Exercise Sheet

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Exercise 1 *Let V be a finite set, let $s \in V$ be an element of V, and let $f : V \mapsto V$ be a function from V to V. We want to certify that f encodes a tree rooted as s. The assumption is that edges are oriented towards s and we have a self-loop at s, i.e., $f(s) = s$.*

1. *Design an algorithm for certifying that f encodes a tree with root s.*
2. *What piece of additional information would simplify the task?*

Exercise 2 *Let $G = (V,E)$ be a directed graph and $s \in V$ be a vertex. Let ℓ be an edge labeling with* positive *reals. We assume that all vertices are reachable from s. Let d map the vertices into reals satisfying*

$$d(s) = 0 \text{ and } d(v) = \min_{e=(u,v)} d(u) + \ell(u,v) \quad \text{for } v \neq s$$

1. *Prove that d are the shortest path distances from s.*
2. *Is this still true if we only request that edge length are non-negative? If your answer is YES, make sure that your proof for part 1) still works. If your answer is NO, give a counter example and point out, where your proof for the first part breaks down.*

Exercise 3 *Consider any algorithm you have studied so far in your education. Make it certifying.*

Exercise 4 *We start with a list of length n and repeatedly split lists into two until all lists have length one. The cost of splitting a list of length n into sublists of length n_1 and n_2, where $n = n_1 + n2$ and $n_1, n_2 < n$ is $f(n_1, n_2)$. You are not allowed to make any additional assumptions about n_1 and n_2. IN particular, you cannot guarantee that the split is balanced. What is the total cost of splitting?*

1. *$f(n_1, n_2) = n$: Show that the cost is $O(n^2)$ and that this can be attained.*
2. *$f(n_1, n_2) = \min(n_1, n_2)$. Show that the cost is $O(n \log n)$ and that this can be attained. In class, I showed you an amortization argument. Give an inductive proof, i.e., study the recurrence*

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \max_{1 \leq n_1 \leq n/2} T(n_1) + T(n - n_1) + n_1 & \text{if } n > 1. \end{cases}$$

3. $f(n_1, n_2) = \log\min(n_1, n_2)$. *Show that the cost is* $O(n)$ *and that this can be attained. Hint: Set up a recurrence and use the induction hyptothesis* $T(n) \leq cn - 2\log dn$ *for suitable constants c and d.*

Exercise 5 *We consider a market with two buyers and two sellers. The utilitities are* $u_1 = (u_{11}, u_{12}) = (5, 1)$ *and* $u_2 = (u_{21}, u_{22}) = (1, 2)$.

1. *First consider the Fisher market where the budgets of the buyers are* $m_1 = 10$ *and* $m_2 = 5$. *What are the equilibrium prices?*

2. *Then consider the Arrow-Debreu market, where buyer 1 owns good 1 and buyer 2 owns good 2, i.e.,* $m_1 = p_1$ *and* $m_2 = p_2$. *Determine non-zero equilibrium prices.*