The Angel Problem, Positional Games, and Digraph Roots

Strategies and Complexity

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Preface

This thesis is about combinatorial games—mostly. It is also about graphs, directed graphs and hypergraphs, to a large extent; and it deals with the complexity of certain computational problems from these two areas. We study three different problems that share several of the above aspects, yet, they form three individual subjects and so we treat them independently in three self-contained chapters:

The angel-devil game. In the first chapter, we present improved strategies for an infinite game played on an infinite chess board, which has been introduced by Berlekamp, Conway, and Guy [**8**]. The *angel*, a chess person who jumps from square to square, tries to escape his opponent, the *devil*, who intends to strand the angel by placing obstacles on the board.

The open question about this game is, whether some angel who is allowed to make sufficiently large but bounded steps in each move, will be able to escape forever. Conway [**11**] has shown that certain quite natural escape attempts are bound to fail.

We attack this problem from the devil's perspective, trying to improve upon a result from [**8**], which established that the ordinary chess king, who can be considered as an angel of minimal power, cannot escape. A reformulation of the game which focuses on the angel's speed as the crucial quantity, allows us to show that certain faster "chess kings" can still be trapped. A second part of this chapter deals with angels on a three-dimensional board. We show that the new dimension grants the angel enough freedom to escape forever.

Weak positional games on hypergraphs. The games in the second chapter are very general versions of the well-known game of Tic-Tac-Toe. Two players alternately claim vertices of a hypergraph, the first player trying to get all vertices within some edge, his opponent striving to prevent this from happening.

Such *weak positional games* are known to be PSPACE-complete, but the respective hardness result from [**39**] utilizes edges of size up to 11. We analyze the restricted class of hypergraphs whose edges contain no more than three vertices each, trying to find optimal strategies for both players. We almost succeed. Under the additional restriction of almost-disjointness, that is, any two edges may share at most one vertex, we obtain a classification of such hypergraphs into those that yield a first player win and those who don't, which immediately leads to efficiently computable optimal strategies for either player. Eventually, a new framework is introduced for describing values that individual parts of a hypergraph contribute to a game that is played on the whole hypergraph.

The complexity of digraph root computation. The final chapter is not about games. A kth *root* of a square Boolean matrix A is some other matrix R with $R^k = A$. Interpreting A as a the adjacency matrix of a directed graph (digraph), we get an induced notion of powers for digraphs: the digraph D^k has an arc from a to b iff there is a walk of length exactly k from a to b in the digraph D.

The computational complexity of deciding whether a given Boolean matrix or, equivalently, a given digraph has a kth root, has been an open problem for twenty years. We answer this question by proving the problem NP-hard for every single integer $k \geq 2$. Our NP-completeness proof takes the graph-theoretic view, using basic concepts like paths, cycles, and vertex neighborhoods.

Besides the phenomena that make root finding hard, we discover a relation between digraph roots and graph isomorphism which materializes in form of an isomorphism-completeness result: For a special class of digraphs defined through arc subdivisions, root finding is of the same complexity as deciding whether two digraphs are isomorphic. This may come as a surprise since all problems known to be of this complexity are more or less obviously isomorphism problems. In abridged form, the results from this chapter have already appeared in [**28**].

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