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Basic Facts about Electrical Networks

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Kirchoff's Laws

- Let $G = (V, E)$ be an undirected graph; fix an orientation of each edge.
- $R_e =$ resistance of edge $e = (u, v)$
- $b_u =$ external current provided (extracted) at u; $\sum_u b_u = 0$.
- $Q_e =$ current through e in the direction from u to v (might be negative)
- Current Law: $\sum_{e=(u,v)} Q_e - \sum_{e=(v,u)} Q_e = b_u$ for every u
- Ohm's Law: If R_e is the resistance of $e = (u, v)$, then

  $$Q_e = \Delta_e / R_e$$

  where Δ_e is the potential difference between u and v.
- Voltage Law: Potential differences sum to zero around any cycle and hence can assign potentials p_u to the vertices

Superposition Principle

Additivity of Solutions

- Assume $b = b^{(1)} + b^{(2)}$ and $b^{(i)}$ legal
- Let $Q^{(i)}$ be an electrical flow for $b^{(i)}$. Then $Q^{(1)} + Q^{(2)}$ is electrical flow for b.
- Potentials also add.

Thompson's Principle

Electrical Flows are Optimal

Let Q be the electrical flow satisfying the demand vector b and let f be any flow satisfying it. Then

$$\mathcal{E}(Q) = \sum_e \Delta_e Q_e = \sum_e R_e Q_e^2 \le \sum_e R_e f_e^2$$

Let $g = f - Q$. Then g is a circulation and $f = g + Q$. Then

$$\sum_e R_e f_e^2 = \sum_e R_e (g_e^2 + 2 g_e Q_e + Q_e^2) \ge \sum_e R_e Q_e^2 + \sum_e 2 \Delta_e g_e.$$

The last term is zero since g is a sum of circular flows and for any cycle the potential differences sum to zero.

Effective Resistance

Effective Resistance

Let Q be an electrical flow of 1 from s to t.

The effective resistance of the network is the potential difference Δ between s and t.

This is also the energy of the flow.

$$\mathcal{E}(Q) = \Delta \cdot 1 = \Delta.$$

$$\mathcal{E}(Q) = \sum_e \Delta_e Q_e = \sum_P \sum_{e \in P} \Delta_e Q_P = \Delta \sum_P Q_P = \Delta.$$

Computing the Currents

Let p_u be the (unknown) potential of node u. For any edge $e = (u, v)$ the current from u to v is $(p_u - p_v)/R_{uv}$. The net-current at u is equal to b_u:

$$\sum_{v \in \delta(u)} \frac{p_u - p_v}{R_{uv}} = b_u.$$

or equivalently

$$\left(\sum_{v \in \delta(u)} \frac{1}{R_{uv}}\right) p_u + \sum_{v \in \delta(u)} \frac{-1}{R_{uv}} p_v = b_u$$

Coefficient matrix is symmetric diagonally dominant (SDD).

Methods for Solving SSD-system $Ax = b$

- Cholesky factorization: $A = LL^T$ where L is a lower triangular matrix
- Gauss-Seidel Iteration: compute $x^{(k)}$ for $k = 1, 2, 3, \ldots$. For fixed k, compute $x_i^{(k+1)}$ for $i = 1, 2, 3, \ldots$:

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j>i} a_{ij} x_j^{(k)} - \sum_{j<i} a_{ij} x_j^{(k+1)} \right)$$

- Preconditioning: solve $BAx = Bb$ for a suitable B.
- Recursive Preconditioning + Partial Cholesky + Chebyshev Iteration (Spielman/Teng, Koutsis/Miller/Peng): $\widetilde{O}(m \log n \log 1/\epsilon)$.

Kirchhoff's Spanning Tree Theorem

- Assume $b_s = 1 = -b_t$ and $b_v = 0$ otherwise (Superposition)
- for a spanning tree T: $c(T) = \prod_e 1/R_e$
- $N = \sum_T c(T)$
- for an edge $e = (a, b)$: $S(a, b) =$ all spanning trees that contain a and b (in this order) on path from s to t.

$$N(a, b) = \sum_{T \in S(a,b)} c(T).$$

- $$Q_{(a,b)} = \frac{N(a, b) - N(b, a)}{N}$$

Q is an electrical flow of value 1 from s to t.

Current Law Holds

For simplicity, multiply all currents by N.

$$Q^T_{(a,b)} = \begin{cases} c(T) & \text{if } T \text{ contains } s \ldots ab \ldots t \\ -c(T) & \text{if } T \text{ contains } s \ldots ba \ldots t \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$Q_{ab} = \sum_T Q^T_{ab}$$

T induces a current of $c(T)$ from s to t along its path from s to t. So total current is N as desired and flow conservation holds.

Voltage Law Holds

thicket = spanning forests with two components F_0 and F_1 such that $s_i \in F_i$. $F = F_0 \cup F_1$.

$$Q^F_{(a,b)} = \begin{cases} Q^{F \cup ab}_{(a,b)} & \text{if } F \cup ab \text{ is a spanning tree} \\ 0 & \text{otherwise.} \end{cases}$$

Then $Q_e = \sum_F Q^F_e$. Let C be a cycle.

$$\begin{aligned} \sum_{e \in C} d(e,C)\Delta_e &= \sum_{e \in C} d(e,C) R_e Q_e = \sum_{e \in C} d(e,C) R_e \sum_F Q^F_e \\ &= \sum_F \sum_{e \in C, e \text{ extends } F} d(e,C) R_e \prod_{e' \in F \cup e} 1/R_{e'} \\ &= \sum_F \prod_{e' \in F} 1/R_{e'} \sum_{e \in C, e \text{ extends } F} d(e,C) = 0 \end{aligned}$$