Computing Geometric Minimum-Dilation Graphs is NP-Hard

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Abstract. Consider a geometric graph G, drawn with straight lines in the plane. For every pair a, b of vertices of G, we compare the shortest-path distance between a and b in G (with Euclidean edge lengths) to their actual Euclidean distance in the plane. The worst-case ratio of these two values, for all pairs of vertices, is called the vertex-to-vertex *dilation* of G. We prove that computing a minimum-dilation graph that connects a given n-point set in the plane, using not more than a given number m of edges, is an NP-hard problem, no matter if edge crossings are allowed or forbidden. In addition, we show that the minimum dilation tree over a given point set may in fact contain edge crossings.

Keywords: dilation, geometric network, plane graph, spanning ratio, stretch factor, NP-hardness

1 Introduction

Given a set P of n points in $\mathbb{R}^2$, one of the basic problems is to find a geometric network $G = (P, E)$ that provides good connections between the points in P, at low cost.

Often, the quality of connections is measured as follows. For any two points, a and b, of P, let $\pi_G(a, b)$ be a shortest path from a to b in G, where the length of a path is given by the sum of the Euclidean lengths $|p_i p_{i+1}|$ of its edges $e_i = \{p_i, p_{i+1}\}$. Then

$$\delta_G(a, b) := \frac{|\pi_G(a, b)|}{|ab|}$$

denotes the dilation of a, b in G, and

$$\delta(G) := \max_{a \neq b \in P} \delta_G(a, b)$$

is the *vertex-to-vertex dilation* of G. This value is also known as the stretch factor or the spanning ratio of G; it should not be confused with the geometric

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dilation that takes all points of the network into account, vertices and interior edge points. In this paper, the *cost* of a network G will be measured by the number of its edges, $|E|$. Alternative cost measures would be the weight, i. e., the sum of all edge lengths, the diameter, the maximum degree, etc.

Clearly, the complete graph over P has optimal dilation 1 but its number of edges is, in general, in $\Omega(n^2)$.[3] On the other hand, spanning trees realize the minimum edge number $n-1$, but they cannot offer good connections. In fact, each tree T containing the vertex set of the regular n-gon has dilation $\delta(T) > 1.57$ for $n = 5$, while $\delta(T) \in \Omega(n)$ holds for larger n; see Ebbers-Baumann et al. [7] and Aronov et al. [1].

In the framework of spanners it has been shown that one can (almost) combine the merits of both solutions, and efficiently construct networks of dilation $1+\epsilon$ that have only $O(\epsilon^{-2} \cdot n)$ many edges; for surveys, see the handbook chapter by Eppstein [8] or the forthcoming monograph by Narasimhan and Smid [13].

In this paper we are interested in the computational complexity of constructing good geometric spanners. More precisely, we study the following problems.

Definition 1. *Given a finite point set P in the plane, a threshold $\delta > 1$ and a parameter $m \geq |P| - 1$,*

- *the decision problem* DilationGraph *asks, whether there exists a geometric graph with vertex set P, that has dilation at most δ and contains at most m edges*
- *the decision problem* PlaneDilationGraph *asks if there exists a crossing-free geometric graph with the same properties.*

In this paper we prove that both DilationGraph and PlaneDilationGraph are NP-hard. It is interesting to observe that the number $|E|$ of edges we need, in order to prove NP-hardness, is only slightly larger than the minimum number $|P| - 1$. This fits nicely to a recent result by Aronov et al. [1], which also states, in a different way, that few extra edges matter a lot in constructing spanners. They proved that with $n-1+k$ edges, where $0 \leq k < n$, a dilation of $O(n/k + 1)$ can be achieved, which is optimal.

A lot of work has been done on the complexity of finding spanners of low dilation and weight in general graphs. Closely related to our work is a result by Brandes and Handke [2]. Building on previous work by Cai [3], they proved the following fact for weighted graphs. For each fixed rational number $\delta \geq 4$, it is an NP-complete problem to decide if a given graph H contains a planar subgraph G, whose weight does not exceed a given bound W, such that for any two vertices v, w of H the relation $|\pi_H(v, w)| \leq \delta \cdot |\pi_G(v, w)|$ holds, where the length of a path is given by the sum of its edge weights.

Our paper extends this result to the (more restrictive) geometric case where H is the complete graph over n points and edge weights are Euclidean lengths. It implies one of the recent results by Gudmundsson and Smid [11] that it is NP-hard to find a δ-spanner with $\leq m$ edges in a given geometric graph.

[3] If m points are collinear we need only $m - 1$ edges to build a connecting chain of dilation 1.

Cai [3] and Cai and Corneil [4] have studied the problem of finding tree spanners of dilation $\leq \delta$ in weighted graphs. They proved that the decision problem is NP-complete for any $\delta \geq 4$, but polynomially solvable for $\delta = 2$, while the case $\delta = 3$ seems to be open. The corresponding geometric problem appears to be rather complicated. This may be due to the surprising fact that the minimum dilation tree over n points in the plane may contain edge crossings, as we shall prove in Section 4, thus solving an open problem stated by Eppstein in [8], p. 444.

While working on the revision of this paper we learned that this fact has also been observed by Cheong et al. [6]. They can show that constructing the minimum dilation tree is NP-hard, too. On the other hand, Eppstein and Wortman [9] showed how to compute, in expected time $O(n \log n)$, a star of minimum dilation for n points.

The rest of this paper is organized as follows. In Section 2 we derive some technical results that will be needed in the reduction; Section 3, the main part, contains the reduction from PARTITION to DILATIONGRAPH; and Section 4 provides a point set whose minimum-dilation tree has a crossing. We close with some open problems in Section 5.

2 Technical Lemmata

Throughout this section, P denotes a finite set of points in the plane.

Definition 2. *(i) A geometric network $G = (P, E)$ with dilation $\delta(G) \leq \delta$ will be called a* δ-graph *for P. A δ-graph $G = (P, E)$ with $|E| \leq m$ edges will be called a* (δ, m)-graph *for P.*
(ii) The δ-ellipse *of two points a, b in the plane is the set of all points x satisfying $|ax| + |bx| \leq \delta \cdot |ab|$.*

Lemma 1. *Each shortest path $\pi_G(a, b)$ in a δ-graph G for P is contained in the δ-ellipse of a, b.*

Proof. For each vertex v on $\pi_G(a, b)$ the inequality

$$\delta \geq \frac{|\pi_G(a, b)|}{|ab|} \geq \frac{|av| + |vb|}{|ab|}$$

implies that v is contained in the δ-ellipse of a, b. □

Now we show how to enforce that certain edges are contained in minimum-weight (δ, m)-graphs for P, using geometric properties.

Lemma 2. *Let a, b be two points in P such that all points of P that are contained in the δ-ellipse around a, b, lie on the line L through a and b, but not between a and b. Then the edge ab is contained in any (δ, m)-graph for P of minimum weight.*

Proof. Assume for contradiction that there is no edge between a and b in some minimum-weight (δ, m)-graph G for P. Let π be a shortest path from a to b in G. By Lemma 1 and by assumption, all vertices of π lie on L, but none of them between a and b. Let $q_1, \ldots, q_t$ be the sequence of these vertices, sorted by their order on L (which is not the order in which they occur on π; for example, q_1 need not be equal to a or b!). Let G' result from G by replacing the edges of π with the edges (q_i, q_{i+1}) for $1 \leq i \leq t-1$. This transformation does not increase the G-distance of any point pair in P because for each edge of π there exists a concatenation of collinear edges in G'. Thus, G' is a (δ, m)-graph for P. On the other hand, it is clearly of smaller weight than G—a contradiction. □

3 The Reduction

We shall prove the NP-hardness of DilationGraph and PlaneDilationGraph by a reduction from the Partition problem:

Given a set S of n positive integers with $\sum_{r \in S} r$ even, decide whether there exists a subset $T \subseteq S$ such that $\sum_{r \in T} r = \sum_{r \in S \setminus T} r$.

Presented with an instance of Partition involving n integers, we are going to construct a planar point set P of size $5919 \cdot n - 4214$. Roughly, this point set P results from densely sampling a plane straight line drawing that consists of $O(n)$ segments, as shown in Figure 6. It takes $|P| - n - 2$ small edges to connect adjacent sample points on the long segments. If a partition exists for the given instance, we can carefully add $2n$ further edges, two in each of the n bubbles depicted in Figure 6, to ensure that the resulting graph is of dilation ≤ 7.

Conversely, suppose that the class of $(7, |P| + n - 2)$-graphs for P is not empty; then it contains a graph of minimum weight. By Lemma 2, $|P| - n - 2$ of its edges are forced to form the long segments. Since the dilation is ≤ 7, the remaining $2n$ edges must be placed inside the bubbles, and their positions must correspond to a partition of the integer set S. In particular, the graph must be plane. These properties will become evident below.

Our construction depends on n and on the size of the maximum element $r_{\max}$ of S, and it uses some scaling factors that will be stated as negative powers of 10. Let λ be the smallest integer greater than or equal to 8 for which

$$\max(10^5 r_{\max}, 2nr_{\max}^2) < 10^\lambda$$

holds. In particular, this ensures $r \cdot 10^{-\lambda} < 10^{-5}$ for all $r \in S$. Observe that exponent λ depends linearly on the bit length of the partition instance (which is bigger than n and the bit length of $r_{\max}$).

The basic idea behind our reduction is to arrange points along two long U-shaped paths like in Figure 1.

The vertical baselines of the two U's will be interrupted by horizontal gadgets—the bubbles depicted in Figure 6, one for each element $r \in S$. Each gadget will stretch horizontally over both U's. It can offer a short cut of $\approx r \cdot 10^{-\lambda}$ to either the left U, or to the right U—but not to both, since this would cause the inner part of the gadget to have a dilation > 7.

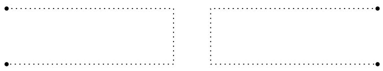

Fig. 1. A double U of intended dilation 7.

Consequently, both U's can receive the same total short cut, approximately

$$P := \frac{1}{2} \sum_{r \in S} r \cdot 10^{-\lambda},$$

if and only if set S admits a partition. After carefully adjusting the lengths of the horizontal edges of the U's, this will become equivalent to both $U's$ having a dilation ≤ 7 at their respective endpoints.

3.1 The choice gadget

The core part of our reduction is the choice gadget, which realizes the selection of an element $r \in S$ for the subset T from the given PARTITION instance. Basically, such a gadget consists simply of two horizontal densely sampled lines, with a larger gap in the upper row. Figure 2 shows the relative lengths and distances of the respective parts. The three segments au, vc, and bd are sampled with a regular spacing of 10^{-2}, giving a total of 1703 points.

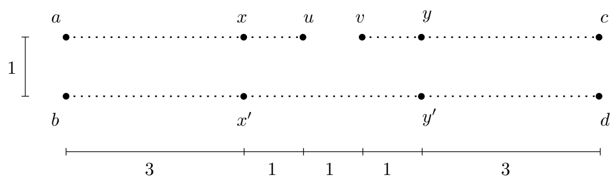

Fig. 2. The choice gadget (with lengths annotated).

Assume we want to connect this point set to a tree, i.e., with $|P| - 1$ edges so that the dilation is exactly 7—the same threshold as is intended on the global scale. By Lemma 2, we know that, because the three line segments are very densely sampled, in a minimum weight graph we must have an edge between any pair of direct neighbors on those segments. This leaves just two more edges for connecting the segments.

There must be at least one edge from top to bottom, so let's assume that there is such an edge, e, incident to a point on the line between a and u. Then the second edge cannot touch segment au, too, because otherwise the resulting path from c to d would be more than 10 units long. Hence, there must also be an edge f between the upper right segment vc and the bottom segment bd.

Taking the dilation of the points u and v into consideration, too, we see that a dilation of 7 can only be achieved if the edges e and f connect the points x, x' and y, y', respectively. Shifting these links to the left or right would impair the dilation between one of the pairs $\{a, b\}$, $\{c, d\}$, or $\{u, v\}$. Tilting e or f would have a similar effect because slanted edges are longer than vertical ones.

However, allowing only one solution is not what we desire. For the configuration to work as a choice gadget, we apply two minor modifications to leave some very restricted room for the precise placement of the links e and f. On the bottom line we introduce two extra points: a point $\hat{x}$ exactly $r \cdot 10^{-\lambda}$ to the left of x', where r is the given integer from the set S that we want to encode into this gadget, and a point $\hat{y}$ located $r \cdot 10^{-\lambda}$ to the right of y'.[4]

Moreover, we shift the middle points u and v on the upper line slightly outwards, each by a distance of $r \cdot 10^{-\lambda-1}$, so that the width of the gap increases by $2r \cdot 10^{-\lambda-1}$. See Figure 3 for a close-up on the relevant parts of the modified point set.

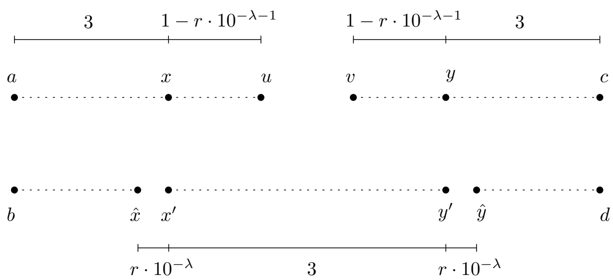

Fig. 3. A non-proportional drawing of the crucial parts of the final choice gadget.

What is the effect of these modifications? First of all, it is easy to see that a dilation-7 tree on the new point set cannot deviate much from the optimum that we have determined for the original set above. There still have to be all edges between direct neighbors along the three segments and there have to be the two edges e and f somewhere around x, x', y, and y'. Increasing the distance from u to v by $2r \cdot 10^{-\lambda-1}$ does not give us enough room to fix e or f to any upper vertex other than the designated x and y, respectively, since any shift of these edges would immediately increase one of the relevant distances by 10^{-2}. Moving the upper endpoint of e some k points to the left while moving the lower endpoint k steps to the right would not work either because it would increase the edge length to at least $\sqrt{1^2 + .02^2} \approx 1.0002$, which yields an increase in the distance that cannot be compensated by the comparatively small shift of u and v.

It turns out that the only freedom for placing the connections e and f lies in choosing x' or $\hat{x}$, respectively y' or $\hat{y}$, as their lower endpoints. What if we

[4] Just for the record: we have now used 1705 points per choice gadget.

connect e to $\hat{x}$ but f to y', i.e., the left connection tilts slightly to the left, while the right one stays perfectly vertical? The resulting dilation of u and v would then be

$$\frac{7 - 2r \cdot 10^{-\lambda-1} + r \cdot 10^{-\lambda} + \sqrt{1 + (r \cdot 10^{-\lambda})^2} - 1}{1 + 2r \cdot 10^{-\lambda-1}}.$$

The square root, which is due to the slope of the edge e, minus 1 is smaller than the $2r \cdot 10^{-\lambda-1}$ term so that this expression is smaller than

$$\frac{7 + r \cdot 10^{-\lambda}}{1 + \frac{1}{5}r \cdot 10^{-\lambda}} = 7 \cdot \frac{7 + r \cdot 10^{-\lambda}}{7 + \frac{7}{5}r \cdot 10^{-\lambda}} < 7$$

for such r.

Thus we see that slanting one of the two edges e and f outward does not create a dilation of more than 7 in the gadget. (It is obvious that the vertices u and v form the dilation-critical pair in this configuration, all other pairs having better dilation.)

But if we slanted both edges outward, connecting them to $\hat{x}$ and $\hat{y}$, we would get a dilation of

$$\frac{7 - 2r \cdot 10^{-\lambda-1} + 2r \cdot 10^{-\lambda} + 2\sqrt{1 + (r \cdot 10^{-\lambda})^2} - 2}{1 + 2r \cdot 10^{-\lambda-1}},$$

which is lower bounded by 7, as straightforward calculation shows. Therefore it is not possible to slant both edges outward without raising the dilation above 7.

This concludes the construction of our choice gadget. For a given integer r, we built it in such a way that either the path from a to b or that from c to d can be reduced by

$$1 + r \cdot 10^{-\lambda} - \sqrt{1 + r^2 \cdot 10^{-2\lambda}} \;\geq\; r \cdot 10^{-\lambda} - r^2 \cdot 10^{-2\lambda}.$$

We will now insert such gadgets into the big picture of Figure 1 by connecting their left endpoints a, b to the left U there and the right endpoints c, d to the right U.

3.2 Linking the choice gadgets

For a PARTITION instance S of size n, we have to build n individual choice gadgets, one for each $r \in S$. We arrange all these gadgets vertically, one below the other, forcing their left and right endpoints to get connected by paths. Figure 4 shows two choice gadgets linked at their endpoints.

The vertical distance between two choice gadgets is three on each side, left and right, we bridge this gap by a column of points with a regular spacing of 10^{-2}. The highest of these points is placed exactly $1/10$ below the endpoint of the upper gadget and symmetrically, the lowest point sits $1/10$ above the endpoint of the bottom gadget.

Since the internal spacing of such a link is by a factor of 10 smaller than the gaps to the endpoints, Lemma 2 applies and tells us that any dilation-7

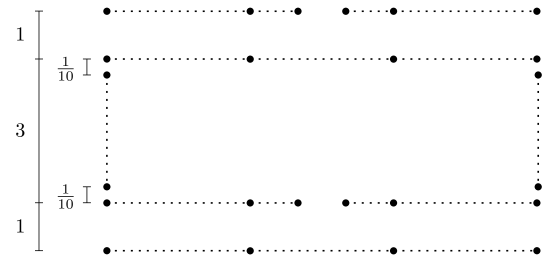

Fig. 4. Two choice gadgets linked at their endpoints.

graph must connect these 281 points by the canonical 280 length-10^{-2} edges. The only question that remains is, how such a vertical segment connects to the gadgets above and below. It clearly has to establish connections there to avoid extremely large dilation between the endpoint of the gadget and the endpoint of the vertical link, which are only 1/10 apart.

Globally, only a connection that minimizes the length of the resulting path along the vertical link and through the gadgets can lead to a dilation ≤ 7 of the corresponding U. Such connections will look roughly like the one shown in Figure 5. The further away the slanted edge is from the endpoints c and w, the better the short cut. However, the diagonal is also restricted by the dilation between c and w.

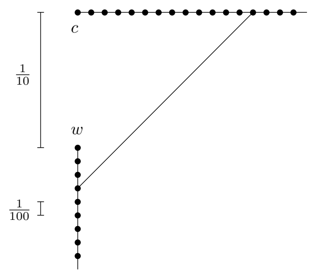

Fig. 5. Efficiently connecting around a corner.

It is not hard to find out that an almost perfect 45°-connection yields the optimal tradeoff between short-cut effect and c-w dilation. One verifies that connecting the 23th point to the right (counting c as the first) with the 15th on the left (counting w as the first) yields the best[5] short cut, giving a c-w dilation

[5] One could as well connect the 25th point to the 13th point and obtain the same values for dilation and shortcut; minimum dilation graphs need not be unique.

of

$$\left(\frac{22}{100}+\frac{14}{100}+\sqrt{\left(\frac{22}{100}\right)^2+\left(\frac{14}{100}+\frac{1}{10}\right)^2}\right)/\frac{1}{10}\ \approx 6.856\ <\ 7.$$

On path length we save at each corner

$$\left(\frac{22}{100}+\frac{14}{100}+\frac{1}{10}-\sqrt{\left(\frac{22}{100}\right)^2+\left(\frac{14}{100}+\frac{1}{10}\right)^2}\right)\ =\ \frac{23}{50}-\frac{\sqrt{265}}{50}\ \approx\ 0.134.$$

3.3 Putting everything together

We are now prepared to assemble all n choice gadgets and the two big U's into one big point set, as shown in Figure 6.

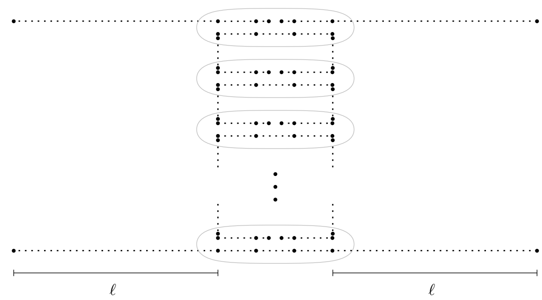

Fig. 6. The whole reduction in one picture; n choice gadgets are marked with bubbles.

Let us consider again how many edges we are going to allow for this point set, in order to enforce a dilation very close to 7. Each of the segments in Figure 6 shall form a long path. Taken alone, every choice gadget shall form a tree, with all its points connected but without any internal cycles. Cycles are only created by the links between gadgets. Precisely, every pair of link paths induces exactly one cycle. For a total number of $|P|$ points, we thus fix the number of edges to

$$m = |P| - 1 + (n-1) = |P| + n - 2,$$

where n again denotes the number of gadgets.

It remains to calibrate the length ℓ of the horizontal segments of the two U's in such a way that only a fair split of the total "short cut potential"

$$P = \sum_{r \in S} r \cdot 10^{-\lambda},$$

can result in a dilation ≤ 7 between the endpoints of each U.

First, let us assume that a partition $S = T \cup (S \setminus T)$ is possible such that the sum of the elements of T equals the sum over the elements of $S \setminus T$. In the choice gadgets associated with T the left edges are slanted, whereas in the other gadgets the right edges are slanted. By the results of Subsections 3.1 and 3.2, the path through the left U has total length

$$\leq\ 2\,\ell + n7 - \sum_{r\in T}(r\cdot 10^{-\lambda} - r^2\cdot 10^{-2\lambda}) + (n-1)\left(3 - 2(\frac{23}{50} - \frac{\sqrt{265}}{50})\right) \quad (1)$$

$$\leq\ 2\,\ell + (n-1)\left(10 - \frac{23-\sqrt{265}}{25}\right) + 7 - \frac{1}{2}P + nr_{\max}^2 10^{-2\lambda} \quad (2)$$

Exactly the same upper bound applies to the length of the path through the right U. We want the value of (2) to be at most 7 times the Euclidean distance of the endpoints of a U, which equals $n\cdot 1 + (n-1)\cdot 3 = 4n-3$, by construction. This can be achieved by letting

$$\ell\ \leq (n-1)\left(9 + \frac{23-\sqrt{265}}{50}\right) + \frac{1}{4}P - \frac{1}{2}nr_{\max}^2 10^{-2\lambda}. \quad (3)$$

Now let us assume that no partition of S is possible. Let L denote the set of all gadgets whose left edges are slanted, and let R be the set of choice gadgets with a slanted right edge. If a gadget lies in both, L and R, it causes a dilation > 7. So assume that L and R are disjoint. For one of these sets—say: L— must $\sum_{r\in L} r < \frac{1}{2}\sum_{r\in S} r$ hold. Then the total length of the path through the left U is at least

$$2\,\ell + n7 - \sum_{r\in L}(r\cdot 10^{-\lambda} - \sqrt{1 + r^2\cdot 10^{-2\lambda}}) + (n-1)\left(3 - \frac{23-\sqrt{265}}{25}\right) \quad (4)$$

$$\geq 2\,\ell + n7 - \frac{1}{2}P + 1\cdot 10^{-\lambda} + (n-1)\left(3 - \frac{23-\sqrt{265}}{25}\right) \quad (5)$$

$$\geq 2\,\ell + (n-1)\left(10 - \frac{23-\sqrt{265}}{25}\right) + 7 - \frac{1}{2}P + 10^{-\lambda}. \quad (6)$$

The left U will give a dilation > 7 if the value of (6) exceeds 7 times the distance of its endpoints, that is, if

$$\ell\ > (n-1)\left(9 + \frac{23-\sqrt{265}}{50}\right) + \frac{1}{4}P - \frac{1}{2}10^{-\lambda}. \quad (7)$$

In order to fulfill conditions (3) and (7) we use Newton's method to approximate $\sqrt{265}$, the only irrational number involved, by a rational number q to an error smaller than

$$10^{-(2\lambda+2)} < \frac{10^{-(\lambda+2)}}{2n} < \frac{10^{-\lambda} - nr_{\max}^2 10^{-2\lambda}}{100n};$$

these estimates hold due to the choice of λ. This takes a number of iterations logarithmic in λ. Then, we compute ℓ from (3), read as an equality, after substituting the root by q.

Observe that the coefficient of $n-1$ in (3) is ≈ 9.1344, and that the additive terms are bounded. If we split ℓ into $913(n-1)$ equal pieces, each of them has a (rational) length close enough to 10^{-2} to make Lemma 2 work for the horizontal segments of the big U. This takes $4 \cdot 913 \cdot (n-1)$ additional points. Now the definition of point set P is complete.

It is clear that that the description complexity of the constructed point set P is polynomial in the size of the Partition instance S, and that all computations can be carried out by a Turing machine. Moreover, S admits a partition if and only if there exists a graph of dilation ≤ 7 over P with $|P| + n - 2$ edges; and each such graph of minimum weight must be plane. Thus, we have shown the following.

Theorem 1. *The decision problems* DilationGraph *and* PlaneDilationGraph *are NP-hard.*

By counting the numbers of points and edges introduced in our construction, one verifies that even the following problem is NP-hard. Given a set of k points, is there a plane graph of dilation ≤ 7 over these points that contains at most $\frac{5920}{5919} \cdot k - \frac{7624}{5919}$ many edges?

4 Crossings in the Minimum Dilation Tree

It is well-known that a (Euclidian) minimum spanning tree on a point set in the plane cannot have any edge crossings. In [8, p. 444], Eppstein asks whether this is also the case for minimum-dilation trees.

We give a negative answer to this question. In fact, it is not too hard to verify that any spanning tree on the 7-point set in Figure 7 has a dilation of at least 2 and that the two trees that attain this value both contain an edge crossing.

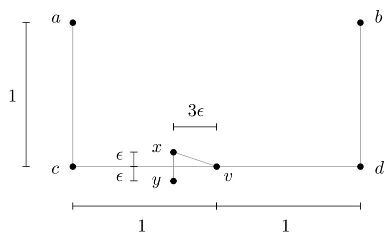

Fig. 7. A 7-point set whose minimum-dilation trees have a crossing.

The reason for the inevitable crossing in our example lies in the overlay of two structures on different scales, that is, of a large U on *acvdb* and a tiny hook on *xyv*. However, we could easily draw a crossing-free tree on the points of Figure 7 if we were allowed to produce a slightly suboptimal dilation.

5 Open Problems

Is there a constant $c > 1$ such that for every point set P, there exists a crossing-free spanning tree on P whose dilation is no more than c times that of a minimum-dilation tree? How fast can such tree be computed?

In view of the recent result by Mulzer and Rote [12] on the minimum weight triangulation, is it also NP-hard to construct the minimum dilation triangulation of a given point set?

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