment gadget has polynomial size in M, there are $n = \mathcal{O}^*(2^{N/2})$ vertices on each curve. Thus, any $\mathcal{O}(n^{2-\delta})$ algorithm for the Fréchet distance would yield an $\mathcal{O}^*(2^{(1-\delta/2)N})$ algorithm for $\mathsf{CNF\text{-}SAT}$, contradicting SETH'.

Organization We start by defining the variants of the Fréchet distance, c-packedness, and other basic notions in Section 2. Section 3 deals with general curves. We prove the main result for the discrete Fréchet distance in less than 3 pages in Section 3.1. This construction also already proves inapproximability. We generalize the proof to the continuous Fréchet distance in Section 3.2 (which is more tedious than in the discrete case) and to $m \ll n$ in Section 3.3 (which is an easy trick). Section 4 deals with c-packed curves. In Section 4.1 we present a new OR-gadget that generates less packed curves; plugging in the curves constructed in the main result proves Theorem 1.3. In Section 4.2 we make use of the fact that in $\geqslant 4$ dimensions there are point sets Q_1, Q_2 of arbitrary size with each pair of points (q_1, q_2) having distance exactly 1. This allows to construct less packed curves that we plug into the OR-gadget from the preceding section to prove Theorem 1.4.

2 Preliminaries

For $N \in \mathbb{N}$ we let $[N] := \{1, \ldots, N\}$. A (polygonal) curve P is defined by its vertices $p_1, \ldots, p_n$. We view P as a continuous function $P\colon [0, n] \to \mathbb{R}^d$ with $P(i+\lambda) = (1-\lambda)p_i + \lambda p_{i+1}$ for $i \in [n-1]$, $\lambda \in [0, 1]$. We write $|P| = n$ for the number of vertices of P. For two curves P_1, P_2 we let $P_1 \circ P_2$ be the curve on $|P_1| + |P_2|$ vertices that first follows P_1, then walks along the segment from $P_1(|P_1|)$ to $P_2(0)$, and then follows P_2. In particular, for two points $p, q \in \mathbb{R}^d$ the curve $p \circ q$ is the segment from p to q, and any curve P on vertices $p_1, \ldots, p_n$ can be written as $P = p_1 \circ \ldots \circ p_n$.

Consider a curve P and two points $p_1 = P(\lambda_1)$, $p_2 = P(\lambda_2)$ with $\lambda_1, \lambda_2 \in [0, n]$. We say that p_1 *is within distance D of p_2 along* P if the length of the subcurve of P between $P(\lambda_1)$ and $P(\lambda_2)$ is at most D.

Variants of the Fréchet distance Let Φ_n be the set of all continuous and non-decreasing functions ϕ from $[0, 1]$ onto $[0, n]$. The *continuous Fréchet distance* between two curves P_1, P_2 with $|P_1| = n$, $|P_2| = m$ is defined as

$$d_{\mathrm{F}}(P_1, P_2) := \inf_{\substack{\phi_1 \in \Phi_n \\ \phi_2 \in \Phi_m}} \max_{t \in [0,1]} \|P_1(\phi_1(t)) - P_2(\phi_2(t))\|,$$

where $\|.\|$ denotes the Euclidean distance. We call (ϕ_1, ϕ_2) a (continuous) *traversal* of (P_1, P_2), and say that it has *width* D if $\max_{t \in [0,1]} \|P_1(\phi_1(t)) - P_2(\phi_2(t))\| \leqslant D$.

In the discrete case, we let Δ_n be the set of all non-decreasing functions ϕ from $[0, 1]$ onto $[n]$. The *discrete Fréchet distance* between two curves P_1, P_2 with $|P_1| = n$, $|P_2| = m$ is then defined as

$$d_{\mathrm{dF}}(P_1, P_2) := \inf_{\substack{\phi_1 \in \Delta_n \\ \phi_2 \in \Delta_m}} \max_{t \in [0,1]} \|P_1(\phi_1(t)) - P_2(\phi_2(t))\|.$$

We obtain an analogous notion of a (discrete) *traversal* and its *width*. Note that any $\phi \in \Delta_n$ is a staircase function attaining all values in $[n]$. Hence, $(\phi_1(t), \phi_2(t))$ changes only at finitely many points in time t. At any such *time step* we jump to the next vertex in P_1 or P_2 or both.

It is known that for any curves P_1, P_2 we have $d_{\mathrm{F}}(P_1, P_2) \leqslant d_{\mathrm{dF}}(P_1, P_2)$ [22].

Realistic input curves As an example of input restrictions that resemble practical input curves we consider the model of [21]. A curve P is *c-packed* if for any point $q \in \mathbb{R}^d$ and any radius $r > 0$ the total length of P inside the ball $B(q, r)$ is at most cr. Here, $B(q, r)$ is the ball of radius r around q. In this paper, we say that a curve P is *$\Theta(c)$-packed*, if there are constants $\alpha > \beta > 0$ such that P is αc-packed but not βc-packed.

This model is well motivated from a practical point of view. Examples of classes of c-packed curves are boundaries of convex polygons and γ-fat shapes as well as algebraic curves of bounded maximal degree (see [21]).

Satisfiability In $\mathsf{CNF\text{-}SAT}$ we are given a formula φ on variables $x_1, \ldots, x_N$ and clauses $C_1, \ldots, C_M$ in conjunctive normal form with unbounded clause width. Let V be any set of variables of φ. Let a be any assignment of T (true) or F (false) to the variables of V. We say that the partial assignment a *satisfies* a clause $C = \bigvee_{i \in I} x_i \vee \bigvee_{i \in J} \neg x_i$ if for some $i \in I \cap V$ we have $a(x_i) = \mathsf{T}$ or for some $i \in J \cap V$ we have $a(x_i) = \mathsf{F}$. We denote by $\mathsf{sat}(a, C)$ whether partial assignment a satisfies clause C. Note that assignments a of V and a' of the remaining variables V' form a satisfying assignment (a, a') of φ if and only if we have $\mathsf{sat}(a, C_i) \vee \mathsf{sat}(a', C_i) = \mathsf{T}$ for all $i \in \{1, \ldots, M\}$.

All bounds that we prove in this paper assume the hypothesis SETH' (see Section 1), which asserts that $\mathsf{CNF\text{-}SAT}$ has no $\mathcal{O}^*((2-\delta)^N)$ algorithm for any $\delta > 0$. Here, $\mathcal{O}^*$ hides polynomials factors in N and M. The following is an easy corollary of SETH'.

Lemma 2.1. *There is no $\mathcal{O}^*((2-\delta)^N)$ algorithm for* $\mathsf{CNF\text{-}SAT}$ *restricted to formulas with N variables and $M \leqslant 2^{\delta' N}$ clauses for any $\delta, \delta' > 0$, unless* SETH' *fails.*

Proof. Any such algorithm would imply an $\mathcal{O}^*((2-\delta)^N)$ algorithm for $\mathsf{CNF\text{-}SAT}$ (with no restrictions on the input), since for $M \leqslant 2^{\delta' N}$ we can run the given algorithm, while for $M > 2^{\delta' N}$ we can decide satisfiability in time $\mathcal{O}(M 2^N) = \mathcal{O}(M^{1+1/\delta'}) = \mathcal{O}^*(1)$. □

3 General curves

We first present a reduction from $\mathsf{CNF\text{-}SAT}$ to the Fréchet distance and show that is proves Theorem 1.1 for the discrete Fréchet distance. In Section 3.2 we then show that the same construction also works for the continuous Fréchet distance. Finally, in Section 3.3 we generalize these results to curves with imbalanced numbers of vertices n, m to show Theorem 1.2.

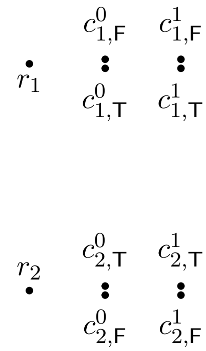
3.1 The basic reduction, discrete case

Let φ be a given $\mathsf{CNF\text{-}SAT}$ instance with variables $x_1, \ldots, x_N$ and clauses $C_1, \ldots, C_M$. We split the variables into two halves $V_1 := \{x_1, \ldots, x_{N/2}\}$ and $V_2 := \{x_{N/2+1}, \ldots, x_N\}$. For $k \in \{1, 2\}$ let A_k be all assignments[3] of T or F to the variables in V_k, so that $|A_k| = 2^{N/2}$. In the whole section we let $\varepsilon := 1/1000$.

We will construct two curves P_1, P_2 such that $d_{\mathrm{dF}}(P_1, P_2) \leqslant 1$ if and only if φ is satisfiable. In the construction we will use gadgets as follows.

[3] In later sections we will replace V_1, V_2 by different partitionings and A_1, A_2 by subsets of all assignments. The lemmas in this section are proven in a generality that allows this extension.

Clause gadgets This gadget encodes whether a partial assignment satisfies a clause. We set for $i \in \{0,1\}$



$$c^i_{1,\mathsf{T}} := \big(i/3, \tfrac{1}{2} - \varepsilon\big), \qquad c^i_{1,\mathsf{F}} := \big(i/3, \tfrac{1}{2} + \varepsilon\big),$$
$$c^i_{2,\mathsf{T}} := \big(i/3, -\tfrac{1}{2} + \varepsilon\big), \quad c^i_{2,\mathsf{F}} := \big(i/3, -\tfrac{1}{2} - \varepsilon\big).$$

Let $k \in \{1,2\}$. For any partial assignment $a_k \in A_k$ and clause C_i, $i \in [M]$, we construct a clause gadget consisting of a single point,

$$CG(a_k, i) := c^{i \bmod 2}_{k,\mathsf{sat}(a_k,C_i)}.$$

Thus, if assignment a_k satisfies clause C_i then the corresponding clause gadget is nearer to the clause gadgets associated with A_{3-k}. Explicitly calculating all pairwise distances of these points, we obtain the following lemma.

Lemma 3.1. *Let $a_k \in A_k$, $k \in \{1,2\}$, and $i,j \in [M]$. If $i \equiv j \pmod 2$ and $\mathsf{sat}(a_1, C_i) \vee \mathsf{sat}(a_2, C_j) = \mathsf{T}$ then $\|CG(a_1,i) - CG(a_2,j)\| \leqslant 1$. Otherwise $\|CG(a_1,i) - CG(a_2,j)\| \geqslant 1 + 2\varepsilon$.*

Assignment gadgets This gadget consists of clause gadgets and encodes the set of satisfied clauses for an assignment. We set

$$r_1 := (-\tfrac{1}{3}, \tfrac{1}{2}), \quad r_2 := (-\tfrac{1}{3}, -\tfrac{1}{2}).$$

The assignment gadget for any $a_k \in A_k$ consists the starting point r_k followed by all clause gadgets of a_k,

$$AG(a_k) := r_k \circ \bigcirc_{i \in [M]} CG(a_k, i).$$

The figure to the right shows an assignment gadget on $M = 2$ clauses at the top and an assignment gadget on $M = 4$ clauses at the bottom.

Lemma 3.2. *Let $a_k \in A_k$, $k \in \{1,2\}$. If (a_1,a_2) is a satisfying assignment of φ then $d_{\mathrm{dF}}(AG(a_1), AG(a_2)) \leqslant 1$. If (a_1,a_2) is not satisfying then $d_{\mathrm{dF}}(AG(a_1), AG(a_2)) > 1 + \varepsilon$, and we even have $d_{\mathrm{dF}}(AG(a_1) \circ \pi_1, AG(a_2) \circ \pi_2) > 1 + \varepsilon$ for any curves π_1, π_2.*

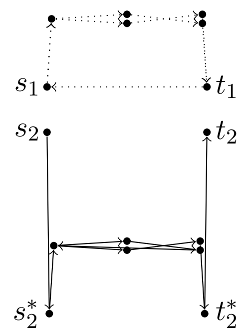
Proof. If (a_1,a_2) is satisfying then the parallel traversal

$$(r_1, r_2), (CG(a_1,1), CG(a_2,1)), \ldots, (CG(a_1,M), CG(a_2,M))$$

has width 1 by Lemma 3.1.

Assume for the sake of contradiction that (a_1,a_2) is not satisfying but there is a traversal of $(AG(a_1) \circ \pi_1, AG(a_2) \circ \pi_2)$ with width $1+\varepsilon$. Observe that $\|r_1 - r_2\| = 1$ and $\|r_k - c^i_{3-k,x}\| \geqslant 1 + 2\varepsilon$ for any $k \in \{1,2\}, i \in \{0,1\}, x \in \{\mathsf{T},\mathsf{F}\}$. Thus, the traversal has to start at positions (r_1, r_2) and then step to positions $(CG(a_1,1), CG(a_2,1))$, as advancing in only one of the curves leaves us in distance larger than $1+\varepsilon$. Inductively and using Lemma 3.1, the same argument shows that in the i-th step we are at positions $(CG(a_1,i), CG(a_2,i))$ for any $i \in [M]$. Since there is an unsatisfied clause C_i, so that $\|CG(a_1,i) - CG(a_2,i)\| \geqslant 1 + 2\varepsilon$ by Lemma 3.1, we obtain a contradiction. $\square$

Construction of the curves The curve P_k will consist of all assignment gadgets for assignments A_k, $k \in \{1,2\}$, plus some additional points. The additional points implement an OR-gadget over the assignment gadgets, by enforcing that any traversal of (P_1, P_2) with width $1+\varepsilon$ has to traverse two assignment gadgets in parallel, and traversing one pair of assignment gadgets in parallel suffices.



We define the following control points,

$$s_1 := (-\tfrac{1}{3}, \tfrac{1}{5}), \quad t_1 := (\tfrac{1}{3}, \tfrac{1}{5}),$$
$$s_2 := (-\tfrac{1}{3}, 0), \quad t_2 := (\tfrac{1}{3}, 0), \quad s_2^* := (-\tfrac{1}{3}, -\tfrac{4}{5}), \quad t_2^* := (\tfrac{1}{3}, -\tfrac{4}{5}).$$

Finally, we set

$$P_1 := \bigcirc_{a_1 \in A_1} \big(s_1 \circ AG(a_1) \circ t_1\big),$$
$$P_2 := s_2 \circ s_2^* \circ \Big(\bigcirc_{a_2 \in A_2} AG(a_2) \Big) \circ t_2^* \circ t_2.$$

See the figure to the right for an example with $M = 2$ clauses and two assignments drawn per half.

Let Q_k be the vertices that may appear in P_k, i.e., $Q_1 = \{s_1, t_1, r_1, c^0_{1,\mathsf{F}}, c^0_{1,\mathsf{T}}, c^1_{1,\mathsf{F}}, c^1_{1,\mathsf{T}}\}$ and $Q_2 = \{s_2, t_2, r_2, s_2^*, t_2^*, c^0_{2,\mathsf{F}}, c^0_{2,\mathsf{T}}, c^1_{2,\mathsf{F}}, c^1_{2,\mathsf{T}}\}$. Explicitly calculating all pairwise distances of all points, we obtain the following lemma.

Lemma 3.3. *No pair $(q_1, q_2) \in Q_1 \times Q_2$ has $\|q_1 - q_2\| \in (1, 1+\varepsilon]$. Moreover, the set $\{(q_1,q_2) \in Q_1 \times Q_2 \mid \|q_1 - q_2\| \leqslant 1\}$ consists of the following pairs:*

$(q, s_2), (q, t_2)$ *for any* $q \in Q_1$,
(s_1, q) *for any* $q \in Q_2 \setminus \{t_2^*\}$,
(t_1, q) *for any* $q \in Q_2 \setminus \{s_2^*\}$,
(r_1, r_2),
$(c^i_{1,x}, c^i_{2,y})$ *for* $x \vee y = \mathsf{T}$ *where* $i \in \{0,1\}, x, y \in \{\mathsf{T}, \mathsf{F}\}$.

Correctness We show that if φ is satisfiable then $d_{\mathrm{dF}}(P_1, P_2) \leqslant 1$, while otherwise $d_{\mathrm{dF}}(P_1, P_2) > 1 + \varepsilon$.

Lemma 3.4. *If $d_{\mathrm{dF}}(P_1, P_2) \leqslant 1+\varepsilon$ then $A_1 \times A_2$ contains a satisfying assignment.*

Proof. By Lemma 3.3 any traversal with width $1+\varepsilon$ also has width 1. Consider any traversal of (P_1, P_2) with width 1. Consider any time step T at which we are at position s_2^* in P_2. The only point in P_1 that is within distance 1 of s_2^* is s_1, say we are at the copy of s_1 that comes right before assignment gadget $AG(a_1)$, $a_1 \in A_1$. Following time step T, we have to start traversing $AG(a_1)$, so consider the first time step T' where we are at the point r_1 in $AG(a_1)$. The only points in P_2 within distance 1 of r_1 are s_2, t_2, and r_2. Note that we already passed s_2^* in P_2 by time T, so we cannot be in s_2 at time T'. Moreover, in between T and T' we are only at s_1 and r_1 in P_1, which have distance larger than 1 to t_2^*. Thus, we cannot pass t_2^*, and we cannot be at t_2 at time T'. Hence, we are at r_2, say at the copy of r_2 in assignment gadget $AG(a_2)$ for some $a_2 \in A_2$. The yet untraversed remainder of P_k is of the form $AG(a_k) \circ \pi_k$ for $k \in \{1,2\}$. Since our traversal of (P_1, P_2) has width 1, we obtain $d_{\mathrm{dF}}(AG(a_1) \circ \pi_1, AG(a_2) \circ \pi_2) \leqslant 1$. By Lemma 3.2, (a_1, a_2) forms a satisfying assignment of φ. □

Lemma 3.5. *If $A_1 \times A_2$ contains a satisfying assignment then $d_{\mathrm{dF}}(P_1, P_2) \leqslant 1$.*

Proof. Let $(a_1, a_2) \in A_1 \times A_2$ be a satisfying assignment of φ. We describe a traversal through P_1, P_2 with width 1. We start at $s_2 \in P_2$ and the first point of P_1. We stay at s_2 and follow P_1 until we arrive at the copy of s_1 that comes right before $AG(a_1)$ (note that s_2 has distance 1 to any point in P_1). Then we stay at s_1 and follow P_2 until we arrive at the copy of r_2 in $AG(a_2)$ (note that the only point that is too far away from s_1 is t_2^*, but this point comes after all assignment gadgets in P_2). In the next step we go to positions (r_1, r_2) (in $AG(a_1), AG(a_2)$). Then we follow the clause gadgets $(CG(a_1, i), CG(a_2, i))$ in parallel, always staying within distance 1 by Lemma 3.1. In the next step we stay at $CG(a_2, M)$ and go to t_1 in P_1 (which has distance 1 to any point in P_2 except for s_2^*, which we will never encounter again). We stay at t_1 in P_1 and follow P_2 completely until we arrive at its endpoint t_2. Since t_2 has distance 1 to any point in P_1, we can now stay at t_2 in P_2 and follow P_1 to its end. □

Proof of Theorem 1.1, discrete case Note that we have

$$n = \max\{|P_1|, |P_2|\} = \mathcal{O}(M) \cdot \max\{|A_1|, |A_2|\} = \mathcal{O}(M \cdot 2^{N/2}).$$

Moreover, the instance (P_1, P_2) can be constructed in time $\mathcal{O}(NM2^{N/2})$. Any $(1+\varepsilon)$-approximation can decide whether $d_{\mathrm{dF}}(P_1, P_2) \leqslant 1$ or $d_{\mathrm{dF}}(P_1, P_2) > 1 + \varepsilon$, which by Lemmas 3.4 and 3.5 yields an algorithm that decides whether φ is satisfiable. If such an algorithm runs in time $\mathcal{O}(n^{2-\delta})$ for any small $\delta > 0$, then the resulting CNF-SAT algorithm runs in time $\mathcal{O}(M^2 2^{(1-\delta/2)N})$, contradicting SETH′.

3.2 Continuous case

The construction from the last section also works for the continuous Fréchet distance. However, for unsatisfiable formulas it becomes tedious to argue that continuous traversals are not much better than discrete traversals. For instance, we have to argue that we cannot stay at a fixed point between the clause gadgets $c^0_{1,\mathsf{T}}$ and $c^1_{1,\mathsf{T}}$ while traversing more than one clause gadget in P_2.

To adapt the proof from the last section, we have to reprove Lemmas 3.4 and 3.5. We will make use of the following property. Here, we set $\mathrm{sym}(CG(a_1, i)) := CG(a_2, i)$ and $\mathrm{sym}(r_1) := r_2$ and interpolate linearly between them to obtain a symmetric point in $AG(a_2)$ for every point in $AG(a_1)$ (for any fixed $a_1 \in A_1$, $a_2 \in A_2$). We also set $\mathrm{sym}(\mathrm{sym}(p_1)) := p_1$, to obtain a symmetric point in $AG(a_1)$ for every point in $AG(a_2)$.

Lemma 3.6. *Consider any points p_k in $AG(a_k)$, $k \in \{1, 2\}$, with $\|p_1 - p_2\| \leqslant 1 + \varepsilon$. Then we have $\|p_2 - \mathrm{sym}(p_1)\| \leqslant \frac{1}{9}$ and $\|\mathrm{sym}(p_2) - p_1\| \leqslant \frac{1}{9}$.*

Proof. Let $p_k = (x_k, y_k)$ and note that we have $|y_1 - y_2| \geqslant 1 - 2\varepsilon$. Thus, if $|x_1 - x_2| > \frac{1}{9} - 2\varepsilon$ then we have (recall that $\varepsilon = 1/1000$)

$$\|p_1 - p_2\| > \sqrt{(\tfrac{1}{9} - 2\varepsilon)^2 + (1 - 2\varepsilon)^2} > 1 + \varepsilon,$$

a contradiction. Since $\mathrm{sym}(p_1) = (x_1, y_1')$ with $|y_1' - y_2| \leqslant 2\varepsilon$, we obtain

$$\|p_2 - \mathrm{sym}(p_1)\| \leqslant \sqrt{(\tfrac{1}{9} - 2\varepsilon)^2 + (2\varepsilon)^2} \leqslant \tfrac{1}{9}.$$

and the same bound holds for $\|\mathrm{sym}(p_2) - p_1\|$. □

Lemma 3.7. (Analogue of Lemma 3.4) *If $d_{\mathrm{F}}(P_1, P_2) \leqslant 1 + \varepsilon = 1.001$ then $A_1 \times A_2$ contains a satisfying assignment.*

Proof. In this proof, we say that two points $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$ have *y-distance* D if $|y_1 - y_2| \leqslant D$.

Consider any traversal of (P_1, P_2) with width $1+\varepsilon$. Consider any time step T where we are at position s_2^* in P_2. The only points in P_1 that are within distance $1+\varepsilon$ of s_2^* are within distance $1/20$ and y-distance ε of s_1 (since no point in P_1 has lower y-value than s_1 and $\sqrt{1+(1/20)^2} > 1+\varepsilon$). Say we are near the copy of s_1 that comes right before assignment gadget $AG(a_1)$, $a_1 \in A_1$. Following time step T, we have to start traversing $AG(a_1)$, so consider the first time step T' where we are at the point r_1 in $AG(a_1)$. The only points in P_2 within distance $1+\varepsilon$ of r_1 are near s_2, t_2, or r_2. Note that we already passed s_2^* in P_2 by time T, so we cannot be near s_2 at time T'. Moreover, in between T and T' we are always near s_1 or between s_1 and r_1 in P_1, so we are always above and to the left of $s_1 + (1/20, 0)$, which has distance larger than $1+\varepsilon$ to t_2^*. Thus, we cannot pass t_2^*, and we cannot be near t_2 at time T'. Hence, we are near r_2, more precisely, we are in distance $1/20$ and y-distance ε of r_2 (this is the same situation as for s_1 and s_2^*). After that, the traversal has to further traverse $AG(a_1)$ and/or $AG(a_2)$. Consider the first time step at which we are at $CG(a_1, 1)$ or $CG(a_2, 1)$, say we reach $CG(a_1, 1)$ first. By Lemma 3.6, we are within distance $1/9$ of $CG(a_2, 1)$. Since we were near r_2 at time T', we now passed r_2, and since we did not pass $CG(a_2, 1)$ yet, we are even within distance $1/9$ of $CG(a_2, 1)$ *along the curve* P_2. This proves the induction base of the following inductive claim.

Claim 3.8. *Let T_i be the first step in time at which the traversal is at $CG(a_1, i)$ or $CG(a_2, i)$, $i \in [M]$. At time T_i the traversal is within distance 1/9 of $CG(a_k, i)$ along the curve P_k for both $k \in \{1, 2\}$.*

Proof. Note that at all times T_i (and in between) Lemma 3.6 is applicable, so we clearly are within distance $1/9$ of $CG(a_k, i+1)$ at time T_{i+1} for any $i \in [M]$, $k \in \{1, 2\}$. Since $\|CG(a_k, i) - CG(a_k, i+1)\| \geqslant 1/3$, points within distance $1/9$ of $CG(a_k, i)$ are not within distance $1/9$ of $CG(a_k, i+1)$. Hence, if we are within distance $1/9$ of $CG(a_k, i)$ along P_k for both $k \in \{1, 2\}$ at time T_i, then at time T_{i+1} we passed $CG(a_k, i)$ and did not pass $CG(a_k, i+1)$ yet (by definition of T_{i+1}), so that we are within distance $1/9$ of $CG(a_k, i+1)$ along P_k for both $k \in \{1, 2\}$. ☐

Finally, we show that the above claim implies that (a_1, a_2) is a satisfying assignment. Assume for the sake of contradiction that some clause C_i is not satisfied by both a_1 and a_2. Say at time T_i we are at $CG(a_1, i)$ (if we are at $CG(a_2, i)$ instead, then a symmetric argument works). At the same time we are at some point p in $AG(a_2)$. By the above claim, p is within distance $1/9$ of $CG(a_2, i)$ along P_2. Note that p lies on any of the line segments $c^0_{2,\mathsf{T}} \circ c^1_{2,\mathsf{F}}$, $c^0_{2,\mathsf{F}} \circ c^1_{2,\mathsf{T}}$, $c^0_{2,\mathsf{F}} \circ c^1_{2,\mathsf{F}}$, or $r_2 \circ c^0_{2,\mathsf{F}}$, since $\mathsf{sat}(a_2, C_i) = \mathsf{F}$. In any case, the current distance $\|p - CG(a_1, i)\|$ is at least the distance from the point $c^0_{1,\mathsf{F}}$ to the line through $c^0_{2,\mathsf{F}}$ and $c^1_{2,\mathsf{T}}$. We compute this distance as

$$\frac{\frac{1}{3}(1+2\varepsilon)}{\sqrt{(\frac{1}{3})^2 + (2\varepsilon)^2}} > 1+\varepsilon,$$

which contradicts the traversal having width $1+\varepsilon$. ☐

Lemma 3.9. (Analogue of Lemma 3.5) *If $A_1 \times A_2$ contains a satisfying assignment then* $d_{\mathrm{F}}(P_1, P_2) \leqslant 1$.

Proof. Follows from Lemma 3.5 and the general inequality $d_{\mathrm{F}}(P_1, P_2) \leqslant d_{\mathrm{dF}}(P_1, P_2)$. ☐

3.3 Generalization to imbalanced numbers of vertices

Assume that the input curves P_1, P_2 have different numbers of vertices $n = |P_1|$, $m = |P_2|$ with $n \geqslant m$. We show that there is no $\mathcal{O}((nm)^{1-\delta})$ algorithm for the Fréchet distance for any $\delta > 0$, even for any polynomial restriction of $1 \leqslant m \leqslant n$. More precisely, for any $\delta \leqslant \gamma \leqslant 1-\delta$ we show that there is no $\mathcal{O}((nm)^{1-\delta})$ algorithm for the Fréchet distance restricted to instances with $n^{\gamma-\delta} \leqslant m \leqslant n^{\gamma+\delta}$.

To this end, given a $\mathsf{CNF\text{-}SAT}$ instance φ we partition its variables $x_1, \ldots, x_N$ into $V_1' := \{x_1, \ldots, x_\ell\}$ and $V_2' := \{x_{\ell+1}, \ldots, x_N\}$ and let A_k' be all assignments of V_k', $k \in \{1, 2\}$. Note that $|A_1'| = 2^{|V_1'|} = 2^\ell$ and $|A_2'| = 2^{N-\ell}$. Now we use the same construction as in Section 3.1 but replace V_k by V_k' and A_k by A_k'. Again we obtain that any 1.001-approximation for the Fréchet distance of the constructed curves P_1, P_2 decides satisfiability of φ. Observe that the constructed curves contain a number of points of

$$n = |P_1| = \Theta(M \cdot |A_1'|), \quad m = |P_2| = \Theta(M \cdot |A_2'|).$$

Hence, any 1.001-approximation of the Fréchet distance in time $\mathcal{O}((nm)^{1-\delta})$ for any small $\delta > 0$ yields an algorithm for $\mathsf{CNF\text{-}SAT}$ in time $\mathcal{O}(M^2(2^\ell 2^{N-\ell})^{1-\delta}) = \mathcal{O}(M^2 2^{(1-\delta)N})$, contradicting SETH'.

Finally, we set $\ell := N/(\gamma+1)$ (rounded in any way) so that $|A_1'| = \Theta(2^{N/(\gamma+1)})$ and $|A_2'| = \Theta(2^{N\gamma/(\gamma+1)})$. Using Lemma 2.1 we can assume that $1 \leqslant M \leqslant 2^{\delta N/4}$. Hence, we have

$$\Omega(2^{N/(\gamma+1)}) \leqslant n \leqslant \mathcal{O}(2^{N/(\gamma+1)+\delta N/4}),$$
$$\Omega(2^{N\gamma/(\gamma+1)}) \leqslant m \leqslant \mathcal{O}(2^{N\gamma/(\gamma+1)+\delta N/4}),$$

which implies $\Omega(n^{\gamma-\delta/2}) \leqslant m \leqslant \mathcal{O}(n^{\gamma+\delta/2})$. For sufficiently large n, we obtain the desired polynomial restriction $n^{\gamma-\delta} \leqslant m \leqslant n^{\gamma+\delta}$. This proves Theorem 1.2.

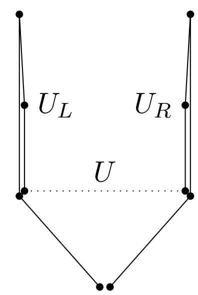
4 Realistic inputs: c-packed curves

4.1 Constant factor approximations

The curves constructed in Section 3.1 are highly packed, since all assignment gadgets lie roughly in the same area. Specifically they are not $o(n)$-packed. In this section we want to construct c-packed instances and show that there is no 1.001-approximation in time $\mathcal{O}((cn)^{1-\delta})$ for any $\delta > 0$ for the Fréchet distance unless SETH' fails, not even restricted to instances with $n^{\gamma-\delta} \leqslant c \leqslant n^{\gamma+\delta}$ for any $\delta \leqslant \gamma \leqslant 1-\delta$. This proves Theorem 1.3.

To this end, we again consider a $\mathsf{CNF\text{-}SAT}$ instance φ, partition its variables $x_1, \ldots, x_N$ into two sets V_1, V_2 of size $N/2$, and consider the set A_k of all assignments of T and F to the variables in V_k. Now we partition A_k into sets $A_k^1, \ldots, A_k^\ell$ of size $\Theta(2^{N/2}/\ell)$, where we fix $1 \leqslant \ell \leqslant 2^{N/2}$ later. Formula φ is satisfiable if and only if for some pair $(j_1, j_2) \in [\ell]^2$ the set $A_1^{j_1} \times A_2^{j_2}$ contains a satisfying assignment. This suggests to use the construction of Section 3.1 after replacing A_1 by $A_1^{j_1}$ and A_2 by $A_2^{j_2}$, yielding a pair of curves $(P_1^{j_1 j_2}, P_2^{j_1 j_2})$. Now, φ is satisfiable if and only if $d_{\mathrm{F}}(P_1^{j_1 j_2}, P_2^{j_1 j_2}) \leqslant 1$ for some $(j_1, j_2) \in [\ell]^2$. For the sake of readability, we rename the constructed curves slightly so that we have curves (P_1^j, P_2^j) for $j \in [\ell^2]$.

OR-gadget In the whole section we let $\rho := 1/\sqrt{2}$. We present an OR-construction over the gadgets (P_1^j, P_2^j) that is not too packed, in contrast to the OR-construction over assignment gadgets that we used in Section 3.1. We start with two building blocks, where for any $j \in \mathbb{N}$ we set

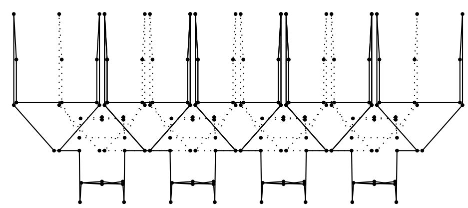


$$U_L(j) := (j\rho, 0) \circ ((j-1)\rho, \rho) \circ ((j-1)\rho, 3\rho) \circ ((j-1)\rho, 2\rho) \circ ((j-1)\rho, \rho),$$
$$U_R(j) := ((j+1)\rho, \rho) \circ ((j+1)\rho, 2\rho) \circ ((j+1)\rho, 3\rho) \circ ((j+1)\rho, \rho) \circ (j\rho, 0).$$

Moreover, we set $U(j) := U_L(j) \circ U_R(j)$. For a curve π and $z \in \mathbb{R}$ we let $\mathrm{tr}_z(\pi)$ be the curve π translated by z in x-direction. The OR-gadget now consists of the following two curves,

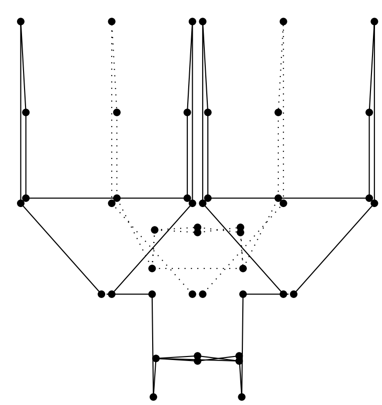
$$R_1 := \bigcirc_{j=1}^{\ell^2} \big(U_L(2j) \circ \mathrm{tr}_{2j\rho}(P_1^j) \circ U_R(2j)\big),$$
$$R_2 := U(1) \circ \bigcirc_{j=1}^{\ell^2} \big(\mathrm{tr}_{2j\rho}(P_2^j) \circ U(2j+1)\big).$$

The figure to the right shows these curves for $\ell^2 = 4$, see below for a figure showing $\ell^2 = 1$ with more details visible.



We denote by R_1^j the j-th "summand" of R_1, i.e., $R_1^j = U_L(2j) \circ \mathrm{tr}_{2j\rho}(P_1^j) \circ U_R(2j)$. Informally, we will use the term *U-shape* for the subcurves R_1^j and $U(2j+1)$, since they resemble the letter U. Moreover, we consider "summands" of R_2, namely $R_2^j := U(2j-1) \circ \mathrm{tr}_{2j\rho}(P_2^j) \circ ((2j+1)\rho, 0)$ and $\tilde{R}_2^j := ((2j-1)\rho, 0) \circ \mathrm{tr}_{2j\rho}(P_2^j) \circ U(2j+1)$.

Analysis In order to be able to replace the curves P_1^j, P_2^j constructed above by other curves in the next section, we analyse the OR-gadget in a rather general way. To this end, we first specify a set of properties and show that the curves P_1^j, P_2^j constructed above satisfy these properties. Then we analyse the OR-gadget using only these properties of P_1^j, P_2^j.



Property 4.1. *(i) If φ is satisfiable then for some $j \in [\ell^2]$ we have $d_{\mathrm{dF}}(P_1^j, P_2^j) \leqslant 1$.*

(ii) If φ is not satisfiable then for all $j \in [\ell^2]$ and curves $\sigma_1, \sigma_2, \pi_1, \pi_2$ such that σ_1 stays to the left and above $(-\rho, \rho)$ and π_1 stays to the right and above (ρ, ρ), we have $d_{\mathrm{F}}(\sigma_1 \circ P_1^j \circ \pi_1, \sigma_2 \circ P_2^j \circ \pi_2) > \beta$, for some $\beta > 1$.

(iii) P_k^j is $\Theta(c)$-packed for some $c \geqslant 1$ for all $j \in [\ell^2]$, $k \in \{1, 2\}$.

(iv) $(0, \rho)$ is within distance 1 of any point in P_1^j for all $j \in [\ell^2]$.

(v) $(0, 0)$ is within distance 1 of any point in P_2^j for all $j \in [\ell^2]$.

Lemma 4.2. *The curves (P_1^j, P_2^j) constructed above satisfy Property 4.1 with $\beta = 1.001$ and $c = \Theta(M \cdot 2^{N/2}/\ell)$. Moreover, we have $|P_k^j| = \Theta(M \cdot 2^{N/2}/\ell)$ for all $j \in [\ell^2]$, $k \in \{1, 2\}$.*

Proof. Property 4.1.(i) follows from Lemma 3.5, since at least one pair $(A_1^{j_1}, A_2^{j_2})$ contains a satisfying assignment. Properties (iv) and (v) can be verified by considering all points in the construction in Section 3.1.

Observe that $|P_k^j| = \Theta(M \cdot 2^{N/2}/\ell)$, since P_k^j consists of $|A_k^j| = \Theta(2^{N/2}/\ell)$ assignment gadgets of size $\Theta(M)$. The upper bound of (iii) follows since any polygonal curve with at most m segments is m-packed. The lower bound of (iii) follows from P_k^j being contained in a ball of radius 1 (by (iv) and (v)) and every segment of P_k^j having constant length.

For (ii), note that from any traversal of $(\sigma_1 \circ P_1^j \circ \pi_1, \sigma_2 \circ P_2^j \circ \pi_2)$ with width 1.001 one can extract a traversal of (P_1^j, P_2^j) with width 1.001, by mapping any point in σ_k to the starting point s_k of P_k^j and any point in π_k to the endpoint t_k of P_k^j, $k \in \{1,2\}$. This does not increase the width, since (1) s_2 and t_2 are within distance 1 to all points in P_1^j, and (2) s_1 has smaller distance to any point in P_2^j than any point in σ_1 has, since σ_1 stays above and to the left of s_1 while all points of P_1^j lie below and to the right of s_1. A similar statement holds for t_1 and π_1. Property (ii) now follows from Lemma 3.7. ☐

In the following lemma we analyse the OR-gadget.

Lemma 4.3. *For any curves (P_1^j, P_2^j) that satisfy Property 4.1, the OR-gadget (R_1, R_2) satisfies:*

(i) $|R_k| = \Theta\big(\sum_{j=1}^{\ell^2} |P_k^j|\big)$ *for* $k \in \{1,2\}$.

(ii) R_1 *and* R_2 *are* $\Theta(c)$*-packed,*

(iii) If φ *is satisfiable then* $d_{\mathrm{F}}(R_1, R_2) \leqslant d_{\mathrm{dF}}(R_1, R_2) \leqslant 1$,

(iv) If φ *is not satisfiable then* $d_{\mathrm{dF}}(R_1, R_2) \geqslant d_{\mathrm{F}}(R_1, R_2) > \min\{\beta, 1.2\}$.

Proof. (i) Precisely, we have $|R_k| = \sum_{j=1}^{\ell^2}(|P_k^j| + 10) + 10(k-1)$ for $k \in \{1,2\}$.

(ii) Let $k \in \{1,2\}$ and consider any ball $B = B(q,r)$. If $r \leqslant 1$ then B hits $\mathcal{O}(1)$ of the curves P_k^j. Since these curves are c-packed, their contribution to the total length of R_k in B is at most $\mathcal{O}(cr)$. Moreover, B hits $\mathcal{O}(1)$ segments of U or U_L, U_R, and the connecting segments to P_k^j. Each of these segments has length at most $2r$ inside B. This yields a total length of R_k in B of $\mathcal{O}((c+1)r)$.

Similarly, if $r > 1$ then B hits $\mathcal{O}(r)$ of the curves P_k^j. Note that the total length of P_k^j is at most c, since the curve is c-packed and contained in a ball of radius 1 around $(0,0)$ or $(0,\rho)$ by Property 4.1. Hence, the total length of of the curves P_k^j in B is $\mathcal{O}(cr)$. Moreover, B hits $\mathcal{O}(r)$ segments of U, U_L, U_R, and the connectors to P_k^j, each of constant length. This yields a total length of R_k in B of $\mathcal{O}((c+1)r)$.

In total, the curve R_k is $\mathcal{O}(c+1)$-packed. As $c \geqslant 1$, it is also $\mathcal{O}(c)$-packed. Since for some $\alpha > 0$ the curve P_k^j is not αc-packed, also R_k is not αc-packed, so R_k is even $\Theta(c)$-packed.

(iii) Note that $d_{\mathrm{F}}(R_1, R_2) \leqslant d_{\mathrm{dF}}(R_1, R_2)$ holds in general, so we only have to show that if φ is satisfiable then $d_{\mathrm{dF}}(R_1, R_2) \leqslant 1$. First we show that we can traverse one U-shape in R_1 and one neighboring U-shape in R_2 together.

Claim 4.4. *For any* $j \in [\ell^2]$, *we have* $d_{\mathrm{dF}}(R_1^j, U(2j-1)) \leqslant 1$ *and* $d_{\mathrm{dF}}(R_1^j, U(2j+1)) \leqslant 1$.

Proof. We only show the first inequality, the second is similar. We start by traversing $U_L(2j)$ and the left half of $U(2j-1)$ in parallel, being at the i-th point of $U_L(2j)$ and $U(2j-1)$ at the same time. At any point in time we are within distance ρ. Now we step to $(2j\rho, \rho)$ in $U(2j-1)$. We stay there while traversing $\mathrm{tr}_{2j\rho}(P_1^j)$ in R_1^j, staying within distance 1 by Property 4.1.(iv). Finally, we traverse $U_R(2j)$ and the second half of $U(2j-1)$ in parallel, where again the largest encountered distance is ρ. ☐

We can stitch these traversals together so that we traverse any number j of neighboring U-shapes in both curves together, because the parts in between the U-shapes are near to a single point, as shown by the following claim. Note that $(2j\rho, 0) \circ ((2j+2)\rho, 0)$ is the connecting segment in R_1 between $U_R(2j)$ and $U_L(2j+2)$, while $((2j-1)\rho, 0) \circ \mathrm{tr}_{2j\rho}(P_2^j) \circ ((2j+1)\rho, 0)$ is the part in R_2 between $U(2j-1)$ and $U(2j+1)$.

Claim 4.5. *For any $j \in [\ell^2]$,*

$$d_{\mathrm{dF}}((2j\rho, 0) \circ ((2j+2)\rho, 0), ((2j+1)\rho, 0)) \leqslant 1,$$
$$d_{\mathrm{dF}}((2j\rho, 0), ((2j-1)\rho, 0) \circ \mathrm{tr}_{2j\rho}(P_2^j) \circ ((2j+1)\rho, 0)) \leqslant 1.$$

Proof. The first claim is immediate. The second follows from Property 4.1.(v). □

Thus, we can stitch together traversals of U-shapes in both curves. However, so far we can only traverse the same number of U-shapes in both curves, but R_2 has one more U-shape than R_1. Consider $J \in [\ell^2]$ with $d_{\mathrm{dF}}(P_1^J, P_2^J) \leqslant 1$, which exists since φ is satisfiable, see Property 4.1.(i). Consider the two subcurves (also see the above figure)

$$R_1' := R_1^J = U_L(2J) \circ \mathrm{tr}_{2J\rho}(P_1^J) \circ U_R(2J),$$
$$R_2' := U(2J-1) \circ \mathrm{tr}_{2J\rho}(P_2^J) \circ U(2J+1).$$

We show that $d_{\mathrm{dF}}(R_1', R_2') \leqslant 1$, i.e., we can traverse two U-shapes in R_2 while traversing only one U-shape in R_1, using $d_{\mathrm{dF}}(P_1^J, P_2^J) \leqslant 1$. Adding simple traversals of U-shapes before and after (R_1', R_2'), we obtain a traversal of (R_1, R_2) with width 1, proving $d_{\mathrm{dF}}(R_1, R_2) \leqslant 1$. It is left to show the following claim.

Claim 4.6. *$d_{\mathrm{dF}}(R_1', R_2') \leqslant 1$.*

Proof. We traverse $U_L(2J)$ and $U(2J-1)$ in parallel until we are at point $((2J-1)\rho, 2\rho)$ in $U_L(2J)$. We stay in this point and follow $U(2J-1)$ until its second-to-last point. In the next step we can finish traversing $U_L(2J)$ and $U(2J-1)$. In the next step we go to the first positions of (the translated) P_1^J and P_2^J. We follow any traversal of (P_1^J, P_2^J) with width 1. Finally, we use a traversal symmetric to the one of $(U_L(2J), U(2J-1))$ to traverse $(U_R(2J), U(2J+1))$. □

(iv) Note that the inequality $d_{\mathrm{dF}}(R_1, R_2) \geqslant d_{\mathrm{F}}(R_1, R_2)$ holds in general, so we only have to show that if φ is not satisfiable then $d_{\mathrm{F}}(R_1, R_2) > \min\{\beta, 1.2\}$. Assume for the sake of contradiction that there is a traversal of (R_1, R_2) with width $\min\{\beta, 1.2\}$. Essentially we show that it cannot traverse 2 U-shapes in R_2 while traversing only one U-shape in R_1, which implies a contradiction since the number of U-shapes in R_2 is larger than in R_1.

Let Y_ρ be the line $\{(x, y) \in \mathbb{R}^2 \mid y = \rho\}$. We inductively prove the following claims.

Claim 4.7. *(i) For any $0 \leqslant j \leqslant \ell^2$, when the traversal is in R_2 at the left highest point $(2j\rho, 3\rho)$ of $U(2j+1)$, then in R_1 we fully traversed R_1^j and are above the line Y_ρ.*

(ii) For any $1 \leqslant j \leqslant \ell^2$, when the traversal is in R_1 at the right highest point $((2j+1)\rho, 3\rho)$ of R_1^j, then in R_2 it is in $U(2j-1)$.

Note that claim (i) for $j = \ell^2$ yields the desired contradiction, since after traversing $R_1^{\ell^2}$ the curve R_1 has ended (at the point $(2\ell^2\rho, 0)$), so that we cannot go above the line Y_ρ anymore.

Proof. (i) Note that we have to be above the line Y_ρ because all points below Y_ρ have distance at least $2\rho > 1.2$ to the point $(2j\rho, 3\rho)$. For $j = 0$, claim (i) holds immediately, since there is no subcurve R_1^0 (so this part of the statement disappears). In general, claim (i) for any $1 \leqslant j \leqslant \ell^2$ follows from claim (ii) for j: When we are at $z_1 := ((2j+1)\rho, 3\rho)$ in R_1^j, we are still in $U(2j-1)$. Once we reach the endpoint $z_2 := ((2j-1)\rho, 0)$ of $U(2j-1)$, in R_1 we are at a point p_1 below the line Y_ρ, since all points in R_2 that follow z_1 and lie above Y_ρ have distance more than $2\rho > 1.2$ to z_2. Now we follow R_2 until we reach $p_2 := (2j\rho, 3\rho)$ in $U(2j+1)$. At this point we have to be above the line Y_ρ in R_1, but all points in R_1^j following p_1 lie below Y_ρ. Thus, at this point we have fully traversed R_1^j (and have to be in R_1^{j+1}).

(ii) This claim for any $1 \leqslant j \leqslant \ell^2$ follows from claim (i) for $j-1$. Assume for the sake of contradiction that claim (ii) for some j does not hold. Consider the subcurve R_1' of R_1^j between (the first occurrence of) $((2j-1)\rho, \rho)$ and $((2j+1)\rho, 3\rho)$. Let R_2' be the subcurve of R_2 that the traversal traverses together with R_1'. Since (R_1', R_2') forms a subtraversal of the traversal of (R_1, R_2), which has width $\min\{\beta, 1.2\}$, we have $d_{\mathrm{F}}(R_1', R_2') \leqslant \min\{\beta, 1.2\}$ (*). By claim (i) for $j-1$, the starting point of R_2' lies before $\mathrm{tr}_{2j\rho}(P_2^j)$ along R_2, since we reach $((2j-2)\rho, 3\rho)$ in $U(2j-1)$ only after being in the starting point of R_1'. Moreover, the endpoint of R_2' lies after $\mathrm{tr}_{2j\rho}(P_2^j)$ along R_2. Indeed, while being at the endpoint $((2j+1)\rho, 3\rho)$ of R_1', we cannot be in $U(2j-1)$ since we assumed that claim (ii) is wrong for j. We can also not be in $\mathrm{tr}_{2j\rho}(P_2^j)$, since by Property 4.1.5 all points in this curve lie in a ball of radius 1 around $(2j\rho, 0)$, so their distance to $((2j+1)\rho, 3\rho)$ is at least $\|((2j+1)\rho, 3\rho) - (2j\rho, 0)\| - 1 = \sqrt{5} - 1 > 1.2$. Hence, we already passed $\mathrm{tr}_{2j\rho}(P_2^j)$, and R_2' is of the form $\sigma_2 \circ \mathrm{tr}_{2j\rho}(P_2^j) \circ \pi_2$ for any curves σ_2, π_2. Note that R_1' is of the form $\sigma_1 \circ \mathrm{tr}_{2j\rho}(P_1^j) \circ \pi_1$ with σ_1 staying above and to the left of $((2j-1)\rho, \rho)$ and π_1 staying above and to the right of $((2j+1)\rho, \rho)$. Thus, after translation Property 4.1.(ii) applies, proving $d_{\mathrm{F}}(R_1', R_2') > \beta$, a contradiction to (*). □

□

Proof of Theorem 1.3 Finally, we use the OR-gadget (Lemma 4.3) together with the curves P_1^j, P_2^j we obtained from Section 3.1 (Lemma 4.2) to prove a runtime bound for c-packed curves: Any 1.001-approximation for the (discrete or continuous) Fréchet distance of (R_1, R_2) decides satisfiability of φ. Note that R_1 and R_2 are c-packed with

$$c = \Theta(M \cdot 2^{N/2}/\ell), \quad n = \max\{|R_1|, |R_2|\} = \Theta(\ell^2 M \cdot 2^{N/2}/\ell).$$

Thus, any $\mathcal{O}((cn)^{1-\delta})$ algorithm for the Fréchet distance implies a $\mathcal{O}(M^2 2^{(1-\delta)N})$ algorithm for $\mathsf{CNF\text{-}SAT}$, contradicting SETH'. Moreover, using Lemma 2.1 we can assume that $1 \leqslant M \leqslant 2^{\delta N/4}$. Setting $\ell := \Theta(2^{\frac{1-\gamma}{1+\gamma} N/2})$ for any $0 \leqslant \gamma \leqslant 1$ we obtain

$$\Omega(2^{\frac{2}{1+\gamma} N/2}) \leqslant n \leqslant \mathcal{O}(2^{(\frac{2}{1+\gamma} + \delta/2) N/2}),$$
$$\Omega(2^{\frac{2\gamma}{1+\gamma} N/2}) \leqslant c \leqslant \mathcal{O}(2^{(\frac{2\gamma}{1+\gamma} + \delta/2) N/2}).$$

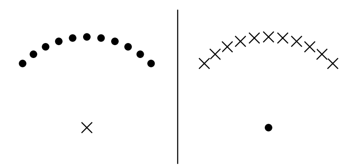
From this it follows that $\Omega(n^{\gamma - \delta/2}) \leqslant c \leqslant \mathcal{O}(n^{\gamma + \delta/2})$, which implies the desired polynomial restriction $n^{\gamma - \delta} \leqslant c \leqslant n^{\gamma + \delta}$ for sufficiently large n.

4.2 Approximation schemes

In this section, we consider the dependence on ε of the runtime of a $(1+\varepsilon)$-approximation for the Fréchet distance on c-packed curves. We show that in $\mathbb{R}^d$ with $d \geqslant 5$ there is no such algorithm in time $\mathcal{O}(\min\{cn/\sqrt{\varepsilon}, n^2\}^{1-\delta})$ for any $\delta > 0$ unless SETH' fails (Theorem 1.4). This matches the dependence on ε of the fastest known algorithm up to a polynomial. The result holds for sufficiently small $\varepsilon > 0$ and any polynomial restriction of $1 \leqslant c \leqslant n$ and $\varepsilon \leqslant 1$.

We will reuse the OR-gadget from the last section, embedded into the first two dimensions of $\mathbb{R}^5$. Specifically, we will reuse Lemma 4.3. However, we adapt the curves P_1^j, P_2^j, essentially by embedding the same set of points in a different way.

In this new embedding we make use of the fact that in $\mathbb{R}^4$ there are point sets Z_1, Z_2 of arbitrary size such that any pair of points $(z_1, z_2) \in Z_1 \times Z_2$ has distance 1. For an example, see the figure to the right, where the left picture shows the projection onto the first two dimensions and the right picture shows the projection onto the last two dimensions. Here, Z_1 (circles) is placed along a quarter-circle in the $(1,2)$-plane and Z_2 (crosses) is placed along a quarter-circle in the $(3,4)$-plane.



Construction As usual, consider a $\mathsf{CNF\text{-}SAT}$ instance φ, partition its variables $x_1, \dots, x_N$ into two sets V_1, V_2 of size $N/2$, and consider any set A_k of assignments of T and F to the variables in V_k. Fix any enumeration $\{a_k^1, \dots, a_k^{|A_k|}\}$ of A_k. Again set $\rho := 1/\sqrt{2}$. For $h \in [|A_k|]$ and $i \in \{0, \dots, M+1\}$ let

$$\mathrm{rot}(a_1^h, i) := \Big(\rho \sin\big(\tfrac{\pi}{4} + \tfrac{\pi}{2}\tfrac{h(M+2)+i}{|A_1|\cdot(M+2)}\big), \rho\cos\big(\tfrac{\pi}{4} + \tfrac{\pi}{2}\tfrac{h(M+2)+i}{|A_1|\cdot(M+2)}\big), 0, 0, 0\Big),$$
$$\mathrm{rot}(a_2^h, i) := \Big(0, 0, \rho \sin\big(\tfrac{\pi}{4} + \tfrac{\pi}{2}\tfrac{h(M+2)+i}{|A_2|\cdot(M+2)}\big), \rho\cos\big(\tfrac{\pi}{4} + \tfrac{\pi}{2}\tfrac{h(M+2)+i}{|A_2|\cdot(M+2)}\big), 0\Big).$$

Note that these points are placed along a quarter-circle in the $(1,2)$-plane or $(3,4)$-plane, respectively, as in the above figure. In particular, $\|\mathrm{rot}(a_1^h, i) - \mathrm{rot}(a_2^{h'}, i')\| = 1$ for all h, h', i, i'. Moreover, let e_5 be the vector $(0,0,0,0,\rho)$. For $a_k \in A_k$ and $i \in [M]$ we set

$$CG(a_k, i) := \begin{cases} (1-2\varepsilon)\,\mathrm{rot}(a_k, i) + (i \bmod 2)\cdot 8\sqrt{\varepsilon}\, e_5, & \text{if } \mathsf{sat}(a_k, C_i) = \mathsf{T} \\ (1+\varepsilon)\,\mathrm{rot}(a_k, i) + (i \bmod 2)\cdot 8\sqrt{\varepsilon}\, e_5, & \text{if } \mathsf{sat}(a_k, C_i) = \mathsf{F} \end{cases}$$

Thus, we align the clause gadgets of A_1 roughly along a quarter-circle in the $(1,2)$-plane, and similarly the clause gadgets of A_2 roughly along a quarter-circle in the $(3,4)$-plane. Moreover, for $a_k \in A_k$ we set

$$\begin{aligned} r_k(a_k) &:= \mathrm{rot}(a_k, 0) - 8\sqrt{\varepsilon}\, e_5, \\ s_1(a_1) &:= (1-400\varepsilon)\,\mathrm{rot}(a_1, 0) + 10\sqrt{\varepsilon}\, e_5, \\ t_1(a_1) &:= (1-400\varepsilon)\,\mathrm{rot}(a_1, M+1) - 10\sqrt{\varepsilon}\, e_5, \\ s_2 = t_2 &:= (0,0,0,0,0), \\ s_2^* &:= (1+9\sqrt{\varepsilon})e_5, \quad t_2^* := -(1+9\sqrt{\varepsilon})e_5. \end{aligned}$$

We define assignment gadgets and the curves P_1, P_2 as in Section 3.1, i.e.,

$$\begin{aligned} AG(a_k) &:= r_k(a_k) \circ \bigcirc_{i\in[M]} CG(a_k, i), \\ P_1 &:= \bigcirc_{a_1 \in A_1} \big(s_1(a_1) \circ AG(a_1) \circ t_1(a_1)\big), \\ P_2 &:= s_2 \circ s_2^* \circ \Big(\bigcirc_{a_2 \in A_2} AG(a_2)\Big) \circ t_2^* \circ t_2. \end{aligned}$$

Analysis Again, we split the considered points into Q_1, Q_2, depending on whether they may appear on P_1 or P_2, i.e., $Q_1 := \{s_1(a_1), t_1(a_1), r_1(a_1), CG(a_1, i) \mid a_1 \in A_1, i \in [M]\}$ and $Q_2 := \{s_2, t_2, s_2^*, t_2^*, r_2(a_2), CG(a_2, i) \mid a_2 \in A_2, i \in [M]\}$. It is easy, but tedious to verify that the constructed points behave as follows.

Lemma 4.8. *The following pairs of points have distance at most 1 for any $a_k \in A_k$:*

$$\begin{aligned}
&(q, s_2), (q, t_2) \text{ for any } q \in Q_1,\\
&(s_1(a_1), q) \text{ for any } q \in Q_2 \setminus \{t_2^*\},\\
&(t_1(a_1), q) \text{ for any } q \in Q_2 \setminus \{s_2^*\},\\
&(r_1(a_1), r_2(a_2)),\\
&(CG(a_1, i), CG(a_2, i)) \text{ if assignment } (a_1, a_2) \text{ satisfies clause } C_i.
\end{aligned}$$

Moreover, the following pairs of points have distance more than $1 + \varepsilon$ for any $a_k \in A_k$:

$$\begin{aligned}
&(q, s_2^*) \text{ for any } q \in Q_1 \setminus \{s_1\},\\
&(q, t_2^*) \text{ for any } q \in Q_1 \setminus \{t_1\},\\
&(r_1(a_1), CG(a_2, i)) \text{ for any } i \in [M],\\
&(CG(a_1, i), r_2(a_2)) \text{ for any } i \in [M],\\
&(CG(a_1, i), CG(a_2, j)) \text{ for any } i, j \in [M], i \not\equiv j \bmod 2,\\
&(CG(a_1, i), CG(a_2, i)) \text{ if assignment } (a_1, a_2) \text{ does not satisfy clause } C_i.
\end{aligned}$$

Proof. Using that ε is sufficiently small, we only have to compute the largest order term of ε for all distances. E.g., for all $a_k \in A_k$

$$\|s_1(a_1) - r_2(a_2)\| = \sqrt{\rho^2((1 - 400\varepsilon)^2 + 1 + (18\sqrt{\varepsilon})^2)} = \sqrt{1 - 476\varepsilon + \mathcal{O}(\varepsilon^2)} \leqslant 1.$$

□

Now we use these curves in the OR-gadget from the last section. To this end, again partition the set of all assignments of V_k into sets $A_k^1, \dots, A_k^\ell$ of size $\Theta(2^{N/2}/\ell)$, where we fix $1 \leqslant \ell \leqslant 2^{N/2}$ later. Use the above construction of P_1, P_2 after replacing A_1 by $A_1^{j_1}$ and A_2 by $A_2^{j_2}$ for any $j_1, j_2 \in [\ell]$ to obtain curves $P_1^{j_1 j_2}, P_2^{j_1 j_2}$. Slightly rename these curves so that we have curves (P_1^j, P_2^j) for $j \in [\ell^2]$. Then these curves satisfy Property 4.1.

Lemma 4.9. *The curves P_1^j, P_2^j satisfy Property 4.1 with $c = \Theta(1 + \sqrt{\varepsilon} M|A_k|)$ and $\beta = 1 + \varepsilon$. Moreover, $|P_k^j| = \Theta(M 2^{N/2}/\ell)$ for any $j \in [\ell^2]$, $k \in \{1, 2\}$.*

Proof. Using Lemma 4.8, we can follow the proof in Section 3.1, since everything that we used about P_1, P_2 is captured by this lemma. This proves that if φ is satisfiable then $d_{\mathrm{dF}}(P_1^j, P_2^j) \leqslant 1$ for some $j \in [\ell^2]$, and if φ is not satisfiable then $d_{\mathrm{dF}}(P_1^j, P_2^j) > 1 + \varepsilon$ for all $j \in [\ell^2]$, i.e., Property 4.1.(i) and (ii) in the discrete case. The same adaptations as in Section 3.2 allow to prove correctness in the continuous case, we omit the details.

It is easy to see that all constructed points lie within distance 1 of $(0, 0, 0, 0, 0)$, showing (iv). For (v) we use that we placed the points along the upper quarter-circle, and not the full circle. This way, all points in P_1^j have a distance to $(0, \rho, 0, 0, 0)$ of at most $\|(0, \rho) - (\frac{1}{2}, \frac{1}{2})\| + \mathcal{O}(\sqrt{\varepsilon}) < 1$, for sufficiently small ε.

For (iii) observe that all segments of P_k^j (except for the finitely many segments incident to s_2^*, t_2^*) have length $\Theta(\sqrt{\varepsilon} + 1/(M|A_k^j|))$, $k \in \{1, 2\}$. Moreover, the $\Theta(M|A_k^j|)$ segments of P_k^j are spread along a quarter-circle. Hence, any ball $B(q,r)$ intersects $\mathcal{O}(1+\min\{1,r\}M|A_k^j|)$ segments of P_k^j. Since each of these segments has length $\mathcal{O}(\min\{r, \sqrt{\varepsilon}+1/(M|A_k^j|)\})$ in $B(q,r)$, the total length of P_k^j in $B(q,r)$ is $\mathcal{O}(r(1+\sqrt{\varepsilon}M|A_k^j|))$. Thus, P_k^j is $\mathcal{O}(1+\sqrt{\varepsilon}M|A_k^j|)$-packed. It is also $\Theta(1+\sqrt{\varepsilon}M|A_k^j|)$-packed, since all $\Theta(M|A_k^j|)$ segments of length $\Theta(\sqrt{\varepsilon}+1/(M|A_k^j|))$ lie in a ball of radius 1 around $(0,0,0,0,0)$ or $(0,\rho,0,0,0)$ by (iv) and (v). Finally, note that $|A_k^j| = 2^{N/2}/\ell$. $\square$

Proof of Theorem 1.4 The above Lemma 4.9 allows to apply Lemma 4.3, which constructs curves R_1, R_2 such that any $(1+\varepsilon)$-approximation for the Fréchet distance of (R_1, R_2) decides satisfiability of φ. Since R_1 and R_2 are c-packed with

$$c = \Theta(1+\sqrt{\varepsilon}M2^{N/2}/\ell), \quad n = \max\{|R_1|, |R_2|\} = \Theta(\ell M 2^{N/2}),$$

we obtain that any $(1+\varepsilon)$-approximation for the Fréchet distance in time $\mathcal{O}((cn/\sqrt{\varepsilon})^{1-\delta})$ yields an algorithm for CNF-SAT in time $\mathcal{O}(M^2 2^{(1-\delta)N})$, as long as $\ell = \mathcal{O}(\sqrt{\varepsilon}M2^{N/2})$. This contradicts SETH′.

Moreover, using Lemma 2.1 we can assume that $1 \leqslant M \leqslant 2^{\delta N/4}$. Setting $\ell := \Theta(\varepsilon^{\frac{1}{2(1+\gamma)}} 2^{\frac{1-\gamma}{1+\gamma}N/2})$ for any $0 \leqslant \gamma \leqslant 1$, we obtain

$$\varepsilon^{\frac{1}{2(1+\gamma)}} 2^{\frac{2}{1+\gamma}N/2} \leqslant n \leqslant \varepsilon^{\frac{1}{2(1+\gamma)}} 2^{(\frac{2}{1+\gamma}+\delta/2)N/2},$$
$$\varepsilon^{\frac{\gamma}{2(1+\gamma)}} 2^{\frac{2\gamma}{1+\gamma}N/2} \leqslant c \leqslant \varepsilon^{\frac{\gamma}{2(1+\gamma)}} 2^{(\frac{2\gamma}{1+\gamma}+\delta/2)N/2}.$$

From this it follows that $\Omega(n^{\gamma-\delta/2}) \leqslant c \leqslant \mathcal{O}(n^{\gamma+\delta})$, which implies the desired polynomial restriction $n^{\gamma-\delta} \leqslant c \leqslant n^{\gamma+\delta}$ for sufficiently large n. Note that this works as long as

$$1 \leqslant \ell \leqslant \mathcal{O}(\sqrt{\varepsilon}M2^{N/2}).$$

Since $\ell = \Theta\big((\sqrt{\varepsilon}n/c)^{1/2}\big)$, the first inequality is equivalent to $cn/\sqrt{\varepsilon} \leqslant n^2$, which is a natural condition, since otherwise the exact algorithm for general curves is faster. Plugging in the definition of $\ell = \Theta(\varepsilon^{\frac{1}{2(1+\gamma)}} 2^{\frac{1-\gamma}{1+\gamma}N/2})$, the second inequality becomes $1/\varepsilon \leqslant \big(2^N M^{(1+\gamma)/\gamma}\big)^2$. Since $(1+\gamma)/\gamma \geqslant 2$, $n = \mathcal{O}(\ell M 2^{N/2}) \leqslant \mathcal{O}(M^2 2^N)$, and $c \geqslant 1$, this is implied by the first condition $cn/\sqrt{\varepsilon} \leqslant n^2$. Hence, we may choose any sufficiently small $\varepsilon = \varepsilon(n)$ with $cn/\sqrt{\varepsilon} \leqslant n^2$.

5 Conclusion

We presented strong evidence that the (continuous or discrete) Fréchet distance has no strongly subquadratic algorithms, by relating this problem to the Strong Exponential Time Hypothesis.

Our extensions of this main result include approximation algorithms and realistic input curves (c-packed curves). These extensions leave three particularly interesting open questions, asking for new algorithms or improved lower bounds. Here, we use $\tilde{\mathcal{O}}$ to ignore any polylogarithmic factors in n, c, and $1/\varepsilon$.

1. Is there a strongly subquadratic $\mathcal{O}(1)$-approximation for the Fréchet distance on general curves?

2. In any dimension $d \in \{2,3,4\}$, is there a $(1+\varepsilon)$-approximation in time $\tilde{\mathcal{O}}(cn)$ for the Fréchet distance on c-packed curves? Or is there even an exact algorithm in time $\tilde{\mathcal{O}}(cn)$?

3. In any dimension $d \geqslant 5$, is there a $(1+\varepsilon)$-approximation in time $\tilde{\mathcal{O}}(cn/\sqrt{\varepsilon})$ for the Fréchet distance on c-packed curves?

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