The Physarum Computer

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SODA 2012
ICALP 2013
J. Theoretical Biology

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informatik

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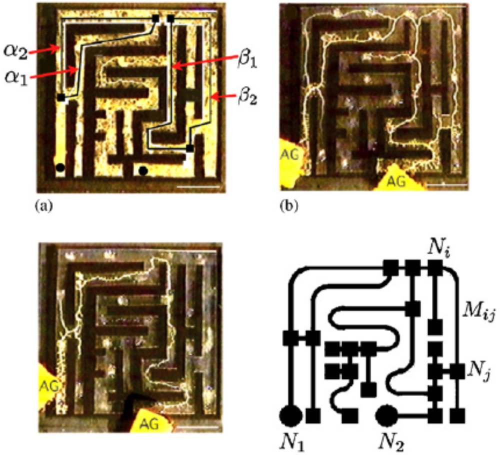

(a) (b) (c) (d)

Physarum, a slime mold,

single cell, several nuclei

builds evolving networks

Nakagaki, Yamada, Tóth, Nature 2000

show video

2008 Ig Nobel Prize

For achievements that first make people LAUGH
then make them THINK

COGNITIVE SCIENCE PRIZE: Toshiyuki Nakagaki, Ryo Kobayashi, Atsushi Tero, Ágotá Tóth
for discovering that slime molds can solve puzzles.

REFERENCE: "Intelligence: Maze-Solving by an Amoeboid Organism," Toshiyuki Nakagaki, Hiroyasu Yamada, and Ágota Tóth, Nature, vol. 407, September 2000, p. 470.

Outline of Talk

- The maze experiment (Nakagaki, Yamada, Tóth). ✓
- A mathematical model for the dynamics of Physarum (Tero et al.).
- The result: convergence against the shortest path.
- Approach:
  - Analytical investigation of simple systems.
  - A simulator.
  - Formulizing conjectures and killing them.
  - Proving the surviving conjecture.
- Beyond shortest paths.
  - Transportation problems.
  - Linear programming (A. Johannson/J. Zou and D. Straszak/N. Vishnoi).
  - Network formation.
- Ideen der Informatik

Mathematical Model (Tero et al.)

- Physarum is a network of tubes (pipes);
- Flow (of liquids and nutrients) through a tube is determined by concentration differences at endpoints of a tube, length of tube, and diameter of tube;
- Tubes adapt to the flow through them: if flow through a tube is high (low) relative to diameter of the tube, the tube grows (shrinks) in diameter.
- Mathematics is the same as for flows in an electrical network with time-dependent resistors.
- Tero et al., J. of Theoretical Biology, 553 – 564, 2007

Mathematical Model (Tero et al.)

- $G = (V, E)$ undirected graph
- Each edge e has a positive length L_e (fixed) and a positive diameter $D_e(t)$ (dynamic).
- Send one unit of current (flow) from s_0 to s_1 in an electrical network where resistance of e equals

$$R_e(t) = L_e / D_e(t).$$

- $Q_e(t)$ is resulting flow across e at time t.
- Dynamics:

$$\dot{D}_e(t) = \frac{dD_e(t)}{dt} = |Q_e(t)| - D_e(t).$$

We will write D_e and Q_e instead of $D_e(t)$ and $Q_e(t)$ from now on.

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The Questions

Does the system convergence for all (!!!) initial conditions?

If so, what does it converge to? Fixpoints?

How fast does it converge?

Beyond shortest paths?

Inspiration for distributed algorithms?

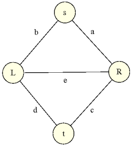
Convergence against Shortest Path

Theorem (Convergence (SODA 12, J. Theoretical Biology))

Dynamics converge against shortest path, i.e.,

- *potential difference between source and sink converges to length of shortest source-sink path,*
- $D_e \to 1$ *for edges on shortest source-sink path,*
- $D_e \to 0$ *for edges not on shortest source sink path*

this assumes that shortest path is unique; otherwise ...



Miyaji/Onishi previously proved convergence for parallel links and Wheatstone graph.

Our Approach

- Analytical investigation of simple systems, in particular, parallel links, and
- experimental investigation (computer simulation) of larger systems,
  - to form intuition about the dynamics,
  - to kill conjectures,
  - to support conjectures.
- Proof attempts for conjectures surviving experiments

Computer Simulation (Discrete Time)

- Electrical flows are driven by electrical potentials; let p_u be the potential at node u at time t. ($p_{s_1} = 0$ always)
- $Q_e = D_e(p_u - p_v)/L_e$ is flow on edge $\{u, v\}$ from u to v.
- Flow conservation gives n equations.

$$\text{for all vertices } u: \quad \sum_{v;\ e=\{u,v\}\in E} D_e(p_u - p_v)/L_e = b_u.$$

- $b_{s_0} = 1 = -b_{s_1}$ and $b_u = 0$, otherwise.
- The equations above define the p_v's and the Q_e's uniquely and can be computed by solving a linear system.
- Discrete Dynamics: $D_e(t+1) = D_e(t) + h \cdot (|Q_e(t)| - D_e(t))$.

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- Remark: linear system best solved by iterative method; simulation requires arbitrary precision arithmetic.

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- We simulated 1000 systems with up to 10000 nodes. Always observed convergence to shortest path. Speed of convergence is determined by length of second shortest path/length of shortest path.

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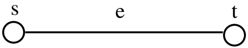
- Discrete Dynamics: $D_e(t+1) = D_e(t) + h \cdot (|Q_e(t)| - D_e(t))$.
- from now on: $\Delta_e = p_u - p_v$ for $e = uv$; potential drop on e.

A Single Link (Miyaji/Ohnishi)



e has length L and diameter D

$$Q = 1 \quad \text{always}$$

$$\dot{D} = 1 - D$$

$$D = D(t) = 1 + (D(0) - 1)e^{-t}$$

Thus

$$D(t) \to 1 \quad \text{for } t \to \infty$$

Diameter of e converges to 1, resistance $\frac{L}{D}$ of e converges to L.

Thus, potential difference between source and sink converges to L (= length of shortest source-sink path)

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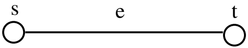

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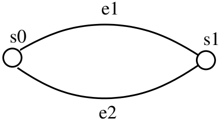
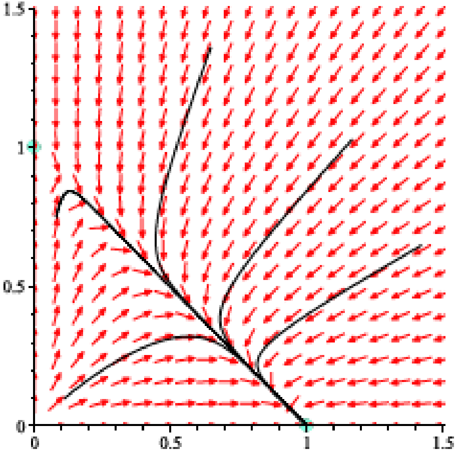
Two Parallel Links (Miyaji/Ohnishi)



e_i has length L_i, $L_1 < L_2$, and diameter D_i

A visualization of the dynamics.
Arrows show the vector $(\dot{D}_1, \dot{D}_2)$.

Trajectories in black.



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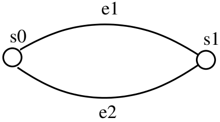

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$\Delta = \Delta(t) =$ potential difference between source and sink

$Q_i = \frac{D_i}{L_i} \cdot \Delta$

$\dot{D}_i = Q_i - D_i = \frac{D_i}{L_i}\Delta - D_i$

Fixpoints: $\dot{D}_i = 0$ for all i:

$\dot{D}_i = 0 \quad \text{iff} \quad D_i = 0$ or $\Delta = L_i$.

Thus $D_2 = 0$ and $\Delta = L_1$ and $D_1 = 1$, or vice versa.

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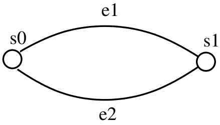

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Convergence

$L_i \frac{\dot{D}_i}{D_i} = \Delta - L_i$

$L_i \frac{d}{dt} \ln D_i = \Delta - L_i$

Let $V = L_2 \ln D_2 - L_1 \ln D_1$. Then

$$\dot{V} = L_1 - L_2$$
$$V(t) = V(0) + (L_1 - L_2)t$$
$$V(t) \to -infty$$
$$D_1 \to \infty \text{ or } D_2 \to 0$$
$$D_1 \to \infty \text{ is impossible.}$$
$$D_2 \to 0 \Rightarrow Q_2 \to 0 \Rightarrow$$
$$Q_1 \to 1 \Rightarrow D_1 \to 1 \Rightarrow \Delta \to L_1$$

V is called a Lyapunov function.

The General Case

Fixpoints: It is easy to verify (quarter page) that the fixpoints are exactly the source-sink paths. This assumes that all paths have different length. Thus, if the system converges, it converges against some source-sink path.

Convergence:

- In order to prove convergence, one needs to find a Lyapunov function.
- In order to prove convergence against the shortest path, one needs some additional arguments.

What did Evolution Optimize?

Evolution optimized dynamics so as to achieve a global objective. Which? (Lyapunov Function)

First idea: the energy of the flow $\sum_e Q_e \Delta_e$ decreases over. time

Not true, even for parallel links.

Theorem

For the case of parallel links:

$$\sum_{i \geq 2} L_i \ln D_i - L_1 \ln D_1, \sum_i Q_i L_i, \quad \frac{\sum_i D_i L_i}{\sum_i D_i}, \quad and\ (p_s - p_t) \sum_i D_i L_i$$

decrease over time

computer experiment: the obvious generalizations to general graphs (replace i by e) do not work.

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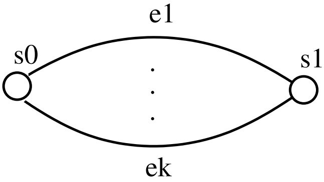
A not so Obvious Generalization



$$\frac{\sum_i D_i L_i}{\sum_i D_i} \quad \Rightarrow \quad \frac{\sum_e D_e L_e}{\text{minimum total diameter of a } s_0\text{-}s_1 \text{ cut}}$$

LEDA came handy.

What did Evolution Optimize?

Computer experiment:

$$V := \frac{\sum_e D_e L_e}{\text{minimum total diameter of a } s_0\text{-}s_1 \text{ cut}} \quad \text{decreases}$$

Theorem (Lyapunov Function)

$$V + \left(\sum_{e \in \delta(\{ s_0 \})} D_e - 1\right)^2 \quad \textit{decreases.}$$

Derivative of V (essentially) satisfies

$$\dot{V} \le -c \cdot \sum_e (D_e - |Q_e|)^2.$$

Proof uses min-cut-max-flow and ...

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Convergence against Shortest Path

Corollary (Convergence)

Dynamics converge against shortest path, i.e.,

- *potential difference between source and sink converges to length of shortest source-sink path,*
- $D_e \to 1$ *for edges on shortest source-sink path,*
- $D_e \to 0$ *for edges not on shortest source sink path.*

this assumes that shortest path is unique; otherwise, ...

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Discretization and Speed of Convergence

$$D_e(t+1) = D_e(t) + h(|Q_e(t)| - D_e(t))$$

Theorem (Epsilon-Approximation of Shortest Path)

Let *opt* be the length of the shortest source-sink path.

Let $\varepsilon > 0$ be arbitrary. Set $h = \varepsilon/(2mL)$, where L is largest edge length and m is the number of edges.

After $\widetilde{O}(nmL^2/\varepsilon^3)$ iterations, solution is $(1+\varepsilon)$ optimal, i.e., $V = \sum_e L_e D_e$ is at most $(1+\varepsilon)opt$.

Arithmetic with $O(\log(nL/\varepsilon))$ bits suffices.

The Transportation Problem

- undirected graph $G = (V, E)$
- $b : V \to \mathbb{R}$ such that $\sum_v b_v = 0$
- v supplies flow b_v if $b_v > 0$
- v extracts flow $|b_v|$ if $b_v < 0$
- find a cheapest flow where cost of sending f units across an edge of length L is $L \cdot f$

Physarum dynamics solves transportation problem.

D_e's converge against a mincost solution of transportation problem.

proof requires a non-degeneracy assumption

Related Work: Directed Physarum

$$\dot{D}_e(t) = Q_e(t) - D_e(t)$$

No biological significance is claimed.

Results

Ito/Johansson/Nakagaki/Tero (2011) prove convergence to shortest directed source-sink path.

Johannson/Zou (2012) and D. Straszak/N. Vishnoi (2016) prove that directed dynamics solves any linear program with monotone objective function (all coefficients of c are positive)

$$\max c^T x \quad \text{subject to} \quad Ax = b \text{ and } x \geq 0.$$

Linear Programming, More Details

Shortest path problem is a linear program (min cost flow), namely

$$\min c^T x \quad \text{subject to} \quad Ax = b \text{ and } x \geq 0,$$

where A is the node-edge incidence matrix.

Physarum Dynamics: Let $X = diag(x_e/c_e)$. Node potentials are defined by

$$\forall u : \sum_{v;\ uv \in E} \frac{x_e}{c_e}(p_v - p_u) = b_u, \text{ i.e.} \qquad p = (AXA^T)^{-1} b.$$

Then

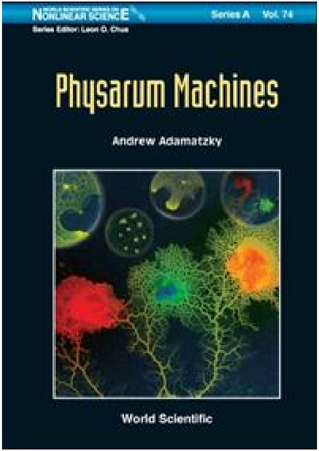
$$Q = XAp = XA(AXA^T)^{-1} b$$

and

$$\dot{x} = Q - x.$$

These formulae make sense for every linear program provided that $c > 0$.

Adamatzky's Book



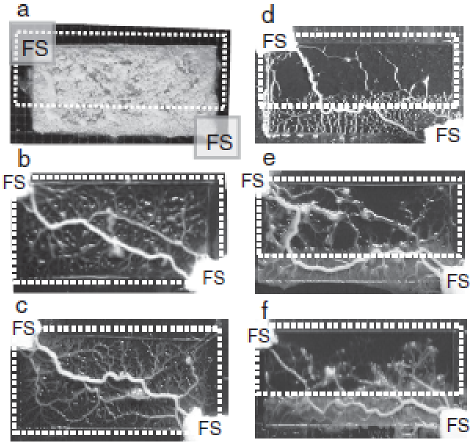
many examples of Physarum computations

- shortest paths
- network design
- Delaunay diagrams
- puzzles

also Youtube-videos: search for Physarum

Open Problems

Nonuniform Physarum



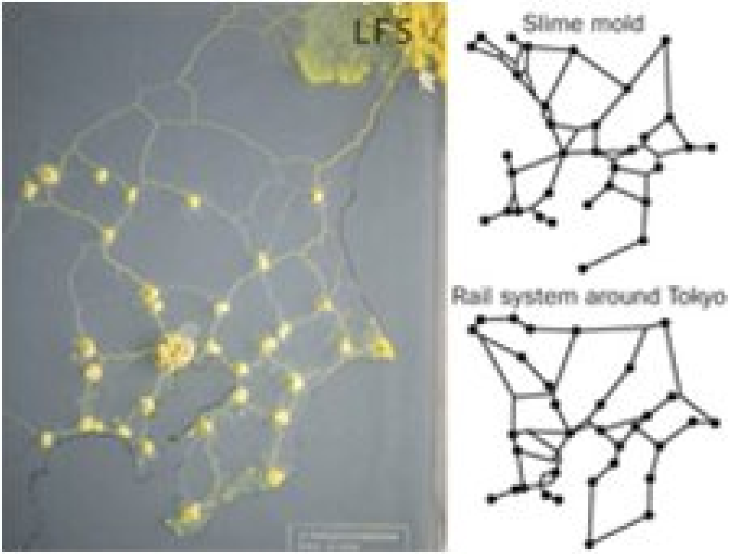
$$\dot{D}_e(t) = a_e(|Q_e(t)| - D_e(t))$$

a_e reactivity of e

We have a heuristic argument for the details of the convergence process. Have verified them in computer simulations.

No convergence proof

Network Design: Science 2010



Observables Demo

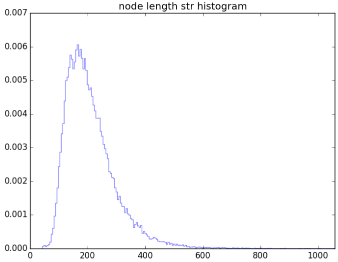


Figure: Normalized node strength histogram for edge lengths. Abscissa shows values in pixel.

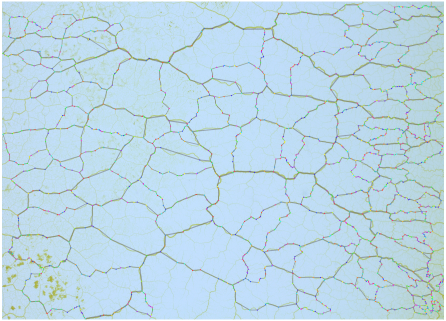


Figure: Input image and extracted graph as overlay (generated with NEFI (Dirnberger/Neumann))

My Current Projects

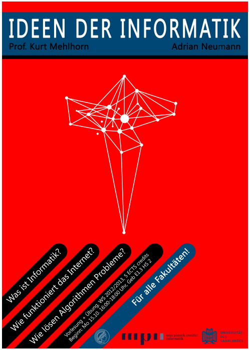
Understand the principles of network formation. What does the network optimize?

Nonuniform Versions of Physarum.

Can I use Physarum as an inspiration for approximation algorithms?

Ideas of Informatics (`http://resources.mpi-inf.mpg.de/departments/d1/teaching/ws14/Ideen-der-Informatik/`)

- An introduction to informatics for non-specialists (Studium Generale).
- Goal: informatics litteracy.
- Presents concepts and their applications, e.g.,
  - cryptography and electronic banking
  - shortest path algs and navigation systems
  - machine learning and image classification.
- Videos are available on my homepage.



Thank You