Angel, Devil, and King ⋆

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Abstract. The Angel-Devil game is played on an infinite chess board. In each turn *the Angel* jumps from his current position to a square at distance at most k. He tries to escape his opponent, *the Devil*, who blocks one square in each move. It is an open question whether an Angel of some power k can escape forever. We consider Kings, who are Angels that can only walk, not jump. Introducing a general notion of speed for such modified pieces, we obtain an improvement on the current best Devil strategy. Our result, based on a recursive construction of dynamic fractal barriers, allows the Devil to encircle Kings of any speed below 2.

1 Introduction

Two players, *the Angel* and *the Devil*, play a game on an infinite chess board whose squares be indexed by pairs of integers. The Angel is an actual "person" moving across the board like some chess piece, while his opponent does not live on the board but only manipulates it. In each move, the Devil blocks an arbitrary square of the board such that this location may no longer be stepped upon by the Angel. The Angel in turn, flies in each move from his current position $(x, y) \in \mathbb{Z}^2$ to some unblocked square at distance at most k for some fixed integer k, i.e., to some position $(x', y') \neq (x, y)$ with $|x' - x|, |y' - y| \leq k$. Note that Devil moves are not restricted to the Angel's proximity or limited by any other distance bounds; he can pick squares at completely arbitrary locations.

The Devil wins if he can stop the Angel, that is, if he manages to get him in a position with all squares in the $(2k + 1) \times (2k + 1)$ area around him blocked. The Angel wins if he succeeds to fly on forever. The open question is, whether for some sufficiently large integer k the Angel with distance bound k, called the *k-Angel*, can win this game.

First variants of this game were discussed by Martin Gardner [7], who names D. Silverman and R. Epstein as original inventors. Though his article deals mainly with finite configurations, i.e., the question whether a chess king (which is simply a 1-Angel) can reach the boundary of a given rectangular board, he also asks for a strategy against a chess knight on an infinite board. In its present

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form the Angel game first appeared in Berlekamp, Conway, and Guy's classic [2] (Chapter 19). Amongst detailed analyses of games with kings and other chess pieces on finite boards against Devils with certain additional restrictions, the authors coin the names "Angel" and "Devil" and give a thorough proof that the chess king can be caught. Then Conway [4] focused entirely on the infinite Angel game, trying to explain possible pitfalls with certain natural escape attempts and pointing out the hardness of the problem. Besides all variants, the central open question remains whether some Angel of sufficient power can escape forever. In his overview article [5], Demaine cites it as a difficult unsolved problem of combinatorial game theory.

In this work, we present an improvement on the current best known Devil strategy. Therefore we introduce a reformulation of the original game, which allows us to focus on speed as the important parameter. We define a k-King to be a k-Angel who cannot fly but runs; that is, a k-King is allowed to make k ordinary chess-king steps per turn, where each single step has to use an unblocked square. We shall see that Kings and Angels are asymptotically equivalent (if some Angel can escape then also some King can, and vice versa) and that the concept of k-Kings naturally extends to fractional and even irrational speed (Definition 1). Our main result is the following:

Theorem 1. *The Devil can catch any α-King with $\alpha < 2$.*

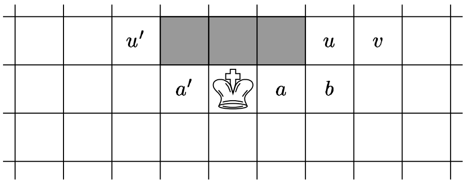
Many proof details, which have to be omitted here due to space constrains, can be found in the first author's thesis [9].

2 Basic Facts and Previous Results

The only case for which the k-Angel problem is solved is $k = 1$, the ordinary chess king. We like to sketch a winning strategy for the Devil, which resembles the analysis in [2]. These key ideas form the starting point for our proof of Theorem 1.

Assume the Devil wants to prevent the king from crossing a certain horizontal line. With three squares above the king already blocked on that line, like in the left of Figure 1, this is easily achieved. The Devil simply answers a king move a to the right with an extension of that triple block by a play at u. A further move to b is countered by v and likewise, a left movement to a' is blocked at u'. Pushing along in this simple fashion ensures that the king cannot cross. It is not difficult to get the three initial blocks placed on a blank line when a king is just approaching. By inspection of cases, one can show that five approach moves suffice for the Devil to create the basic triple.

The right of Figure 1 indicates how to turn the pushing argument into a Devil win. With his first moves, the Devil blocks a finite number of squares in the four corners of an imaginary box around the king, which is chosen large enough to ensure that during this preparatory phase the king does not get too close to the boundary. After that, the Devil plays the above wall-pushing strategy along the dotted lines whenever the king approaches the border. The solid corners are



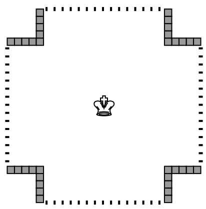


Fig. 1. Pushing the chess king along a line (left) and catching him in a box (right).

there to ensure that the Devil is never forced to play on two fronts at the same time.

The Fool argument. The first general idea for an escape with a k-Angel might be to run away in one direction. With sufficiently high power k, should not the Angel be simply too fast for the Devil? The answer is *no.* Conway [4] defines a *k-Fool* to be a k-Angel who commits himself to strictly increasing his y-coordinate in every move. He shows that a Fool of any power k can be caught. The Fool counter already indicates that devising an escape strategy for some Angel might be a very difficult task. By a dove-tailing argument the result can even be turned into the following surprising fact [4].

Theorem 2 ("Blass-Conway diverting strategy"). *There is a strategy for the Devil with the following property. For each point p of the plane and each distance d, no matter how the Angel moves, there will be two times $t_1 < t_2$ such that at time t_2 the Angel will be d units nearer to p than at time t_1.*

Angels in higher dimensions. For three dimensions, the Angel problem is solved. Independently, the first author [9] and Bollobás and Leader [3] proved that in $\mathbb{Z}^3$, and thus also in all higher dimensions, some Angel can escape.

Winning and losing infinite games. General infinite games may behave a little peculiar in so far as a clear winner need not always exist. The axiom of choice allows the construction of games in which neither player has a winning strategy, even though the game does not allow for draws [8, Sec. 43]. However, it is known [6, 10] that for reasonably well-behaved games this cannot happen: they are *determined.* For the Angel-Devil game it is not hard to show this property directly by a compactness argument, so we know that either the Angel or the Devil must have a winning strategy.

A further useful observation is that in a sense, the game is infinite only from the point of the Angel. If the Devil wins, the game ends, by definition, after finitely many moves. An application of König's lemma shows that in this case the Angel cannot delay his defeat arbitrarily.

Lemma 1. *If the Devil has a winning strategy against some Angel, then there exists a bound N such that the Devil can stop that Angel in at most N moves. Conversely, if the Angel can survive for any arbitrarily large, previously given number of steps then he can escape forever.*

3 The Need for Speed

There is pretty little known about even very weak Angels. Already the destiny of the 2-Angel is not settled and even more, it is unknown whether a chess knight can be caught. We do not have a solution for the 2-Angel, either, but we make a first step in this direction by devising Devil strategies against opponents whose power lies somewhere between that of a 1-Angel and a 2-Angel. The improvement appears rather modest but the new concepts we need to introduce in order to obtain them or even state them, reveal details of the game that seem to lie hidden with Conway's original Angel.

Let us take a closer look at what happens when we upgrade the original chess king to a 2-Angel. This is already a large step; the improvement is actually two-fold. Not only does the 2-Angel move at twice the speed, any barriers must also be twice as thick to hold him back. In a sense, the 2-Angel can be said to be 4 times stronger than the 1-Angel. We focus on the first aspect: *speed.* The Angel's ability to jump over obstacles shall be suppressed as an undesired side effect. Define a *k-King* as a player who in each turn makes exactly k ordinary chess-king moves, while the Devil still gets to place one block per turn. The point is that every single chess-king move must be valid. The k-King cannot fly.

If we want to use Kings for the study of the Angel problem, they should, in some qualitative sense at least, be equivalent to Angels. Obviously, a k-Angel is stronger than a k-King: An escape strategy for a King can be used for an Angel of the same power as well. The converse is, of course, not true—not for trivial reasons at least—but we can show that if you can catch Kings of arbitrary power k then you can also catch any Angel. Of course, the reduction from Angels to Kings requires an increase in speed.

Proposition 1. *If the k-Angel can escape then so can the $99k^2$-King.*

Proof (sketch). We derive an escape strategy for the $99k^2$-King from an escape strategy for the k-Angel. While the King plays against the "real" Devil, we set up an additional, imaginary board with an imaginary k-Angel, where we simulate the action on the King's board through appropriate transformations. The King's board is partitioned into a regular grid of sidelength-$18k^2$ boxes. Likewise, the Angel's board is segmented into blocks of sidelength k. The boxes of the two worlds are in one-to-one correspondence with each other, in the obvious fashion: the starting points lie in corresponding boxes and further, all adjacencies are preserved. These partitions and the correspondences are fixed once and for all.

We play as follows. When the Devil blocks some square in the King's world, we cross out an arbitrary empty square from the corresponding box in the Angel's world or from one of the eight adjacent boxes there. When it is the King's turn, we use our escape strategy for the Angel to get a move in the imaginary world. This move is then translated into the King's plane by a movement of the King into the corresponding box there. If, for example, the Angel jumps from his current box into the next box to the north, then the King runs into the northern box in his world, too. The precise position within that box is completely independent of the Angel's position in his box, however. It depends on technical details which we must skip here for brevity. They have to guarantee that the King only stops at locations from where the four lines into the four axis parallel directions within the current box are completely free. This invariant then ensures a free passage for the King into the next target box, which takes no more than $99k^2$ steps. □

We emphasize again that the quantitative proportion of the above reduction is not our main concern. The purpose of Proposition 1 is to establish the qualitative equivalence between Angels and Kings, as a legitimation to use Kings as a tool to attack the Angel problem.

4 Real Kings

For Theorem 1 to make sense at all, we need to define what Kings of fractional speed shall be. So what is a 3/2-King? On average he should get to make three King steps for 2 Devil steps, which we could realize by a move sequence like $KKKDDKKKDD\ldots$, which shall mean that the King makes 3 steps, then the Devil blocks 2 squares, and so on. However, such a concept would depend on the actual representation of a rational number. The 6/4-King would get a different sequence. We could get around this by demanding reduced fractions but then a 1001/8-King would behave completely different from a 1000/8-King, who should simply be the 125-King. What is worse, the grouping of Devil moves could be lethal for the King. For example, the eight consecutive Devil moves in the sequence $K^{1001}D^8K^{1001}D^8\ldots$ could be used to encircle the King completely, even though his average speed would be greater than 125.

What we want are move sequences that approximate a given speed $\alpha \in \mathbb{R}^+$ as fair as possible, avoiding unnecessarily large chunks of moves for either side. The sequence $(u_n)_{n\in\mathbb{N}}$ defined by

$$u_n = \lfloor (n+1)\gamma + \phi \rfloor - \lfloor n\gamma + \phi \rfloor \in \{0,1\} \quad \text{with } \gamma = \frac{\alpha}{\alpha+1} \in (0,1) \qquad (1)$$

and some constant offset $\phi \in \mathbb{R}$ shows this behavior—if we interpret 1's in the sequence as King and 0's as Devil moves. The basic behavior of such *sturmian sequences* is easy to understand (see [1] for a broad treatment and for historic references). Expression (1) simply compares consecutive elements of the arithmetic progression $(n\gamma + \phi)$. Whenever there lies an integer between the nth and the $(n+1)$st element of $(n\gamma + \phi)$ we have $u_n = 1$, otherwise, when the two elements fall in a common integer gap, (1) evaluates to $u_n = 0$. We conclude that the frequency of 1's in (u_n) is γ; hence the frequency of 0's is $1-\gamma$ and we get (cf. [1])

$$\lim_{n\to\infty} \frac{|\{i \le n : u_i = 1\}|}{|\{i \le n : u_i = 0\}|} = \frac{\gamma}{1-\gamma} = \alpha.$$

Definition 1. *For $\alpha \in \mathbb{R}^+$ we define the α-*King *to be a King whose move sequence is given by (1) with $\phi = 0$. This means that in the n*th *time step the King moves by one square if $u_n = 1$ and the Devil gets to block a new square if $u_n = 0$.*

The choice of the offset ϕ looks arbitrary. For a natural definition it is desirable that the chances of the α-King in the game do not depend on this parameter. And in fact, this can be shown.

Lemma 2. *Any two Kings with move sequences generated by* (1) *with the same speed parameter* α *but different* ϕ*'s either can both escape or can both be caught.*

For integral α, the above definition of an α-King obviously coincides with the previous definition of a k-King. For $\alpha = k \in \mathbb{N}^+$, the defining sequence (1) produces exactly k many 1's between any two consecutive 0's, just as expected. It is also clear that our notion of an α-King fulfills our wish for fairness. Large chunks of Devil moves cannot occur. One easily checks that for $\alpha \geq 1$ the Devil never gets to block two squares at a time. On the other hand, we can guarantee that not only in the long run but also locally, the Devil always gets his share of moves.

Definition 2. *A 0/1-sequence is* (s,t)-bounded, $s,t \in \mathbb{N}^+$, *if every contiguous subword that contains strictly more than* s *occurrences of 1's contains at least* t *occurrences of 0's. We call a King with a given move sequence* (s,t)-bounded *if that sequence is* (s,t)*-bounded.*

Lemma 3. *An* α*-King,* $\alpha \in \mathbb{R}^+$*, is* (s,t)*-bounded for every pair* $s,t \in \mathbb{N}^+$ *with* $\alpha \leq s/t$.

The "strictly" in the definition appears for a technical reason. Namely, starting from any 1 in the sequence, we count 0's until we reach the $(s+1)$st 1. By then we have passed at least t many 0's. When we read on until the $(2s+1)$st 1 shows up, we are sure to have counted at least $2t$ many 0's. And so on. Before the $(rs+1)$st 1 appears, we are guaranteed to read at least rt many 0's.

5 Low-Density Barriers

Let us have a closer look at the Devil strategy against the 1-King from the beginning. It seems we wasted some potential there. After the preparation of the corners, the Devil simply sits and waits for the King to arrive at one of the four sides. We can exploit this potential advantage.

The basic idea for the King counter was our dynamic-wall argument, where we had the King pushing along a line without ever letting him break through. Against a 2-King the Devil would need some blocks already in place in order to carry out the same principle. With every second square blocked in advance, the King cannot break through. Starting from the initial position in Figure 2 with only two additional squares blocked, the Devil can push along with the 2-King by answering the double move a_1, a_2 at u, then b_1, b_2 at v, and so on.

How long would it take the Devil to prepare such a density-1/2 wall against the 2-King? Since he needs to block 1 square out of 2, he can set up such a wall at an absolute speed of 2, which is exactly the speed of the 2-King. In other words, the Devil can build such fences against the 2-King at the same speed the 2-King runs. However, for Theorem 1 this will not be enough, yet; we need barriers of lower, fractional densities.

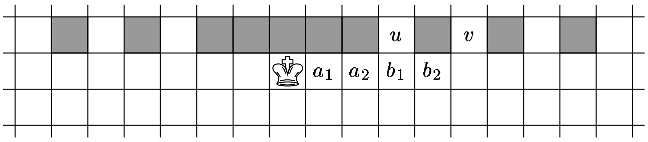


Fig. 2. A wall against the 2-King.

Definition 3. *An* infinite (s,t)-fence *is an infinite horizontal or vertical strip in the plane with some squares blocked such that when an (s,t)-bounded King enters the strip from one side, the Devil can play in a way that prevents the King from leaving it on the other side. Formally, such a fence is a map $F\colon \mathbb{Z} \times [1\,..\,w] \to \{0,1\}$, where $F^{-1}(1)$ is the set of blocked squares. The integer w is called the* width *of F.*

We call such a fence periodic *if there exists some integer λ such that $F(x,y) = F(x+\lambda, y)$ for all $x \in \mathbb{Z}$. Call the minimum such λ the* period *of F. In this case we also define the* density *of the fence, as the ratio*

$$\frac{1}{\lambda}\big|\{(x,y) \mid 1 \le x \le \lambda, 1 \le y \le w, F(x,y) = 1\}\big|.$$

Note that density is measured with respect to length, not area. Width is not the crucial quantity, it appears for merely technical reasons.

Lemma 4. *Against an (s,t)-bounded King, $1 < s/t \le 2$, there exists a periodic infinite fence of density $1 - t/s$ and width $10s + 1$.*

Proof (sketch). We follow the idea of Figure 2 for the 2-King. Define $F\colon \mathbb{Z} \times [1\,..\,10s+1] \to \{0,1\}$ by letting F be everywhere zero except at those points (x,y) with $0 \le x \bmod s < s - t$ and $y = 5s + 1$. In other words, we group the central horizontal line $y = 5s+1$ into segments of s squares and place $s - t$ blocks in each segment, as shown in Figure 3. The density of this pattern is obviously the claimed $(s-t)/s$. We omit the precise mechanism of the fence due to space constraints. The width of $10s+1$ is required to grant the Devil some preparatory moves when the King enters the strip. □

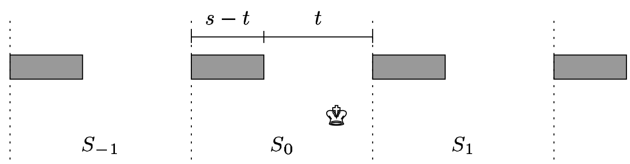


Fig. 3. An infinite (s,t)-fence.

It is important to note that our fences are dynamic, in the sense that the Devil has to play in them while the King tries to break through. So the Devil will not have time to play somewhere else while he defends such a barrier. On the other hand, the density describes the construction cost which has to be spent before the King reaches the fence. So what the Devil wants are fences of low density. Of course, he cannot build infinite structures in finite time. Infinite fences serve as a mere theoretical concept, which is easier to handle than finite fences, whose existence can be easily derived from the infinite ones.

Definition 4. *A* finite (s,t)-fence *is a rectangular box of size $\ell \times w$ in the plane with some squares blocked, such that when an (s,t)-bounded King enters through one of the length-ℓ sides he can only leave through that side again, and such that all squares along the two length-w sides are blocked. Formally, such a fence is a map $F\colon [1\mathinner{..}\ell] \times [1\mathinner{..}w] \to \{0,1\}$, where $F^{-1}(1)$ is the set of blocked squares. The integers ℓ and w are called the* length *and* width *of F, respectively. The* density *of the fence is the ratio*

$$\frac{1}{\ell}\left|\{(x,y) \mid 1 \le x \le \ell, 1 \le y \le w, F(x,y) = 1\}\right|.$$

The following transformation of an infinite fence into a finite fence is not very difficult.

Lemma 5. *If there exists a periodic infinite (s,t)-fence of density σ then there exist finite (s,t)-fences of the same width w and of density no more than $\sigma + 2w/\ell$ for any length $\ell \ge 1$.*

The $2w/\ell$ term comes from the solid walls to the sides, which are of mere technical relevance. It can always be made arbitrarily small by working with sufficiently long fences, only.

6 A Fractal Fence

Lemma 4 provides us with an infinite fence of density $1 - t/s$, which is strictly smaller than $1/2$ for any α-King with $\alpha < 2$. However, this does not yet suffice to catch any such King, yet. The trick is to assemble many such fences into a huge new fence of slightly smaller density. Iterating this process we will eventually produce fences of arbitrarily small density, which will be very cheap for the Devil to build. The key tool is the following lemma.

Lemma 6. *If there exist finite (s,t)-fences, $s/t \le 2$, of any length above some value ℓ_0, all of the same width w and with density bounded by a common $\sigma < 1/2$, then there also exists a periodic infinite (s,t)-fence with density below $(s/t)\sigma^2$.*

Proof (sketch). The basic idea is to assemble infinitely many identical vertical finite density-σ fences to a wide horizontal fence L. (Requiring only fences longer than a lower bound ℓ_0 is a technical necessity. Because of the solid side walls of size $2w$, very short fences can never have low densities.) As the length ℓ of those finite fences we simply pick any multiple of s larger than ℓ_0 and w, and the gaps between the fences be $m := \lfloor t\ell/s\sigma \rfloor \ge \ell$ squares wide. The width of the big infinite fence L be 7ℓ. The left of Figure 4 shows how the vertical fences are placed in the central ℓ-strip of L. Dashed lines depict the open borders, solid lines the solid side walls. The gray areas are the regions that require permanent Devil play as soon as the King enters if a break though to the other side shall be avoided.

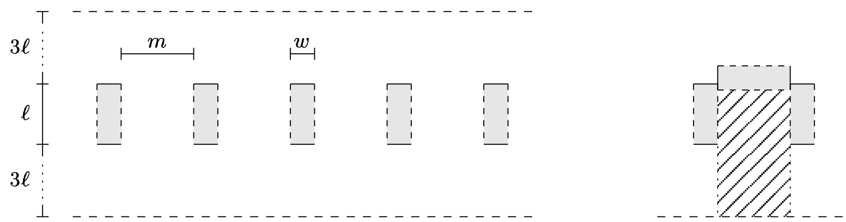


Fig. 4. Assembling many finite vertical fences into one big infinite horizontal fence (left) and blocking a slot (right).

The density of L is easily computed to lie below the required $(s/t)\sigma^2$. Showing that L is indeed an (s,t)-fence is the difficult part. We sketch the key ideas. Assume that the King enters L from the south, so we have to keep him from reaching the upper border. The plan is to build a horizontal fence of length m between the upper ends of two vertical fences whenever the King runs north between them, as indicated in the right of Figure 4. The shaded area there, between two vertical fences and below the (potential) horizontal fence, we call a *slot*. We say that the King is *in standard position* if he is located within a slot whose upper border is already closed or if he sits between two such blocked slots, perhaps within the vertical fence between them.

It is not difficult for the Devil to reach an initial standard position. The first 3ℓ King steps give him enough time to close two or three adjacent slots above the approaching King. The harder part is to keep forcing the King from standard position to standard position as long as he remains in L.

So assume that the King is in standard position. When he enters one of the three surrounding fences, the Devil follows the strategy of that respective fence to make sure that the King does not break through. (Since those fences do not overlap, the Devil is never forced to play in two fences simultaneously.) Hence, the King cannot leave the current slot above line 3ℓ without rebouncing from the fences. If the King leaves the slot that way below, to the left, say, the Devil starts constructing the horizontal fence across the slot to the left. This takes no more than $m\sigma = \lfloor t\ell/s\sigma \rfloor \sigma \leq t\ell/s$ Devil moves. During this time the Devil completely ignores the King's play. In particular, he does *not* respond to the possible King's crossing of any fences, thus rendering them ineffective. The clue is that this period of inactive fences is too short for the King to reach any upper fence or the fence to the right of the old slot, and crossing the vertical fence to the left is useless because soon the Devil has the horizontal fence in place there, too. So after the construction, the King will be in standard position again. $\square$

The proof of Theorem 1 is now straight-forward. Pick positive integers s and t with $\alpha \leq s/t < 2$, so that the α-King is (s,t)-bounded by Lemma 3. Then Lemma 4 provides us with an infinite periodic (s,t)-fence of density $\sigma < 1/2$ and repeated application of Lemmas 5 and 6 yields fences of smaller and smaller densities, which converge to zero. In a game against the α-King, the Devil can now arrange four such finite (s,t)-fences of very small density along the four sides of a huge square around the King, who will not be able to reach the boundary of that square before the fences are ready and thus will never be able to leave that prison.

7 Outlook

The immediate open question is, of course, whether the 2-King can be caught—perhaps with the techniques from this paper. This appears probable but observe that we do not have a simple compactness argument by which we could conclude such a statement directly from Theorem 1.

More generally, we hope that α-Kings allow for further small improvements that might bring us gradually closer to new Angel results. On the other hand, it is not unlikely that—in case some Angel is able to escape at all—King speed 2 is already the threshold between winning and losing.

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