Reliable Algorithmic Software

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State of the Art

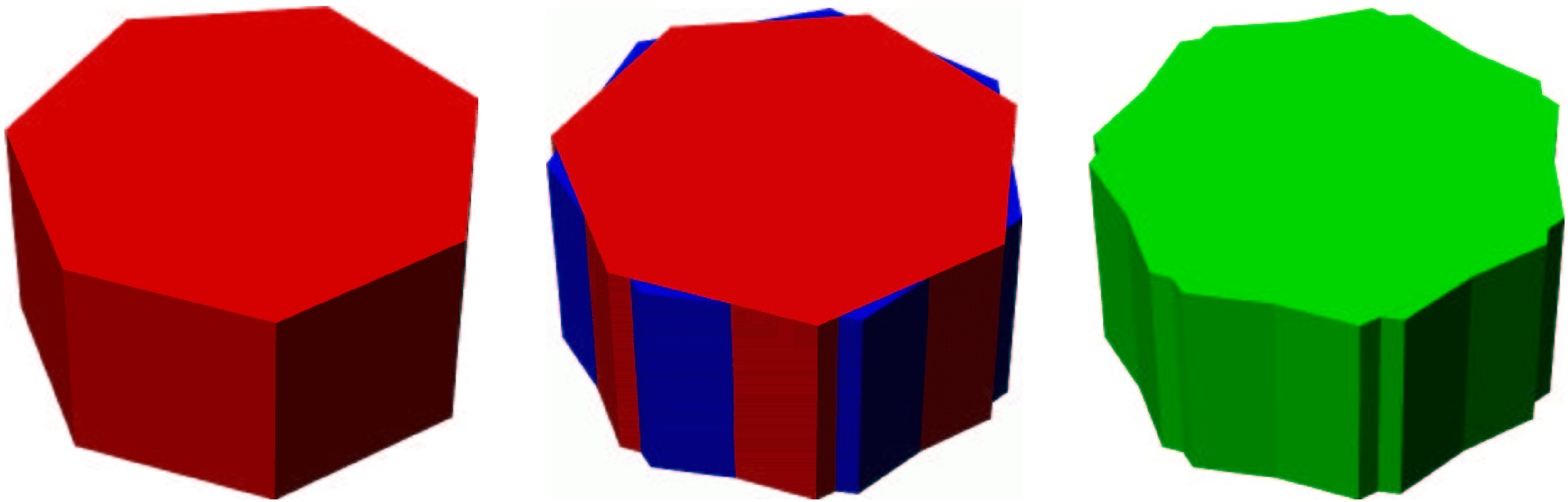
- Popular algorithmic systems: Maple, Mathematica, STL, LEDA, CGAL, ACIS, LAPACK, MATLAB, CPLEX, Xpress, ILOG solver.

  Can you trust any of them?

  Most manuals evade the issue and avoid sentences which could be interpreted as guarantees.

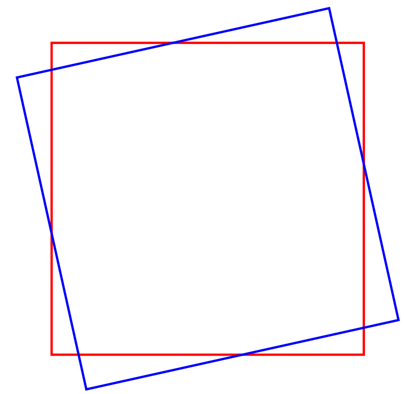
- two basic algorithmic problems with no reliable implementation
  - Computer Aided Design (CAD), Boolean Operations on Solids
  - Linear Programming
- LEDA and CGAL are reliable: Belief or Fact?
  - LEDA = library of efficient algorithms and data types
  - CGAL = computational geometry algorithms library
- details on next slides

State of the Art: Boolean Operations on Solids



- The left-most picture shows a regular cylinder P, $n = 7$.
- The middle picture shows two copies of the cylinder: Q was obtained by rotating P by α degrees about its axis, $\alpha \approx 20^\circ$.
- the right-most picture shows the union of P and Q (= a cylinder whose base is a $4n$-gon).

The State of the Art



- existing CAD-systems are **not** reliable
- construct a regular n-cylinder P,
- obtain Q by rotating P by α degrees,
- and compute the union of P and Q.

System	n	α	time	output
ACIS	1000	1.0e-4	5 min	correct
ACIS	1000	1.0e-6	30 sec	incorrect answer
Rhino3D	200	1.0e-2	15sec	correct
Rhino3D	400	1.0e-2	–	CRASH

- the situation is even worse for objects with curved boundaries

Linear Programming

$$\text{maximize} \quad c^T x \quad \text{subject to} \quad Ax \leq b \quad x \geq 0$$

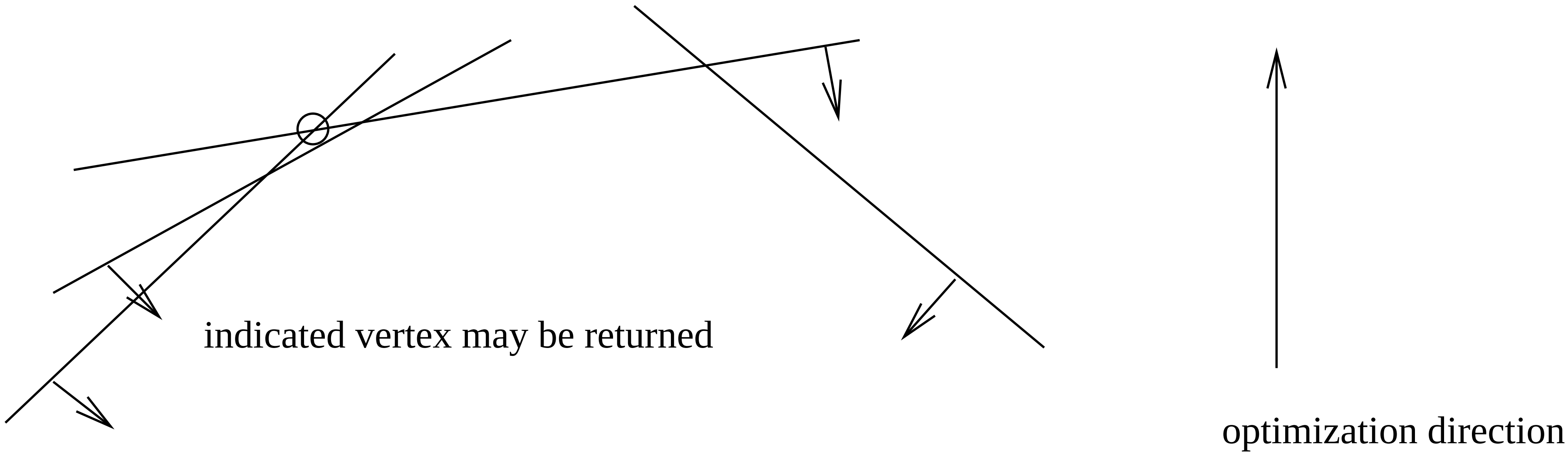
- linear programming is a most powerful algorithmic paradigm
- *There is no linear programming solver that is guaranteed to solve large-scale linear programs to optimality. Every existing solver may return suboptimal or infeasible solutions. There are solvers that solve small problems to optimality.*

Problem				**CPLEX**				**Exact Verification**
Name	C	R	NZ	T	V	Res	RelObjErr	T
degen3	1504	1818	26230	8.08	0	opt	6.91e-16	8.79
etamacro	401	688	2489	0.13	10	dfeas	1.50e-16	1.11
fffff800	525	854	6235	0.09	0	opt	0.00e+00	4.41
pilot.we	737	2789	9218	3.8	0	opt	2.93e-11	1654.64
scsd6	148	1350	5666	0.1	13	dfeas	0.00e+00	0.52
scsd8	398	2750	11334	0.48	0	opt	7.54e-16	1.52

Dhiflaoui/Funke/Kwappik/M/Seel/Schömer/Schulte/Weber: SODA 03

continue with exact LP-solver

Linear Programming II



- indicated vertex is not primal feasible since it violates a constraint
- indicated vertex is not dual feasible since it is not optimal for a subset of the constraints.

Are LEDA and CGAL Reliable?

- I believe so:
  - the authors are trustworthy individuals at least most of the time
  - most programs are carefully documented but not all of them
  - extensively tested
  - underlying algorithms have been shown correct
  - number types give illusion of a Real RAM
  - geometry kernels are model of geometry
  - program result checking is used
- in court the above is called circumstantial evidence
- Am I willing to bet on correctness?
  - yes, in case of the sophisticated algorithms
  - definitely no, in case of support (graphics, windows, IO)
- there are no formal proofs of correctness

First Summary

- no reliable implementations exist for fundamental algorithmic problems such as Linear Programming or Boolean Operations on Solids
- we are lacking principles: CPLEX and ACIS are state of the art.
- CGAL and LEDA are a step forward,
  but by far not the end of the story
- abstract challenge:
  - work out the principles underlying reliable algorithmic software
  - create a comprehensive collection of reliable algorithmic software components.
- concrete challenges:
  - a correct and efficient CAD system
  - a correct and efficient LP solver
  - a certified LEDA
  - to meet either challenge will require new theory

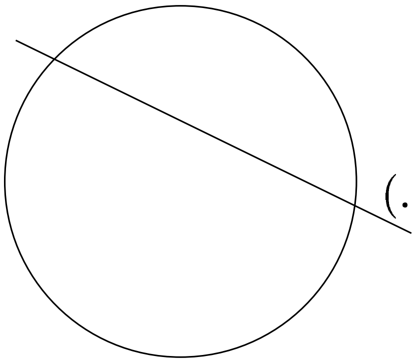
Approaches

- program verification
- exact computation paradigm
- program result checking
- certifying algorithms
- verification of checkers
- cooperation of verification and checking
- a posteriori analysis
- test and repair

Program Verification

- formal program verification is the obvious approach.
- obstacles
  - the mathematics underlying the algorithms must be formalized
  - verification must be applicable to languages in which algorithmicists want to formulate their algorithms
- my opinion: the direct applicability of program verification is doubtful for some time to come
- but see below: verification of checkers

The Exact Computation Paradigm

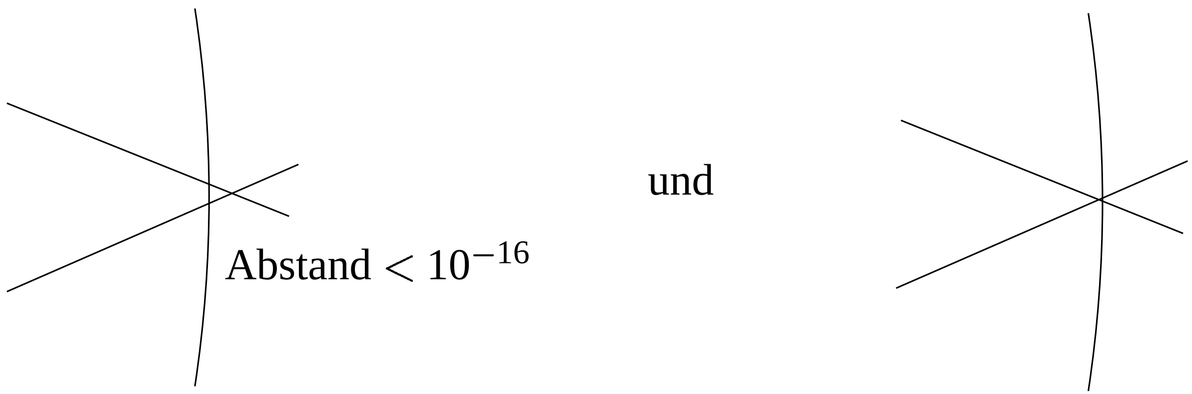
- 

  Circle: $x^2 + y^2 = 100$
  Line: $y = 1 - 0.100000001 \cdot x$

  existing systems approximate the coordinates (usually, 16 digits)

  $$x_0 = 9,999999999999999950000000000000049\ldots$$

  and hence cannot distinguish

  but geometric programs branch on this case distinction $\Longrightarrow$ disaster

- exact computation paradigm: implement an efficient Real RAM

A Separation Bound for Algebraic Expressions

Let E be an expression with operators $+$, $-$, $*$ and $\sqrt{\ }$ and integer operands. Let

- $u(E) =$ value of E after replacing $-$ by $+$.
- $k(E) =$ number of distinct square roots in E.

Then (BFMS, BFMSS)

$$E = 0 \quad \text{or} \quad |E| \geq \frac{1}{u(E)^{2^{k(E)}-1}}$$

Theorem allows us to determine signs of algebraic expressions by numerical computation with precision $(2^{k(E)} - 1)\log u(E)$.

in preceding example: compute the first 25 decimal digits of x_0 and you know how x_0 compares to 10.

related work: Mignotte, Canny, Dube/Yap, Li/Yap, Scheinermann
extensions: division, higher-order roots, roots of univariate polynomials

Discussion I

How small can $A - B\sqrt{C}$ be, if non-zero? $\quad A, B, C \in \mathbb{N}$.

$$|A - B\sqrt{C}| = \left| \frac{(A - B\sqrt{C})(A + B\sqrt{C})}{A + B\sqrt{C}} \right| = \frac{|A^2 - B^2 C|}{|A + B\sqrt{C}|} \geq \frac{1}{|A + B\sqrt{C}|} \geq \frac{1}{|A| + |B|\sqrt{C}}$$

This is a special case of the theorem

- $u(E) = |A| + |B|\sqrt{C}$
- $k = 1$

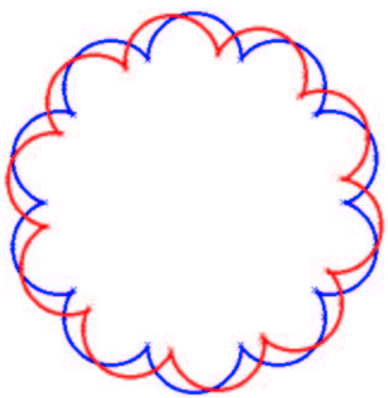
Recent Progress I

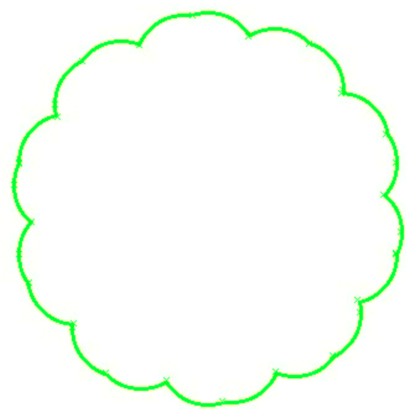
- efficient geometry kernels for linear objects in CGAL and LEDA

  union of n-gons

n	α	time	result
5000	6.175e-06	30 s	correct
20000	9.88e-07	141 s	correct
$\to \infty$	$\to 0$	$\to \infty$ sec	**correct**

- CORE and LEDA offer reasonably efficient computations with radicals
- ESOLID (Manocha): exact boundary evaluation of some curved solids.
- exact boolean operations on 2-dimensional curved objects of low degree



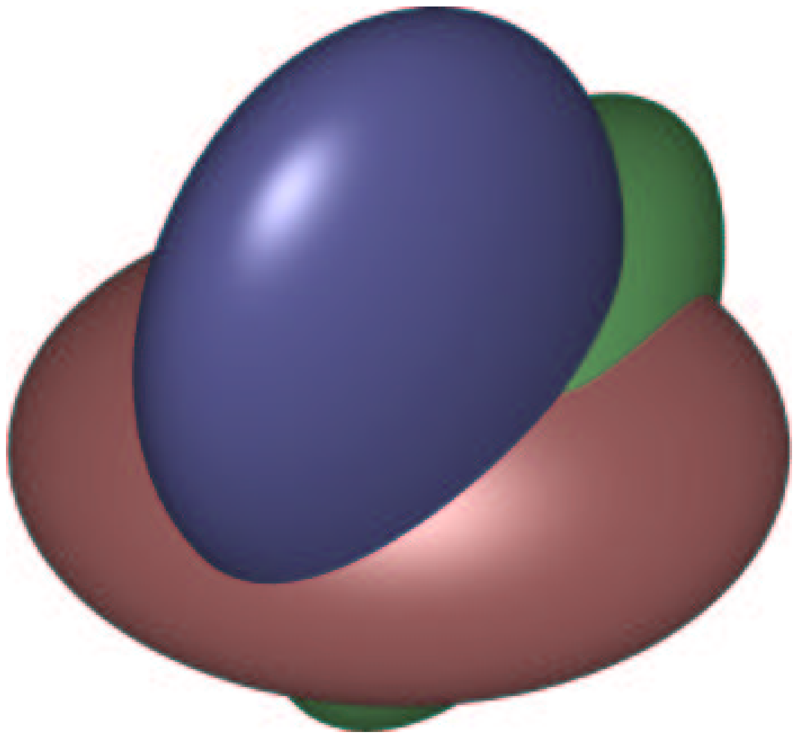


1 min for $n = 1000$

Berberich, Eigenwillig, Hemmer, Hert, M, Schömer: ESA 2002

Recent Progress II

- arrangements of ellipsoids



Geismman, Hemmer, Schömer: CompGeo 2001

Wolpert: PhD-thesis

Program Result Checking

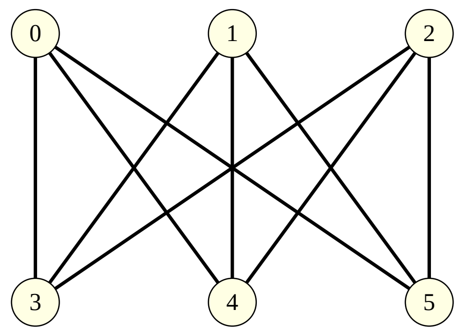
- verification = program works for every input
  result checking = program worked for a specific input
- Blum and Kannan (89): programs that check their work
- a checker for a program computing a function f takes
  - an instance x and an output y, and
  - returns true if $y = f(x)$ and return false, otherwise
- hope: checking is simpler than computing
- example
  - multiplication problem: compute $y = x_1 \cdot x_2$, given x_1 and x_2.
  - a (probabilistic) checker gets x_1, x_2, and y, chooses a small random prime p and
  - verifies that $(x_1 \bmod p) \cdot (x_1 \bmod p) \bmod p = y \bmod p$.
- program result checking is too restrictive to be practical

Certifying Algorithms

- certifying algorithms return additional output (= *a witness*) that simplifies checking.
- on input x, a certifying program for a function f returns a value y and additional information I that makes it easy to check that $y = f(x)$.
- "easy to check" has twofold meaning:
  - there is a simple program C that given x, y, and I checks whether indeed $y = f(x)$.
  - if $y \neq f(x)$ then there should be no I such that (x, y, I) passes checking.
  - simple = correctness is "obvious".
  - the running time of C on inputs x, y, and I should be no larger than the time required to compute $f(x)$ from scratch
  - preferably much much smaller
- observe that certifying program and checker are designed together

Example of a Certifying Algorithms

- planarity testing: given a graph G, decide whether it is planar
  - Tarjan (76): planarity can be tested in linear time
  - Chiba et al (85): planar embedding of a planar graph in linear time
  - a story
  - Hundack/M/Näher (97): Kuratowski subgraph of a non-planar graph in linear time



- many more examples are discussed in LEDA book
- in the LEDA system many programs are certifying.

Verification of Checkers

- the checker should be so simple that its correctness is “obvious”.
- we may hope to formally verify the correctness of the implementation of the checker

  this is a much simpler task than verifying the solution algorithm

  - the mathematics required for the checker is usually much simpler that the one underlying the algorithm for finding solutions and witnesses
  - checkers are simple programs
  - algorithmicists may be willing to code the checkers in languages which ease verification
- **Remark:** for a correct program, verification of the checker is as good as verification of the program itself
- Harald Ganzinger and I are exploring the idea

Cooperation of Verification and Checking

- a sorting routine working on a set S

  (a) must not change S and

  (b) must produce a sorted output.

- I learned the example from Gerhard Goos
- the first property is hard to check (provably as hard as sorting)
- but usually trivial to prove, e.g.,
  if the sorting algorithm uses a *swap*-subroutine to exchange items.
- the second property is easy to check by a linear scan over the output, but hard to prove (if the sorting algorithm is complex).
- give other examples where a combination of verification and checking does the job

A Posteriori Analysis

MAX-PLANCK-GESELLSCHAFT

- there will always be inexact algorithms.
- a-posteriori analysis: analyze the quality of the solution
- example: roots of a univariate polynomial $f(x)$ of degree n
  - given approximate solutions $x_1, \ldots, x_n$, compute

$$\sigma_i = \frac{f(x_i)}{\prod_{j \neq i}(x_i - x_j)} \quad \text{for } 1 \leq i \leq n\ .$$

  - $\Gamma_i =$ disk in the complex plane centered at x_i with radius $n|\sigma_i|$.
  - the union of the disks contains all roots of f
  - a connected component consisting of k disks contains exactly k roots of f.
  - the σ_i are easily computed with controlled error using multi-precision floating point arithmetic

- analogous examples in the combinatorial world, e.g., in approximation algs

Test and Repair

- use solution returned by an inexact algorithm as starting point for an exact algorithm
- example: linear programming

$$\text{maximize} \quad c^T x \quad \text{subject to} \quad Ax = b, \quad x \geq 0$$

A is an $m \times n$ matrix with $m < n$ and rank m (for simplicity)

- a basic solution $x = (x_B, x_N)$ is defined by a $m \times m$ non-singular sub-matrix B of A
  - x_B are the vars corresponding to cols in B, x_N remaining vars
  - $x_N = 0$ and $x_B = B^{-1}b$ solve a linear system
  - a basic solution is *primal feasible* if $x_B \geq 0$
  - a basic solution is *dual feasible* if $c_B^T - c_N^T A_B^{-1} A_N \leq 0$.
  - it is optimal, if it is primal and dual feasible.
- for medium-size linear programs, we can check (exactly !!!) for primal or dual feasibility in reasonable time (details)

An Exact LP-Solver

- use an inexact LP solver to determine an "optimal" basis B
- check the basis for optimality. If optimal, stop.
- if not, use the basis as a starting basis for an exact simplex algorithm
- seems to work reasonably well
- turn this observation into a theorem
- extend to large scale linear programs

- general challenge for optimization problems
  - design (exact) algorithms that start from a given solution x_0 towards an optimal solution.
  - the running time should depend on some natural distance measure between the initial and the optimal solution.
- go back to road map slide

An Example of a Distance Measure

- LP is given as a set of inequalities in d variables, goal is to find the top-vertex
- difficulty of a vertex = number of facets whose top vertex is above the given vertex.
- Kalai (92):
  1. given a vertex v, consider the d facets incident to it
  2. if v is the top vertex of all of them, stop
  3. among the facets incident to v whose top vertex is different from v, choose one at random, say F
  4. find the top vertex of F (by using the same algorithm recursively), call it v, and go to step 1.
- $T(d,n)$, running time for a problem in dimension d and starting with a vertex of difficulty n. Then

$$T(d,n) \leq \exp(O(\sqrt{n \log d}))$$