Superposition: Refutational Completeness

A Σ-interpretation $\mathcal{A}$ is called term-generated, if for every $b \in U_{\mathcal{A}}$ there is a ground term $t \in \mathsf{T}_{\Sigma}(\emptyset)$ such that $b = \mathcal{A}(\alpha)(t)$.

Superposition: Refutational Completeness

Lemma:
Let N be a set of (universally quantified) Σ clauses and let $\mathcal{A}$ be a term-generated Σ-interpretation. Then $\mathcal{A}$ is a model of $\bar{N}$ if and only if it is a model of N.

Proof:
($\Rightarrow$): Let $\mathcal{A} \models \bar{N}$; let $(\forall \vec{x} C) \in N$. Then $\mathcal{A} \models \forall \vec{x} C$
iff $\mathcal{A}(\beta[x_i \mapsto a_i])(C) = 1$ for all β and a_i.
Define θ such that $\mathcal{A}(\beta)(x_i\theta) = a_i$,
then $\mathcal{A}(\beta[x_i \mapsto a_i])(C) = \mathcal{A}(\beta)(C\theta) = 1$ since $C\theta \in \bar{N}$.

($\Leftarrow$): Let $\mathcal{A}$ be a model of N; let $C \in N$ and $C\theta \in \bar{N}$. Then $\mathcal{A}(\beta)(C\theta) = \mathcal{A}(\beta[x_i \mapsto \mathcal{A}(\beta)(x_i\theta)])(C) = 1$ since $\mathcal{A} \models N$.

Superposition: Refutational Completeness

Theorem (Refutational Completeness: Static View):
Let N be a set of clauses that is saturated up to redundancy. Then N has a model if and only if N does not contain the empty clause.

Proof:
If $\bot \in N$, then obviously N does not have a model.
If $\bot \notin N$, then the interpretation R_∞ (that is, $\mathrm{T}_\Sigma(\emptyset)/R_\infty$) is a model of all ground instances in $\bar{N}$ according to part (ii) of the model construction theorem.
As $\mathrm{T}_\Sigma(\emptyset)/R_\infty$ is term generated, it is a model of N.

Superposition: Refutational Completeness

So far, we have considered only inference rules that add new clauses to the current set of clauses (corresponding to the *Deduce* rule of Knuth-Bendix Completion).

In other words, we have derivations of the form $N_0 \vdash N_1 \vdash N_2 \vdash \ldots$, where each N_{i+1} is obtained from N_i by adding the consequence of some inference from clauses in N_i.

Under which circumstances are we allowed to delete (or simplify) a clause during the derivation?

Superposition: Refutational Completeness

A ground clause C is called redundant w. r. t. a set of ground clauses N, if it follows from clauses in N that are smaller than C.

A clause is redundant w. r. t. a set of clauses N, if all its ground instances are redundant w. r. t. $\overline{N}$.

Superposition: Refutational Completeness

A run of the superposition calculus is a sequence
$N_0 \vdash N_1 \vdash N_2 \vdash \ldots$, such that
(i) $N_i \models N_{i+1}$, and
(ii) all clauses in $N_i \setminus N_{i+1}$ are redundant w. r. t. N_{i+1}.

In other words, during a run we may add a new clause if it follows from the old ones, and we may delete a clause, if it is redundant w. r. t. the remaining ones.

For a run, $N_\infty = \bigcup_{i \geq 0} N_i$ and $N_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} N_j$.
The set N_* of all persistent clauses is called the limit of the run.

Superposition: Refutational Completeness

Lemma:
If $N \subseteq N'$, then every inference or clause that is redundant w. r. t. N is redundant w. r. t. N'.

Proof:
Obvious.

Superposition: Refutational Completeness

Lemma:
If all clauses in N' are redundant w. r. t. N, then $N \setminus N' \models N$ and every inference or clause that is redundant w. r. t. N is redundant w. r. t. $N \setminus N'$.

Proof:
Follows from the compactness of first-order logic and the well-foundedness of the multiset extension of the clause ordering.

Superposition: Refutational Completeness

Lemma:

If the conclusion of an inference is contained in a set N of clauses, then the inference is redundant w. r. t. N.

Proof:

Exercise.

Superposition: Refutational Completeness

Lemma:

Let $N_0 \vdash N_1 \vdash N_2 \vdash \ldots$ be a run. If an inference or clause is redundant w.r.t. some N_i, then it is redundant w.r.t. N_∞ and N_*.

Proof:

Exercise.

Superposition: Refutational Completeness

Corollary:
Every clause in N_i is contained in N_* or redundant w. r. t. N_*.

Proof:
If $C \in N_i \setminus N_*$, then there is a $k \geq i$ such that $C \in N_k \setminus N_{k+1}$, so C must be redundant w. r. t. N_{k+1}. Consequently, C is redundant w. r. t. N_*.

Superposition: Refutational Completeness

A run is called fair, if every inference from persistent clauses is redundant w. r. t. some N_i.

Lemma:
If a run is fair, then its limit is saturated up to redundancy.

Proof:
If the run is fair, then every inference from clauses in N_* is redundant w. r. t. some N_i, and therefore redundant w. r. t. N_*. Hence N_* is saturated up to redundancy.

Superposition: Refutational Completeness

Theorem (Refutational Completeness: Dynamic View):
Let $N_0 \vdash N_1 \vdash N_2 \vdash \ldots$ be a fair run, let N_* be its limit.
Then N_0 has a model if and only if $\bot \notin N_*$.

Proof:
($\Leftarrow$): By fairness, N_* is saturated up to redundancy.
If $\bot \notin N_*$, then it has a model.
Since every clause in N_0 is contained in N_* or redundant w. r. t. N_*, this model is also a model of N_0.

($\Rightarrow$): Obvious, since $N_0 \models N_*$.

Superposition: Extensions

Extensions and improvements:

simplification techniques,

selection functions,

basic strategies,

constraint reasoning.

Theory Reasoning

Superposition vs. resolution + equality axioms:

specialized inference rules,
thus no inferences with theory axioms,

computation modulo symmetry,

stronger ordering restrictions,

no variable overlaps,

stronger redundancy criterion.

Theory Reasoning

Similar techniques can be used for other theories:

transitive relations,

dense total orderings without endpoints,

commutativity,

associativity and commutativity,

abelian monoids,

abelian groups,

divisible torsion-free abelian groups.

Theory Reasoning

Observations:

no inferences with theory axioms:
yes, usually possible.

computation modulo theory axioms:
often possible, but requires unification and orderings modulo theory.

stronger ordering restrictions, no variable overlaps:
sometimes possible, but in many cases, certain variable overlaps remain necessary.

stronger redundancy criterion:
depends on the model construction.

Theory Reasoning

Observations:

In many cases, integrating more theory axioms simplifies matters.

Inefficient unification procedures may be replaced by constraints.