Angel, Devil, and King

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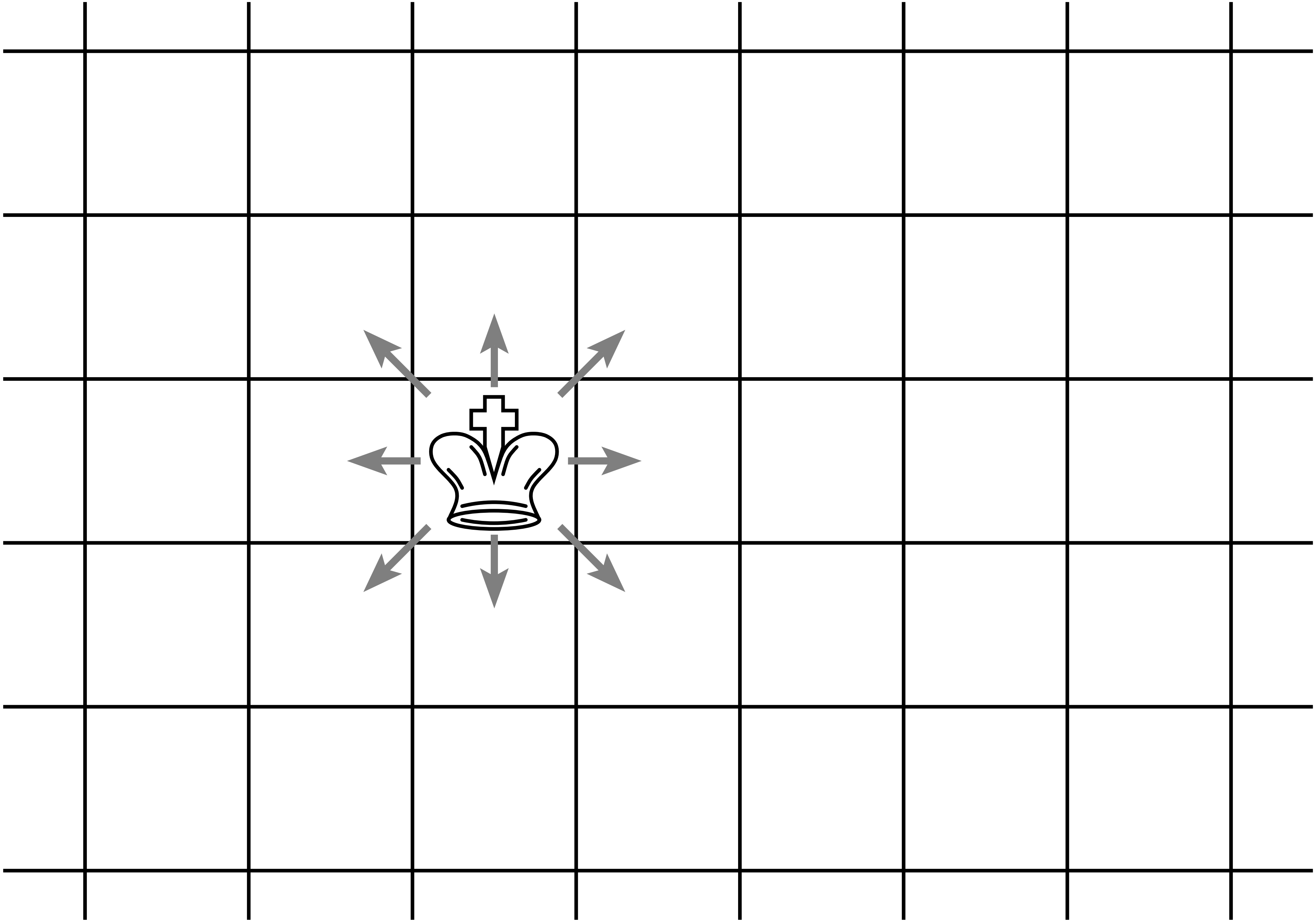

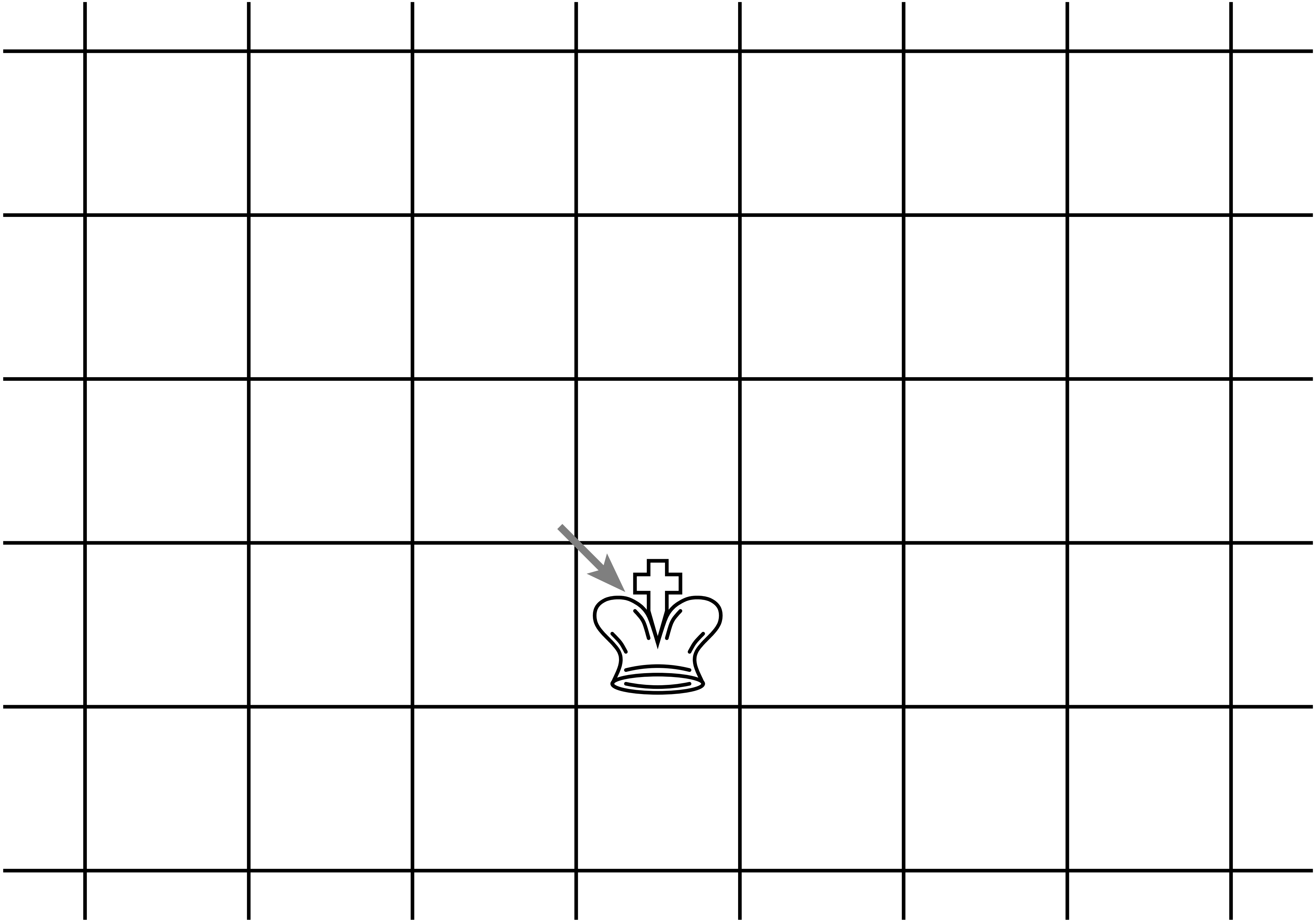

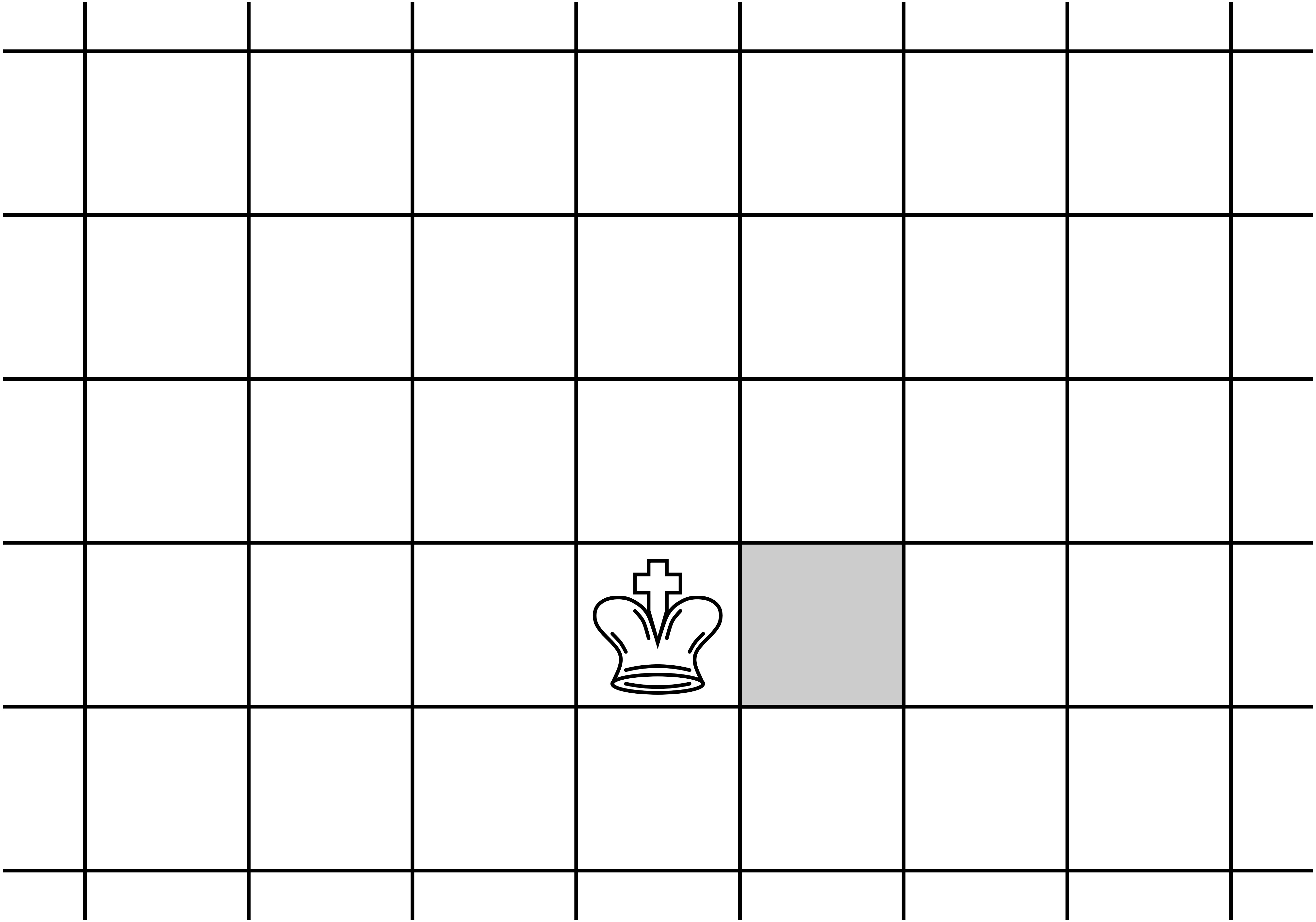

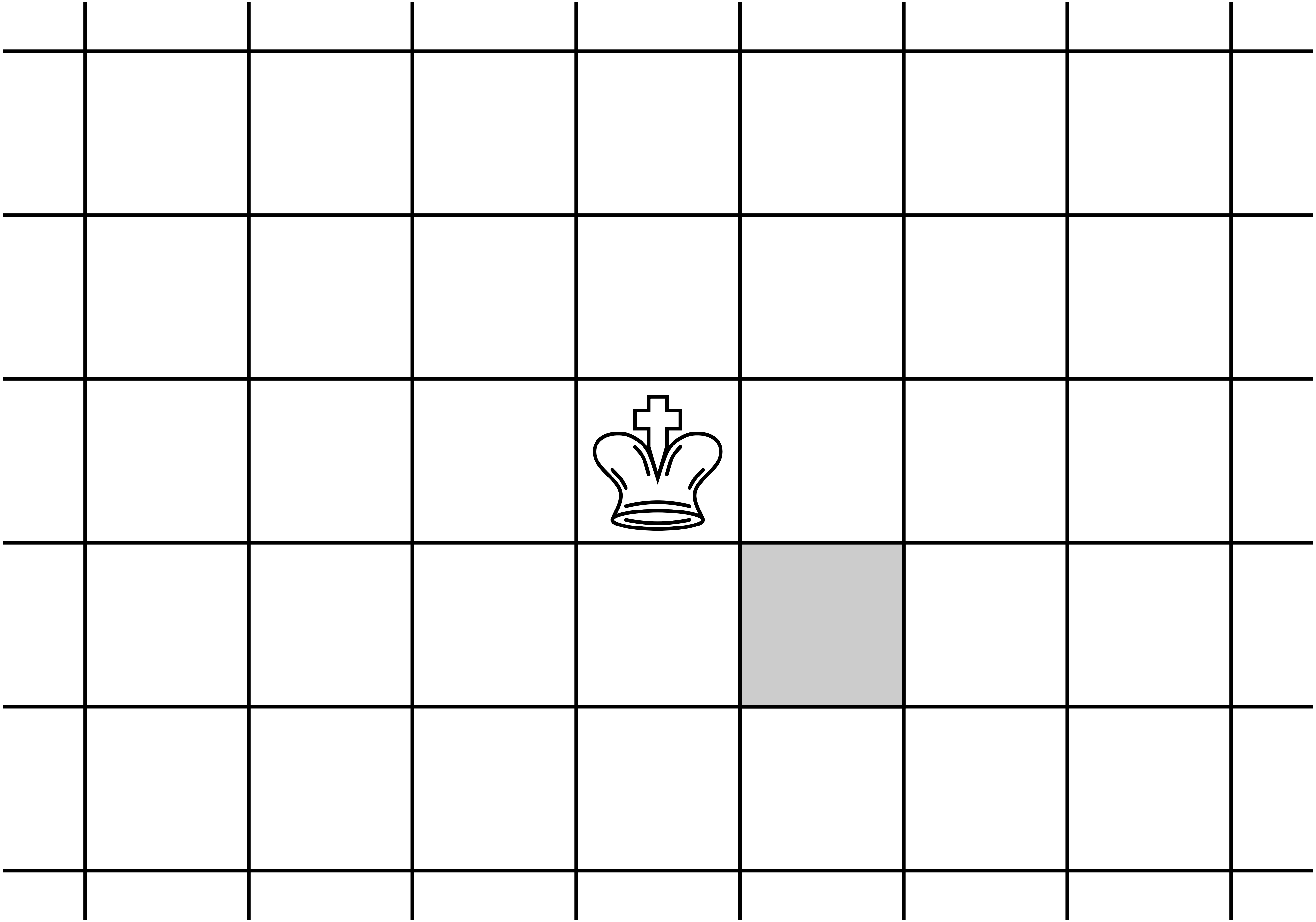

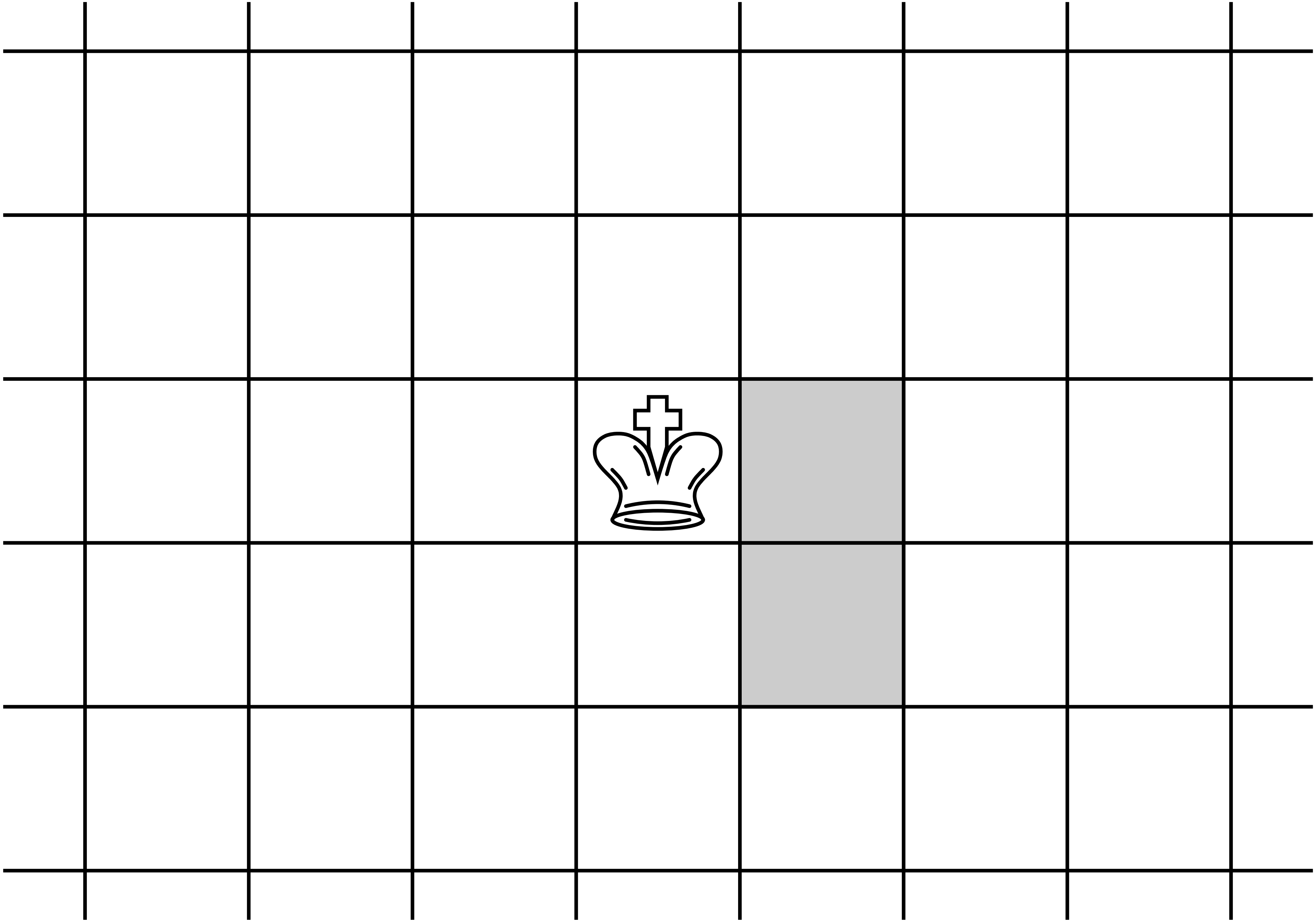

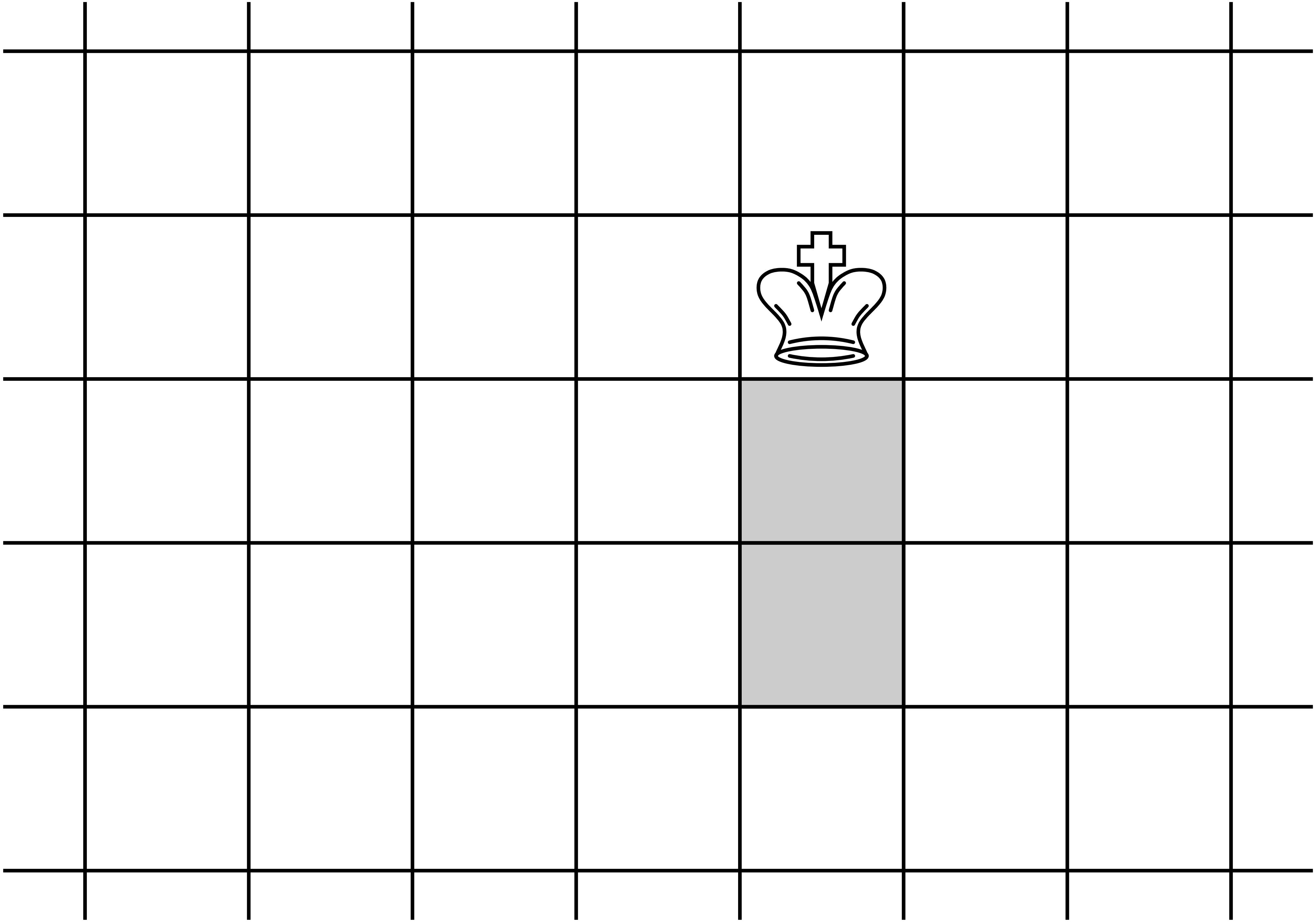

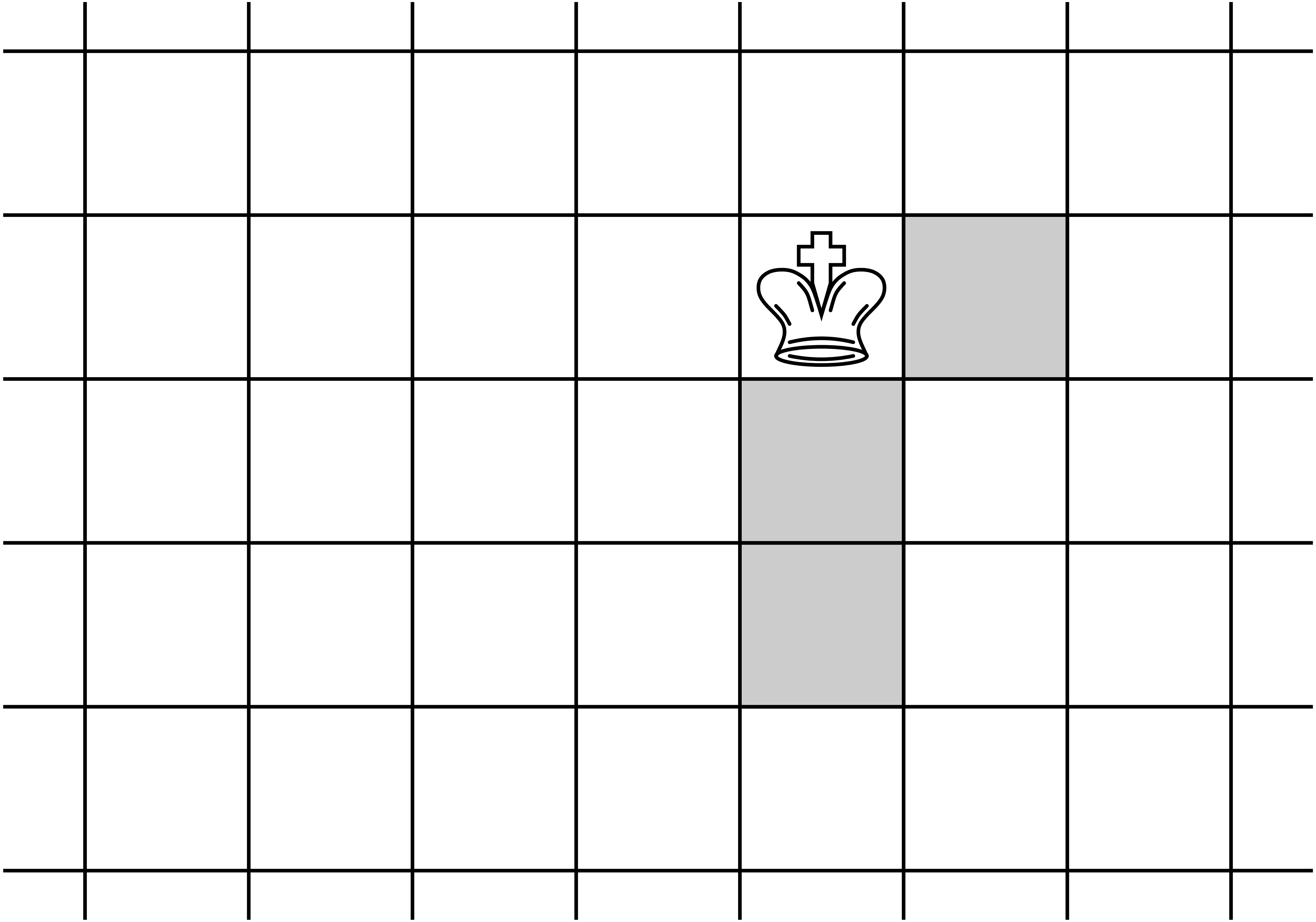

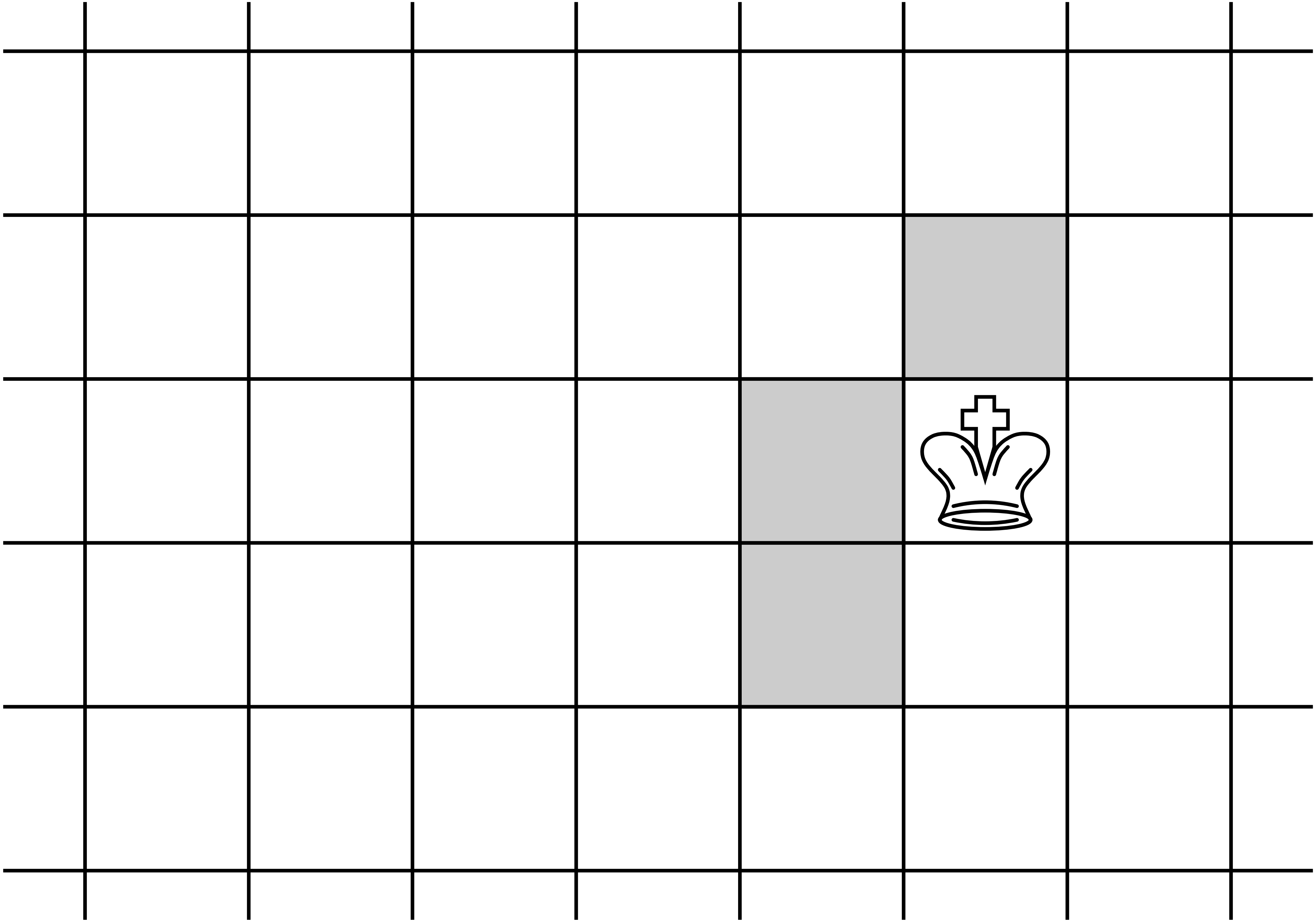

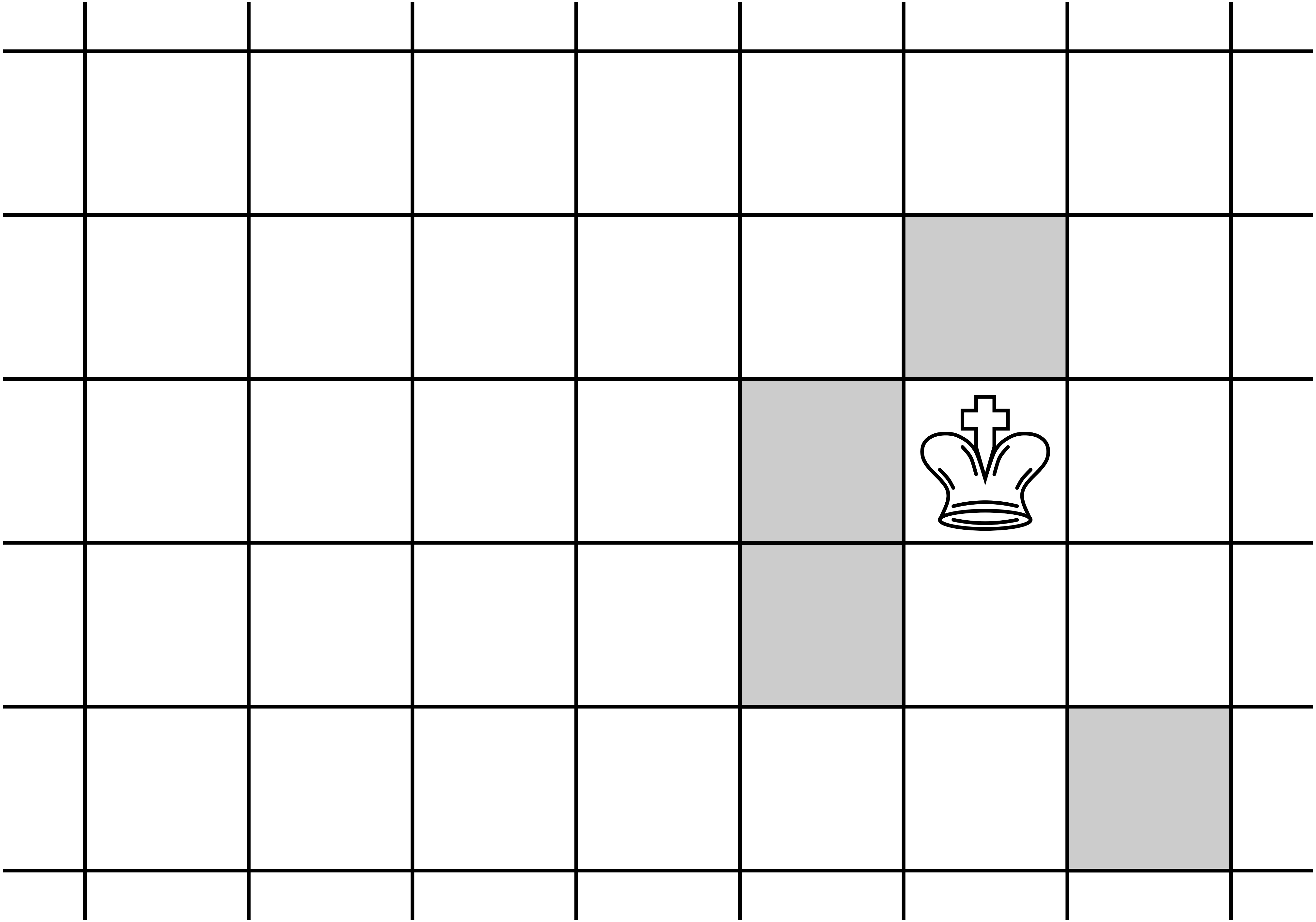

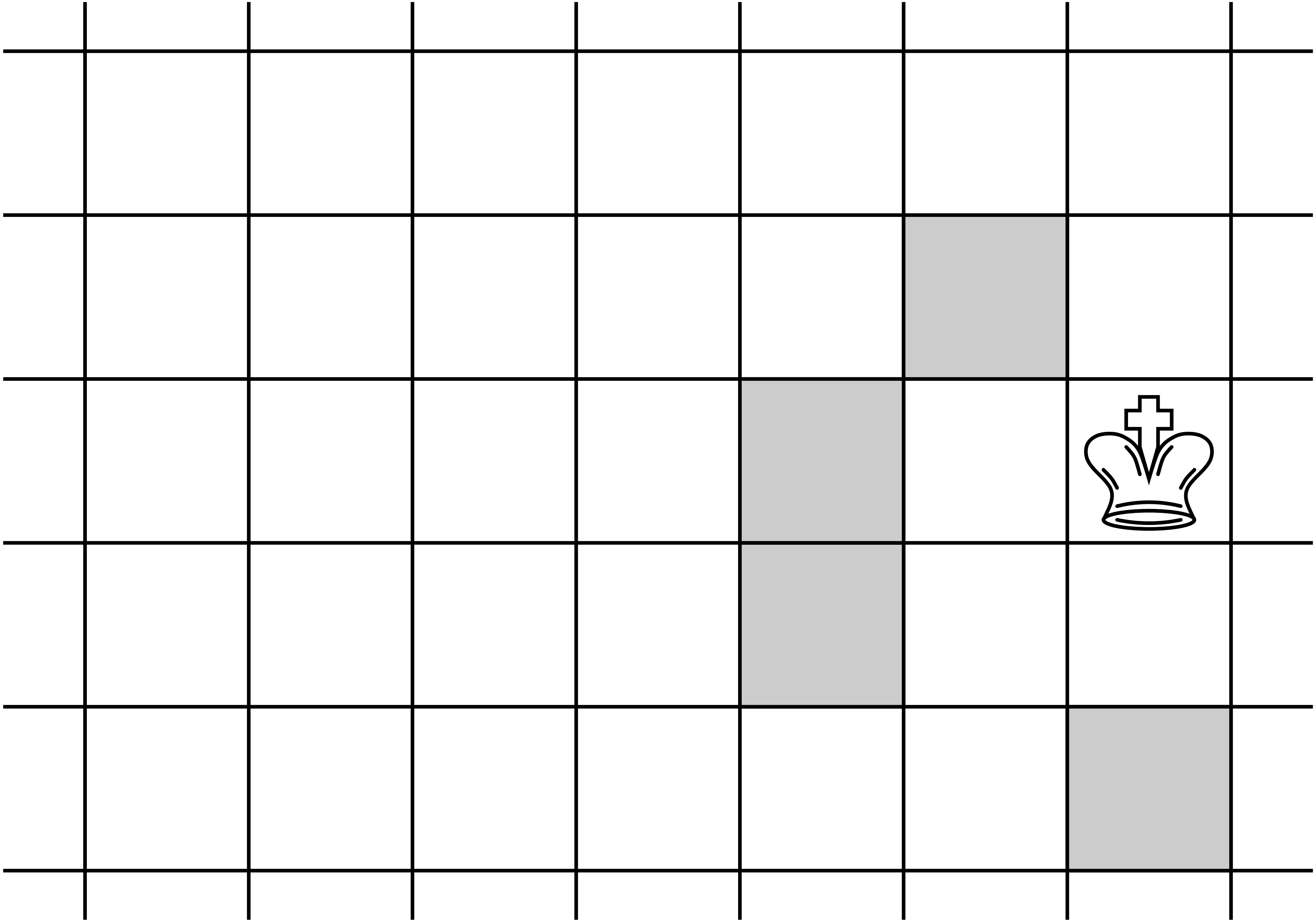

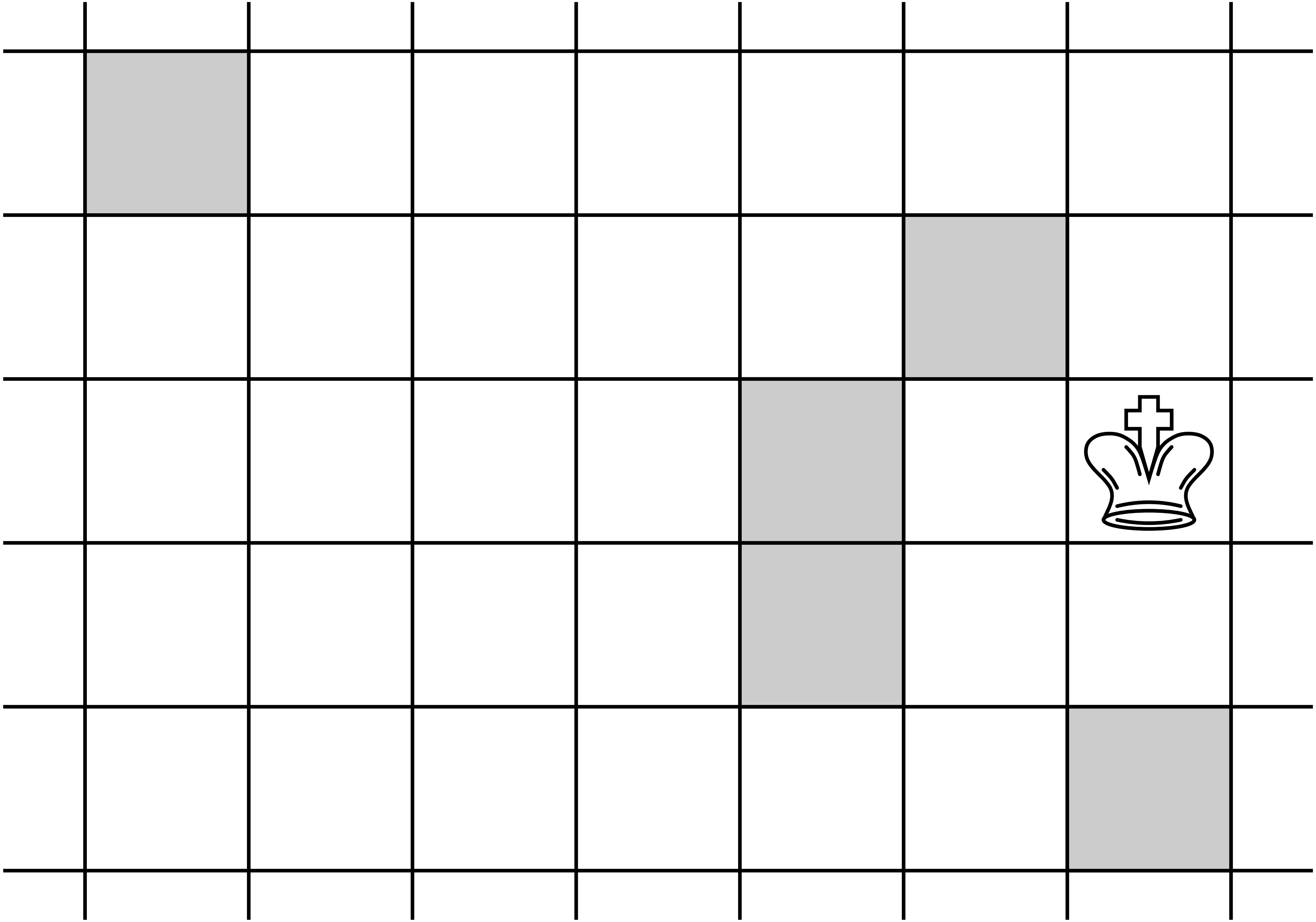

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The chess king can be caught on an infinite checkers board.

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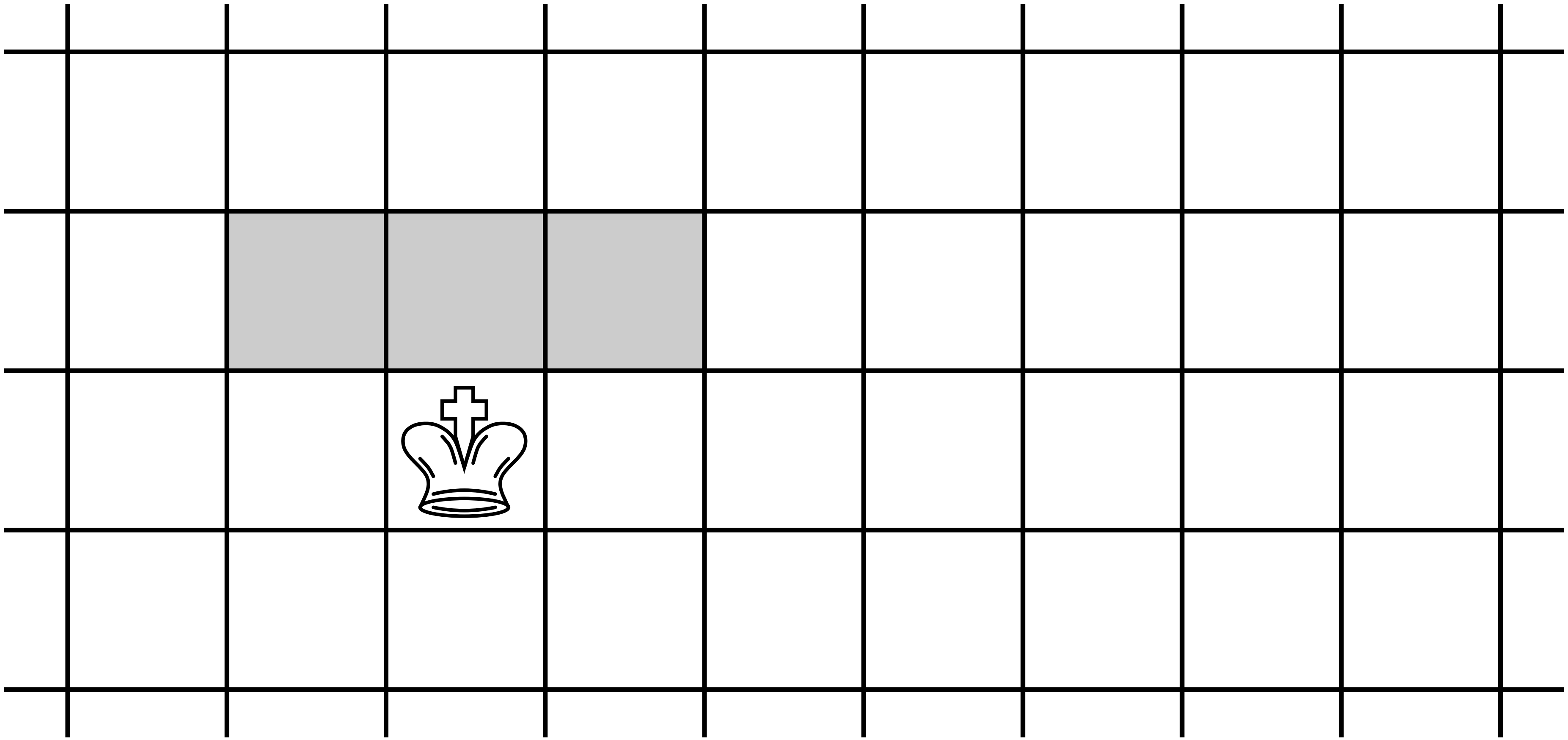

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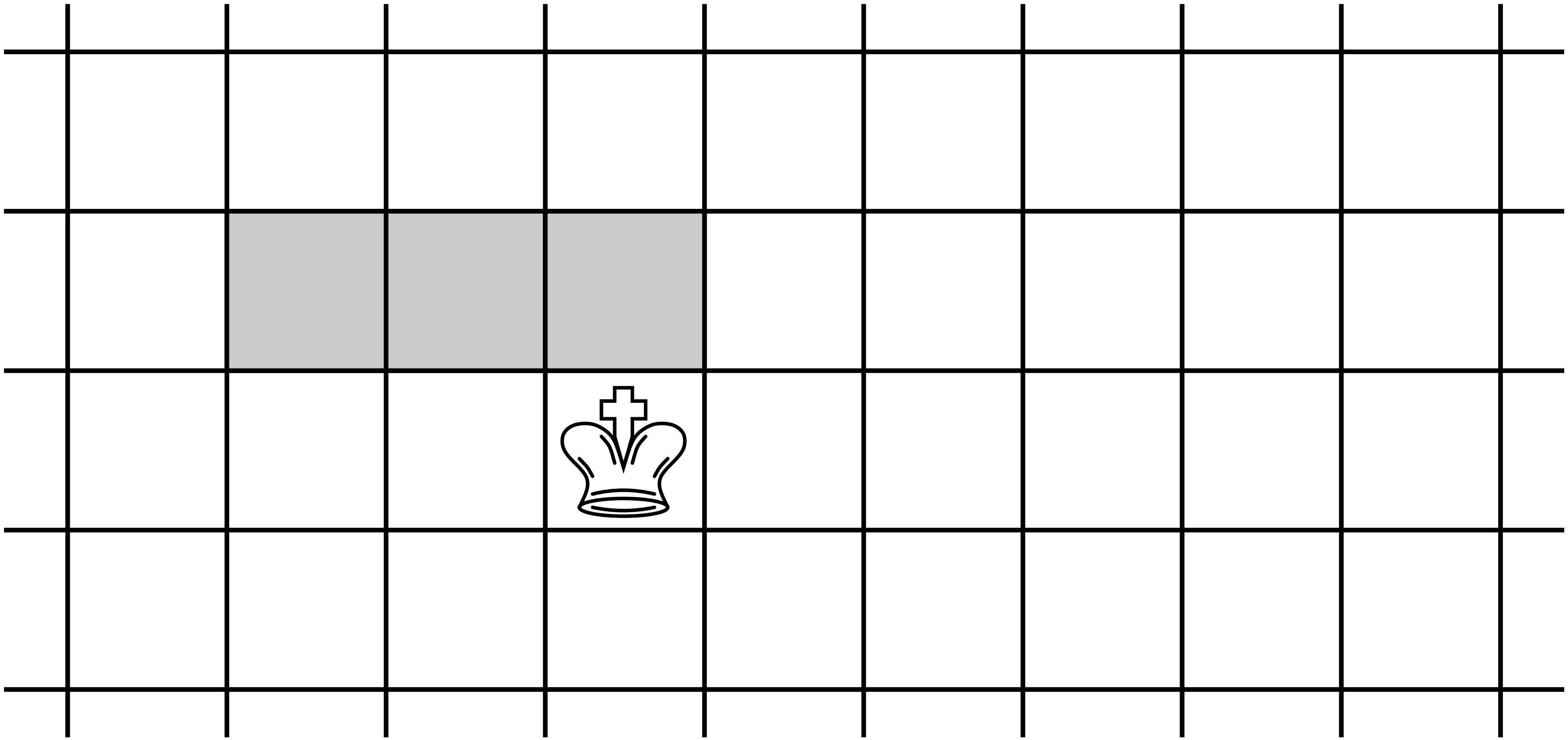

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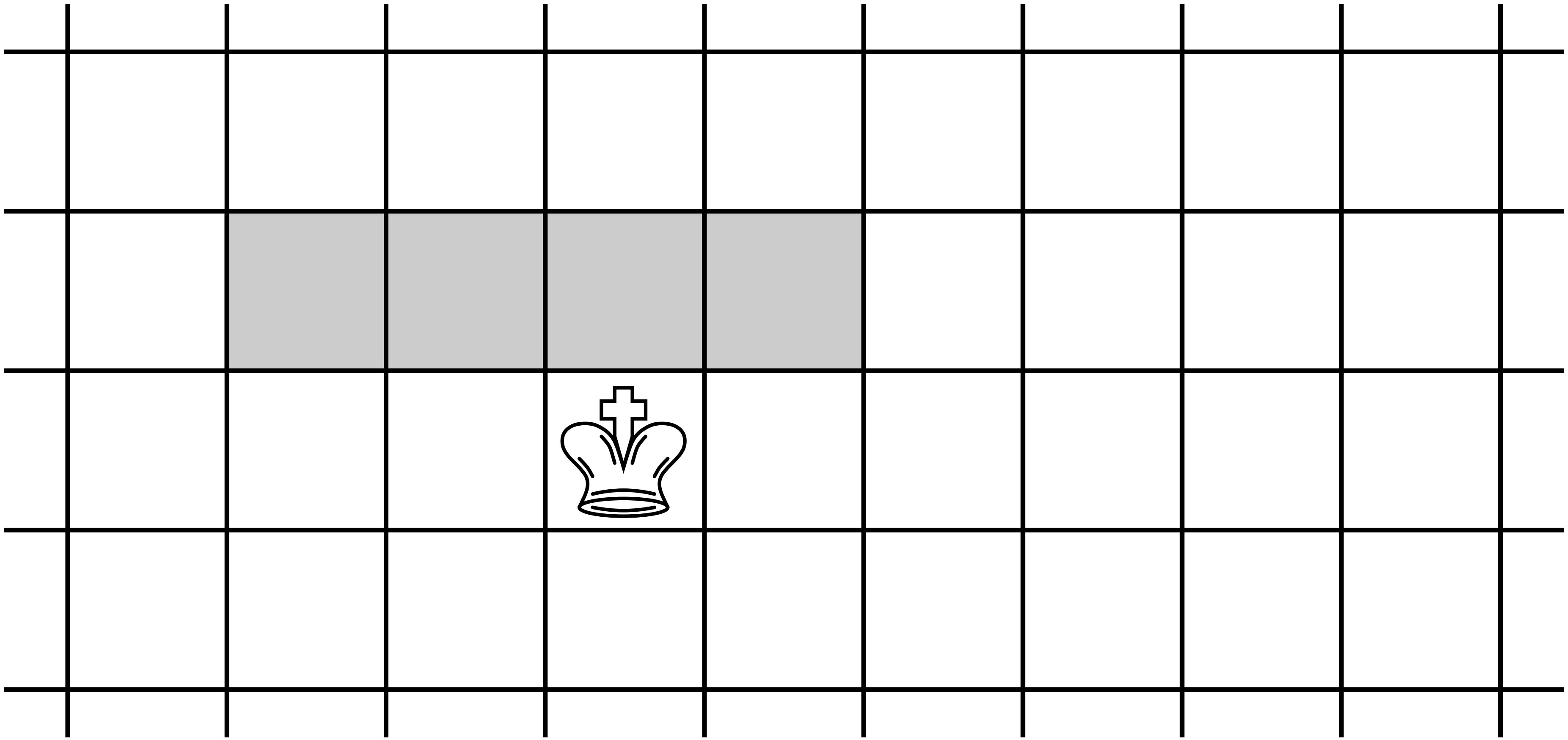

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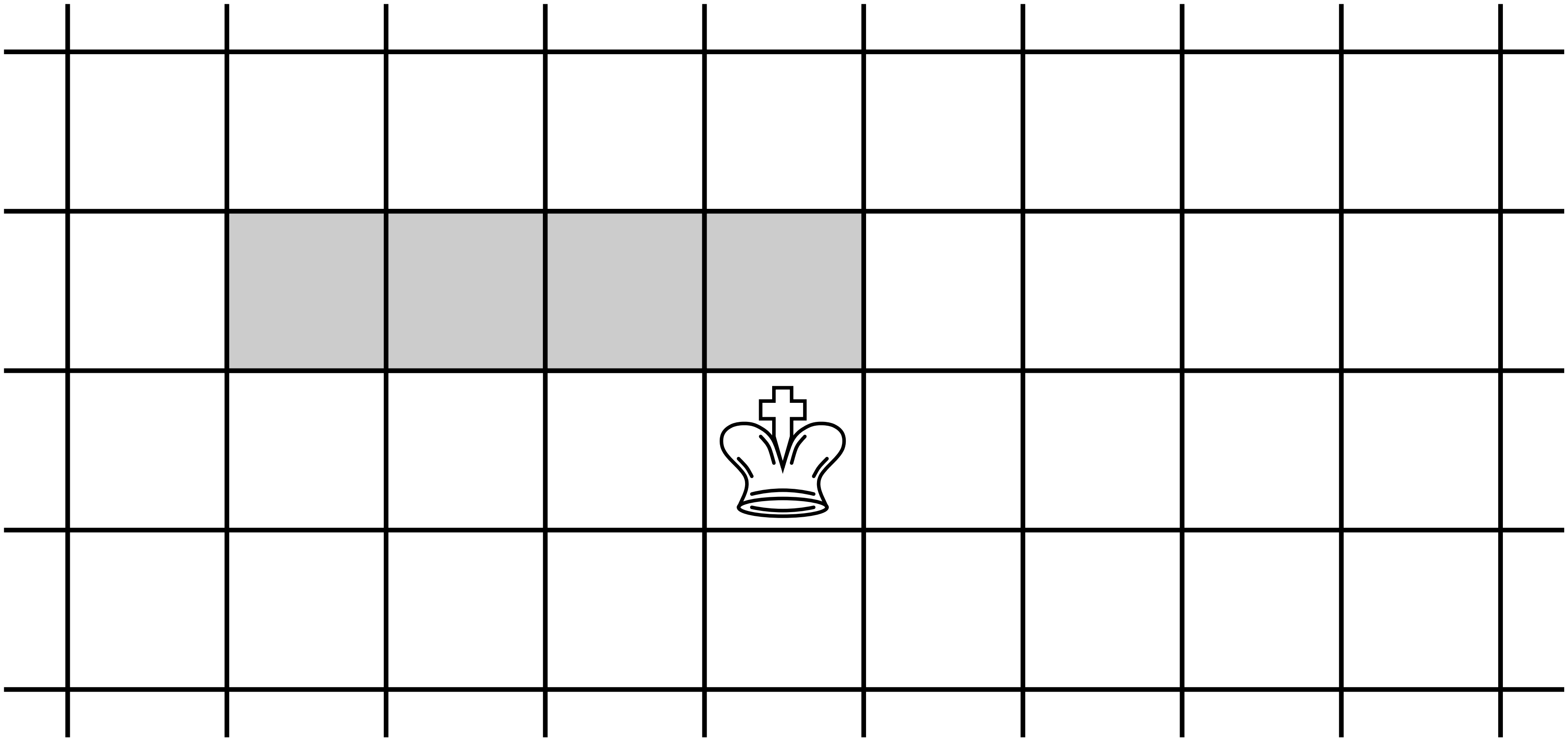

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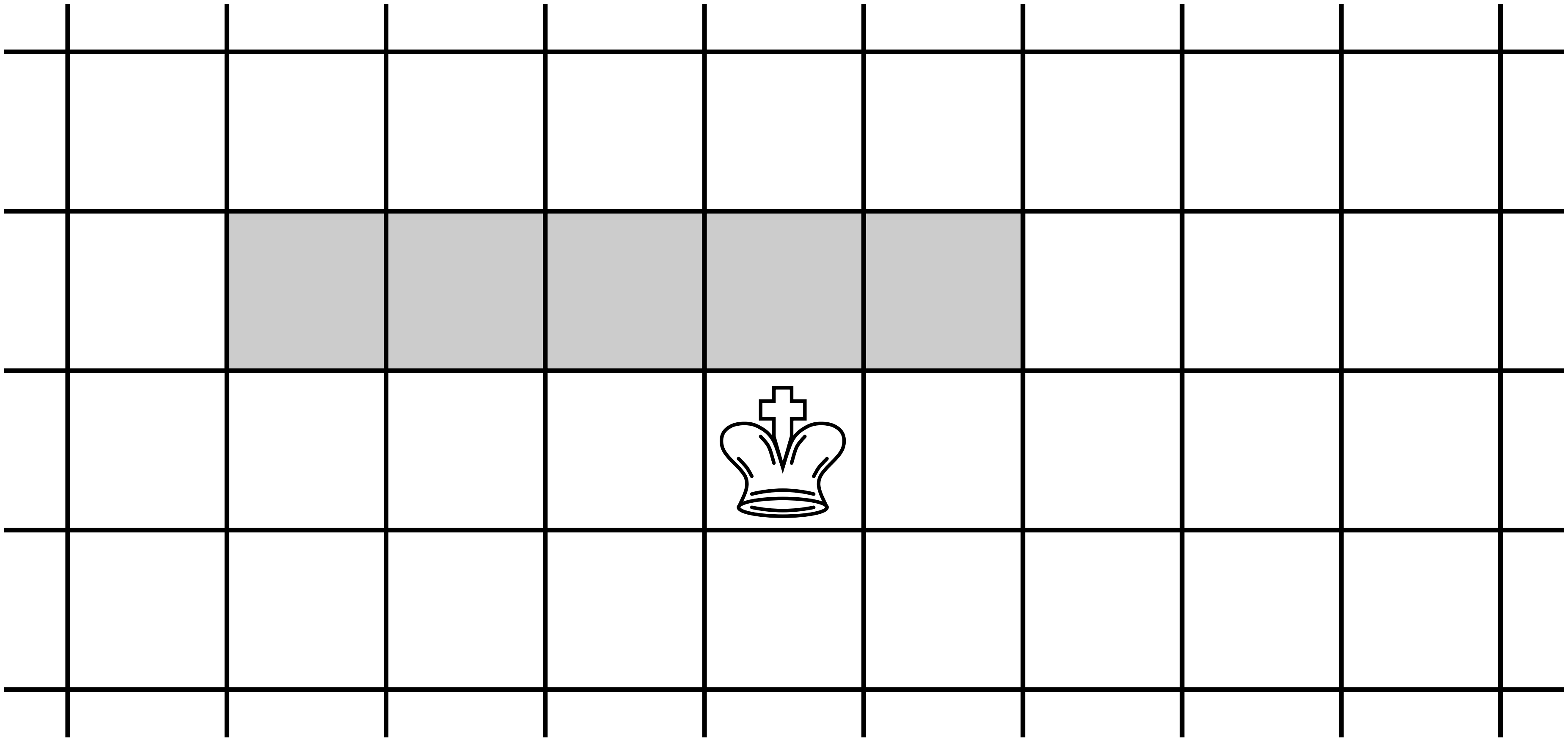

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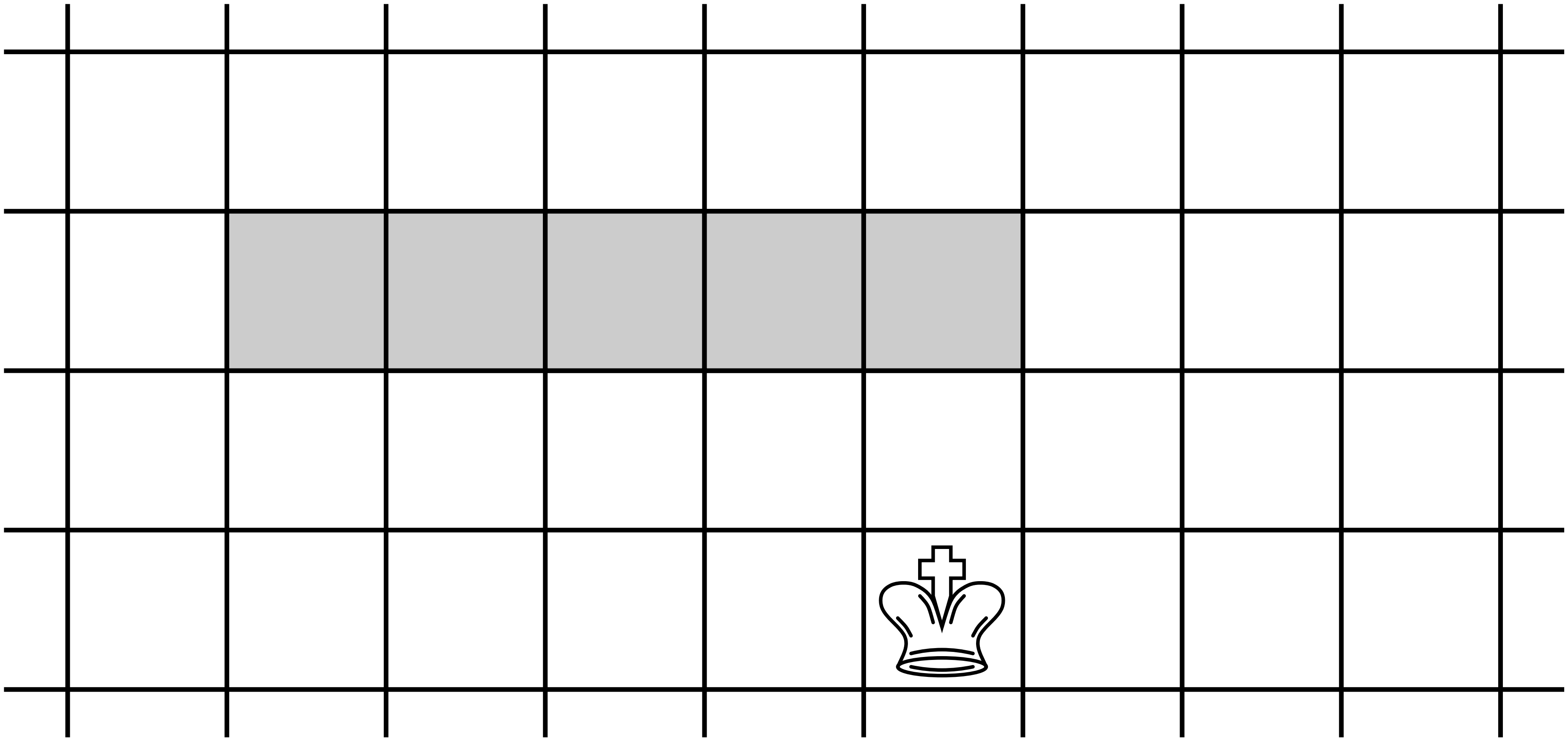

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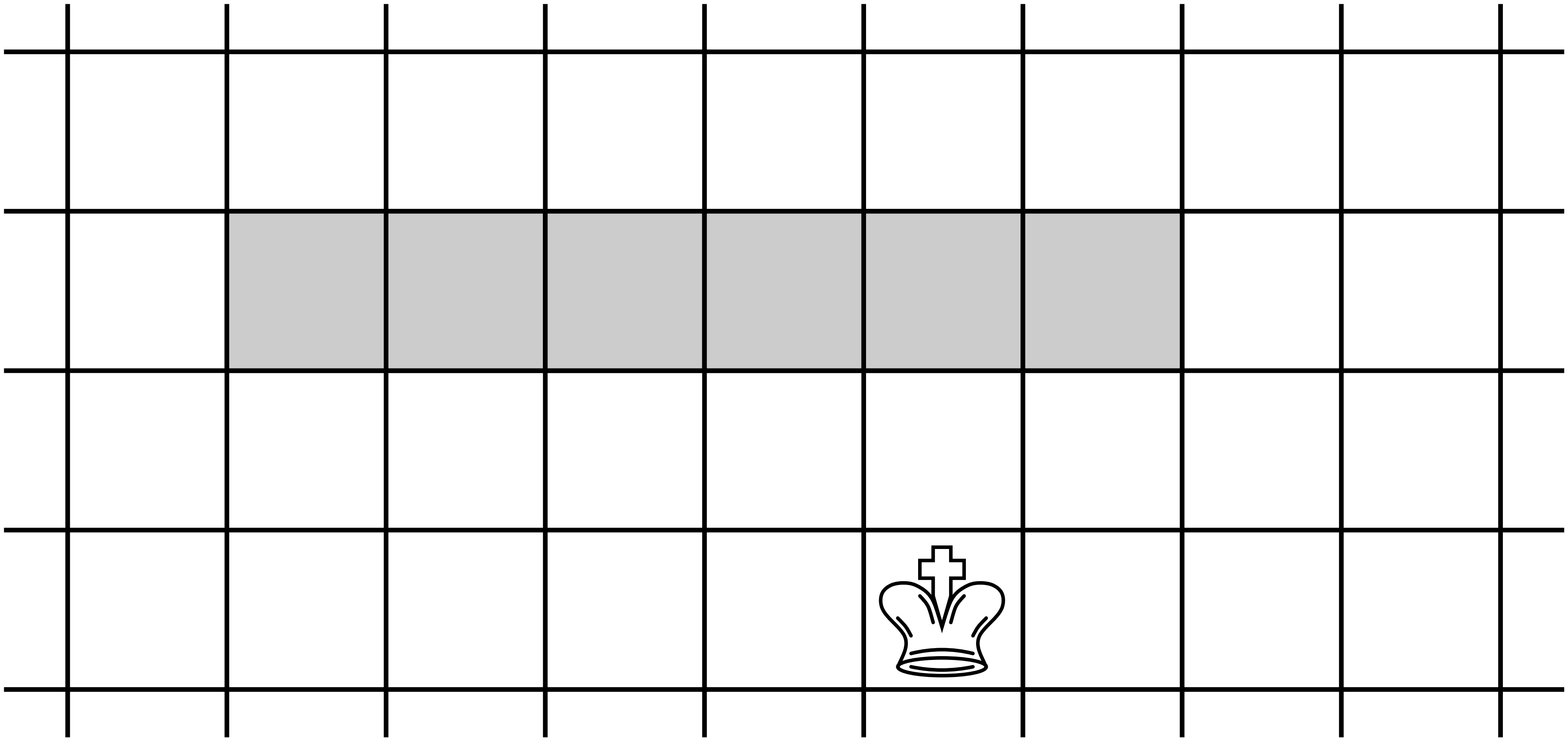
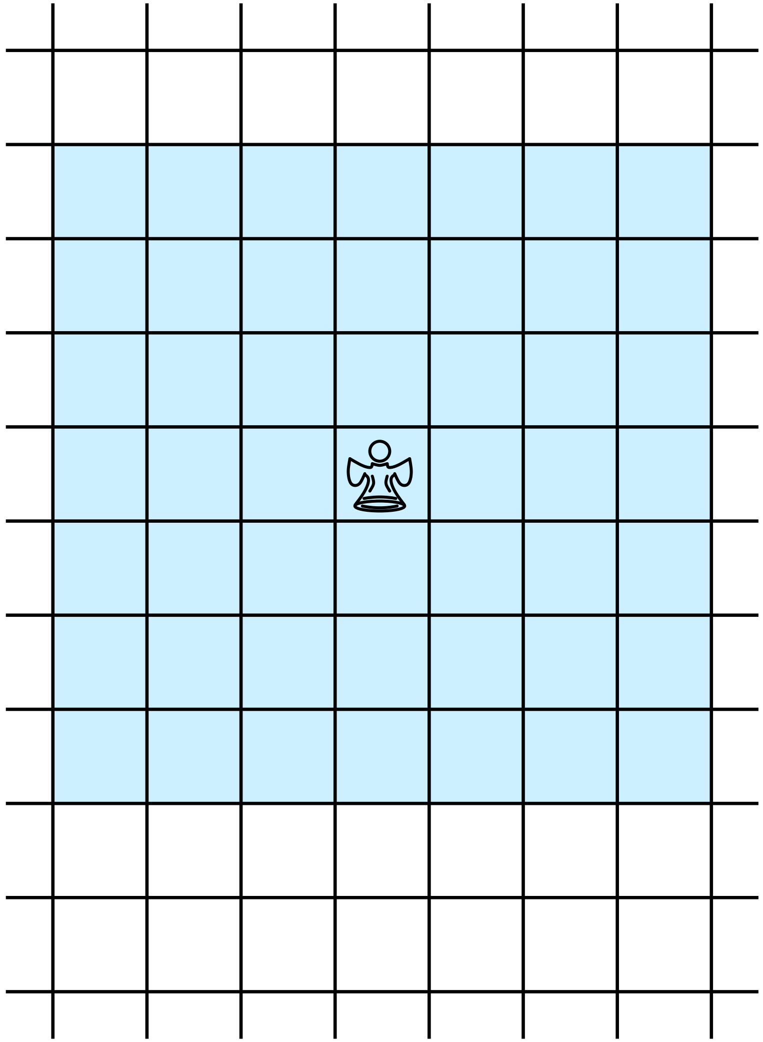


The Angel Problem

Definition
[Berlekamp, Conway, Guy]

A k-Angel can "fly" in one move to any unblocked square at distance at most k.



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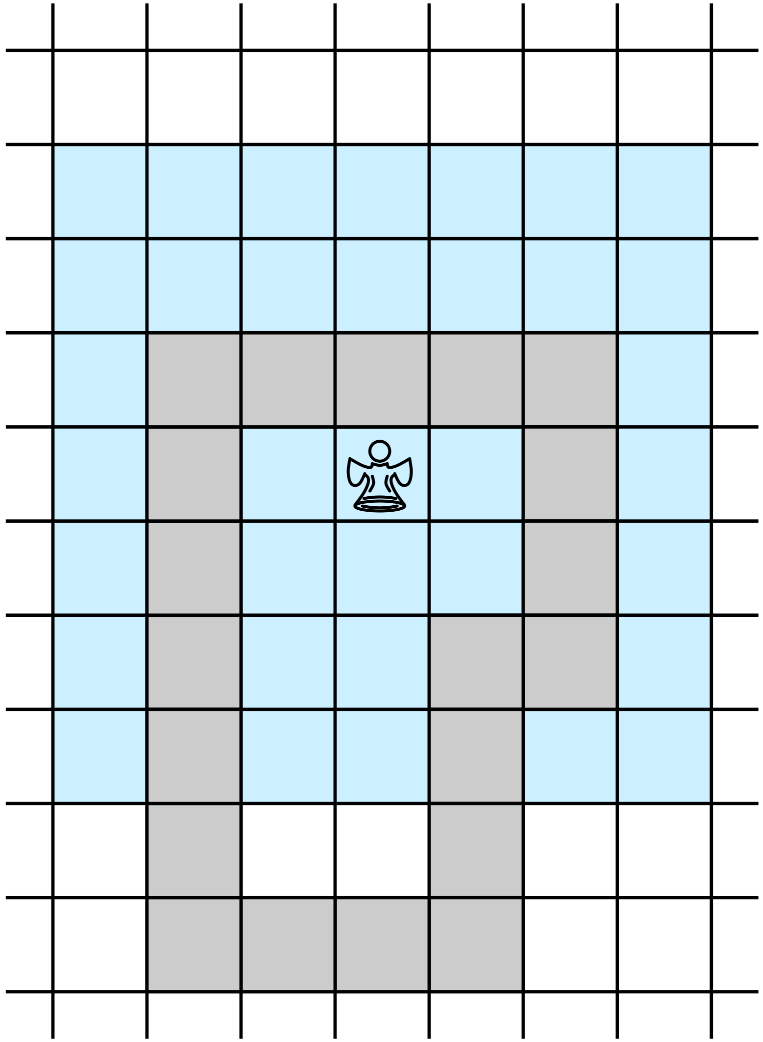

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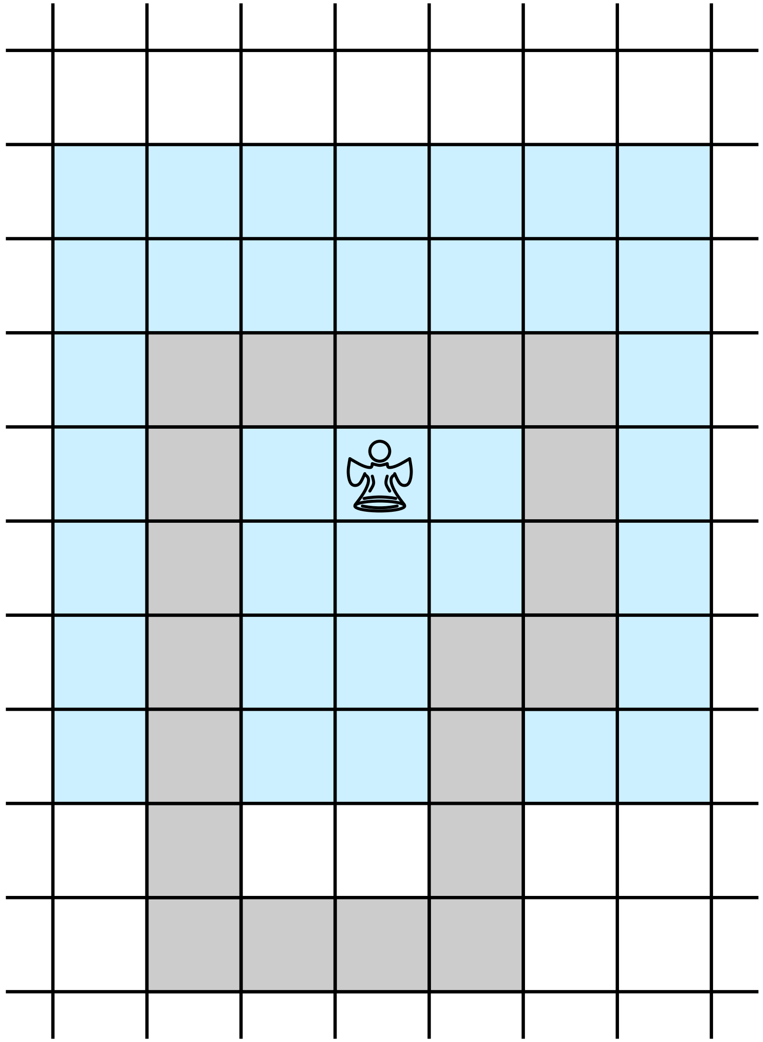
Open Problem

Can some k-Angel of some finite power k escape his opponent, the Devil, forever.



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A Fool is an Angel who commits himself to increasing his y-coordinate in every move.

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Theorem [Conway]

The Devil catches any k-Fool of finite power k.

Between 1-Angel and 2-Angel

Only the destiny of the 1-Angel ($=$ chess king) is known.

For all other k-Angels, $k \geq 2$, the outcome is open.

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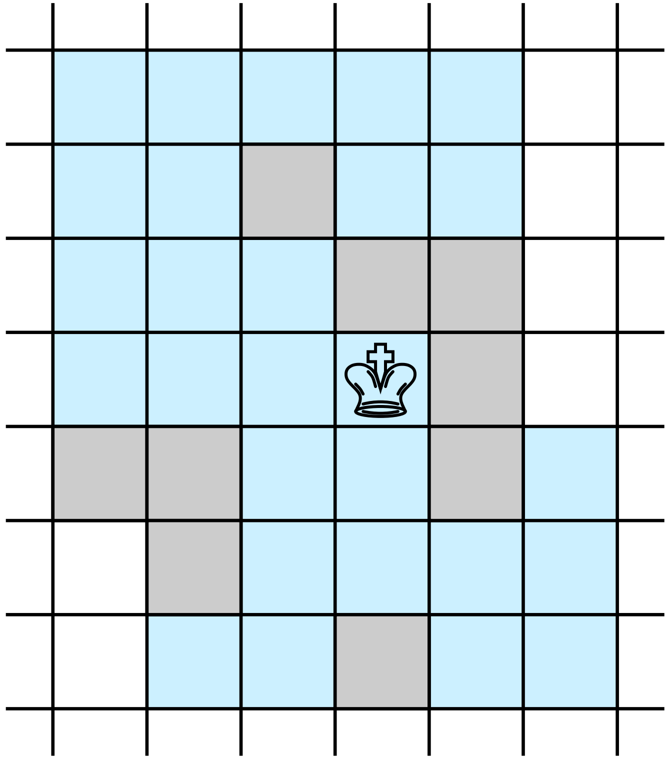
We modify the problem to have speed as the only parameter.

Angels With Broken Wings

Deprive Angels of their ability to fly across obstacles.

Definition

A k-King is a k-Angel who can only run, not fly. In each turn he makes k ordinary chess-king moves.



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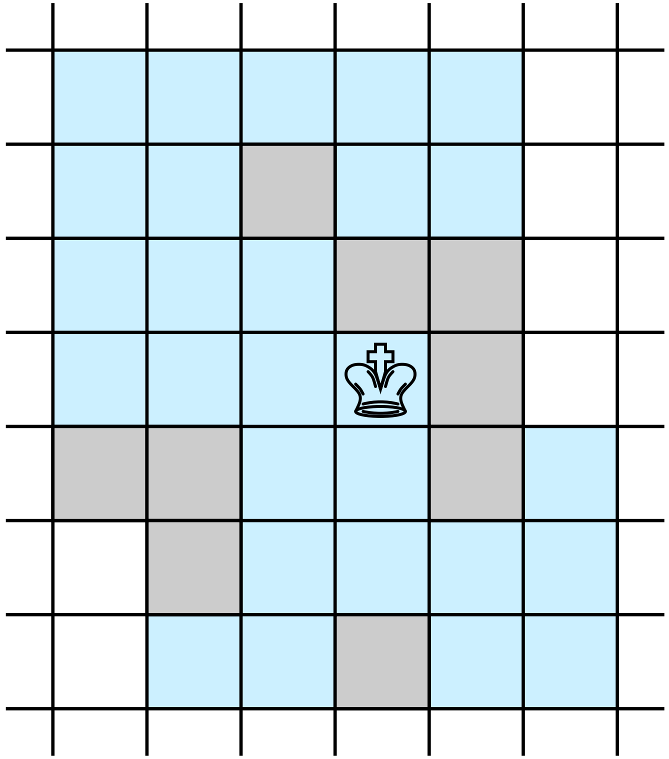

Proposition

If the k-Angel can escape forever then so can the $99k^2$-King.

The Main Result

Theorem The Devil can catch any α-King with $\alpha < 2$.

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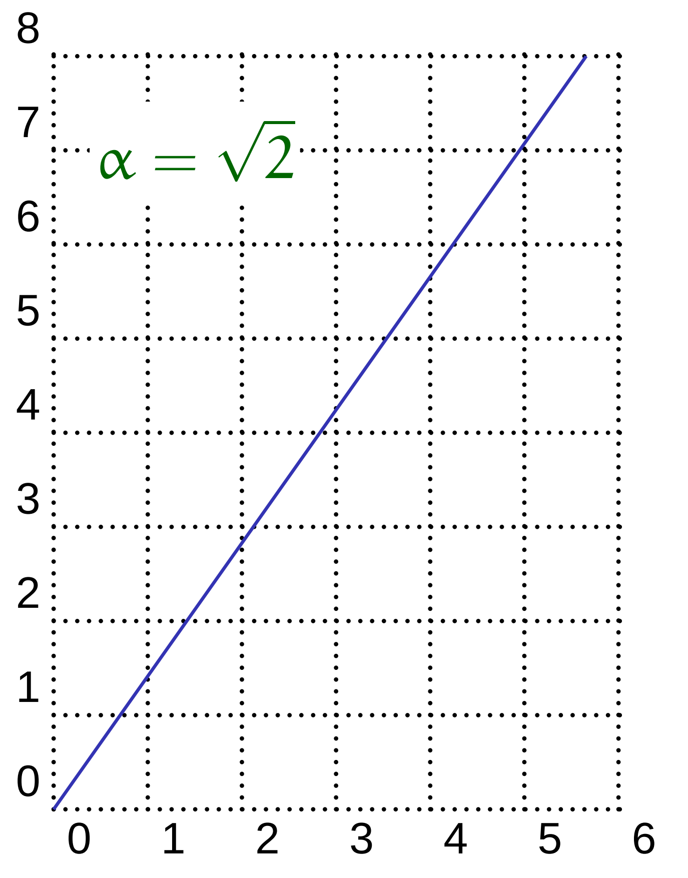
For fractional and irrational speed $\alpha > 1$ define Angel / Devil turns be means of sturmian sequences:

Shoot a ray of slope α from the origin and mark crossings with the integer grid:



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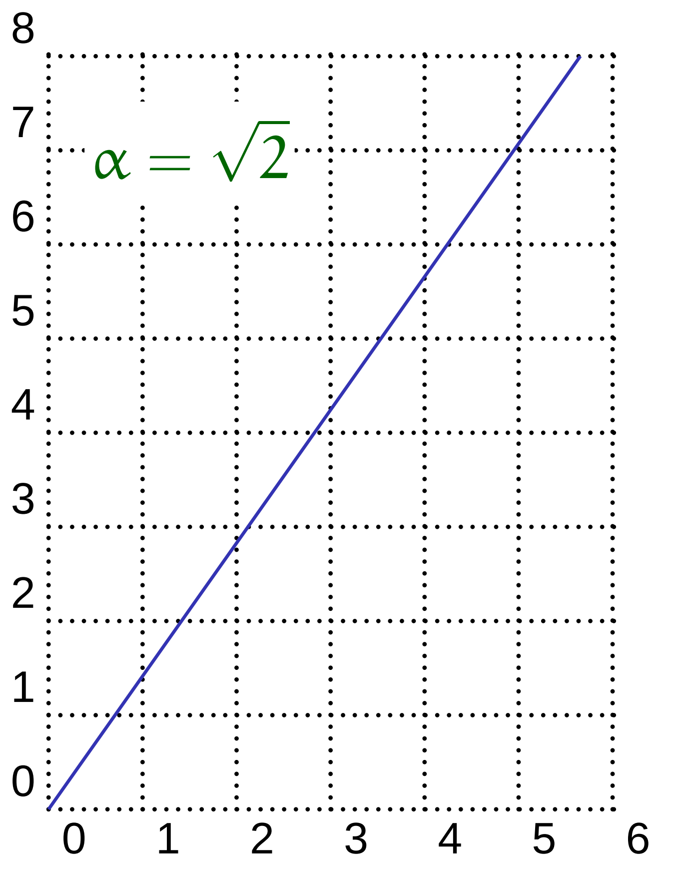

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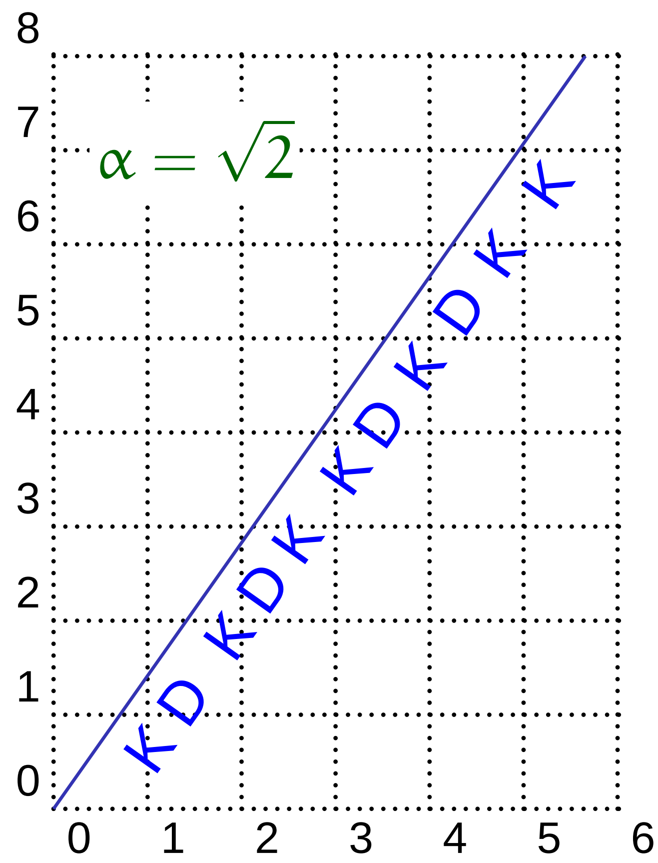

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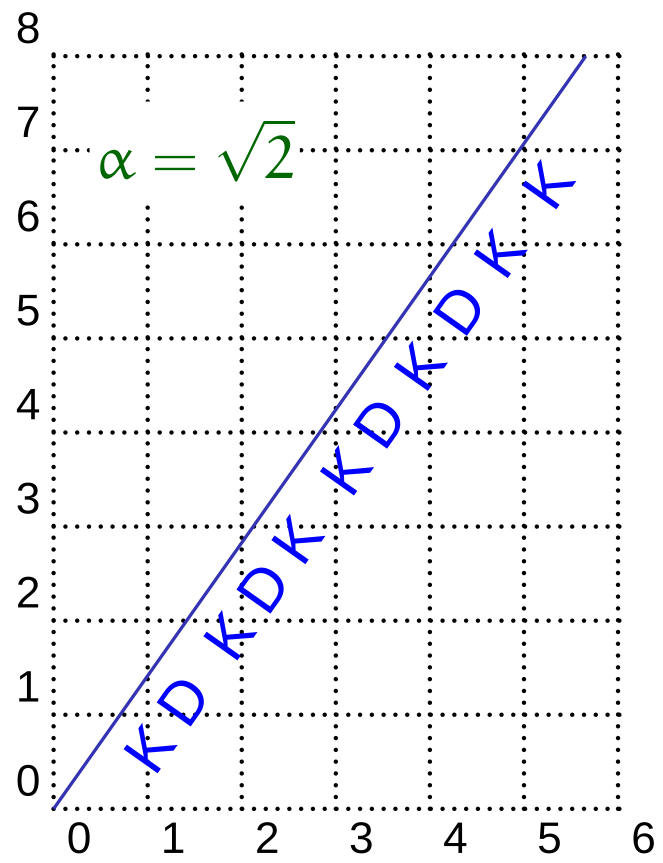

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“Lemma.” This distribution is “fair” and shifting of the grid / origin does not affect winning and losing.

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For speed $\alpha = \frac{s}{t}$ we have exactly s King moves per t Devil moves.

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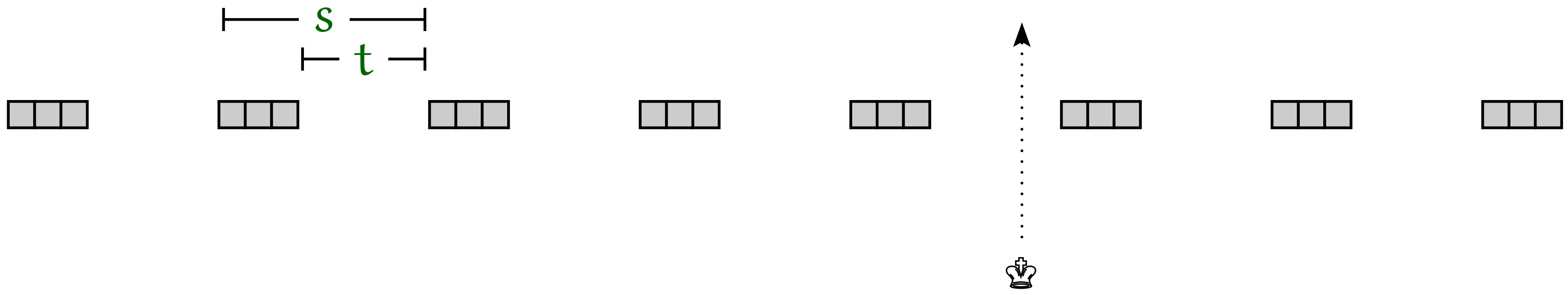

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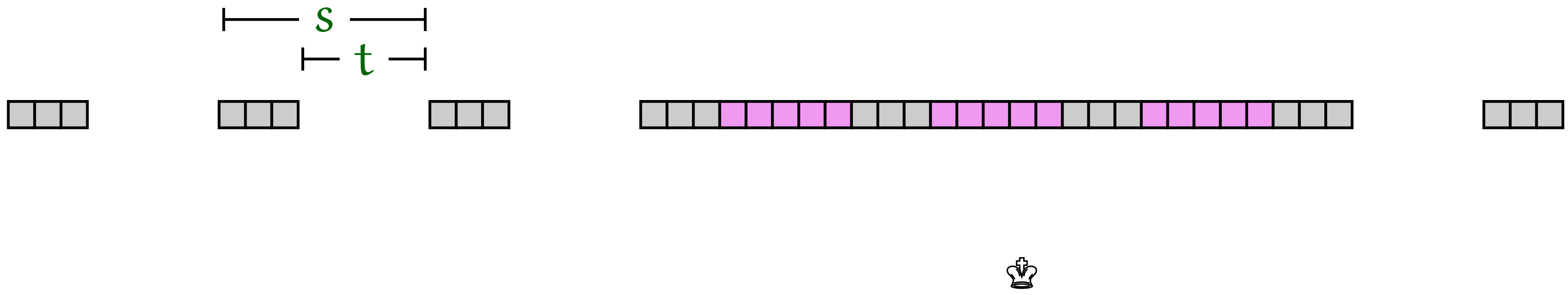

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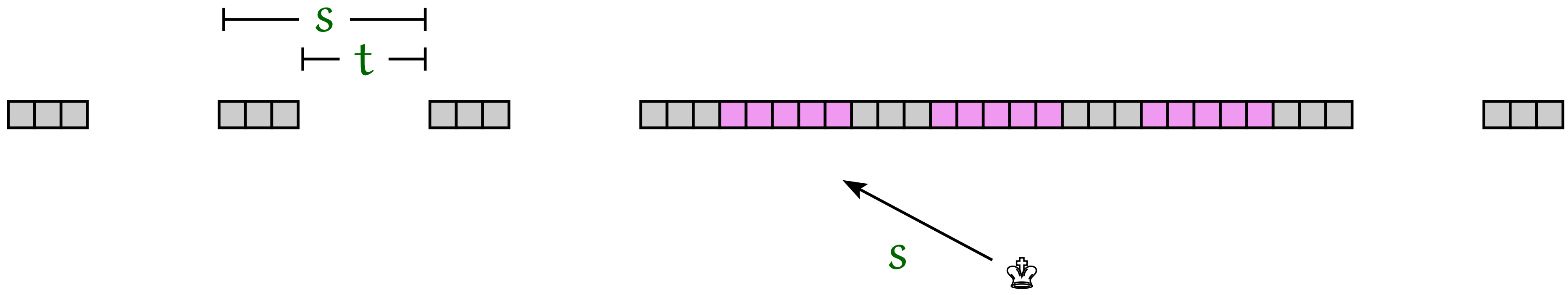

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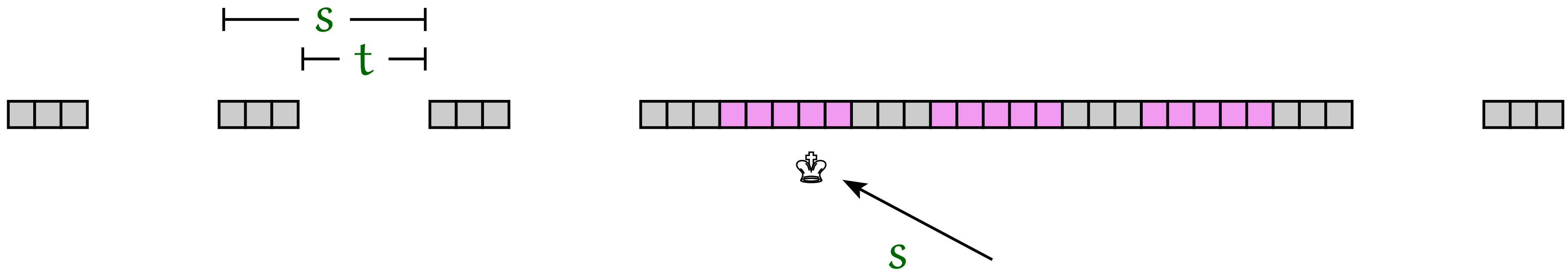

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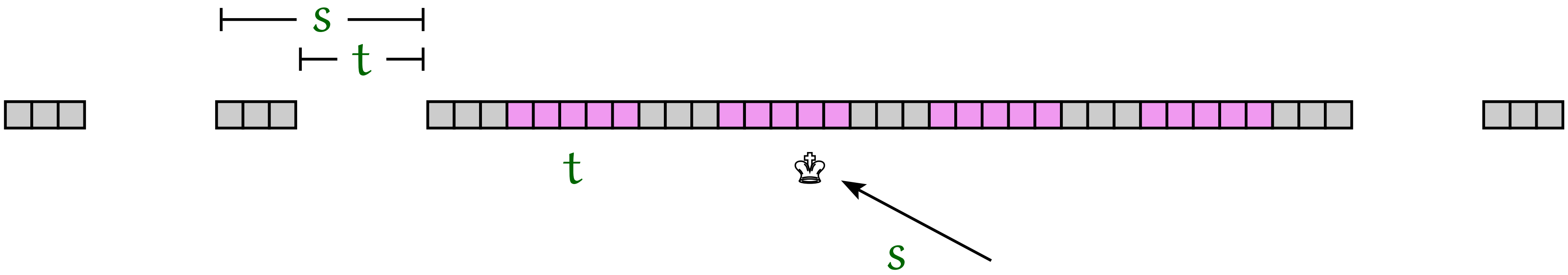

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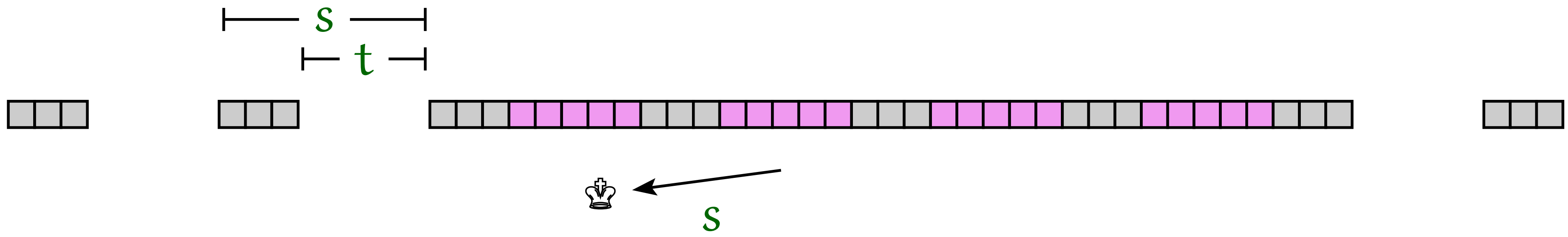

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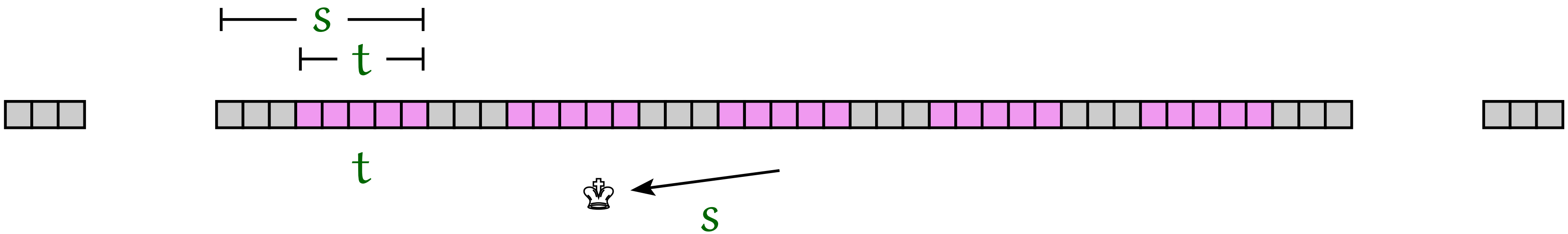

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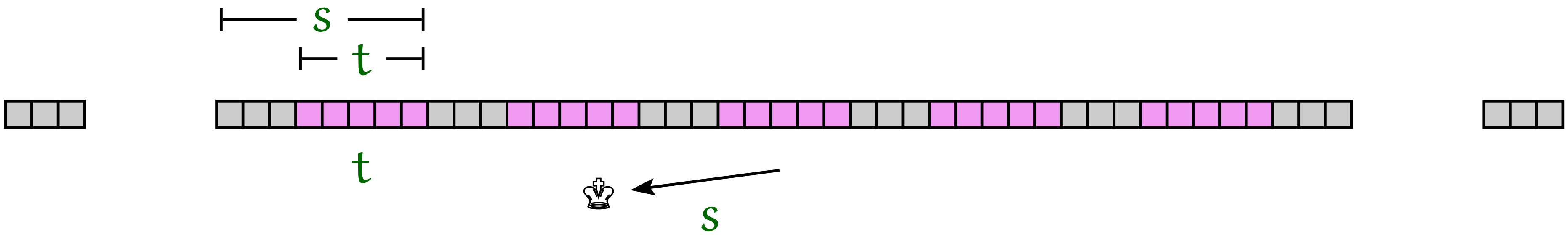

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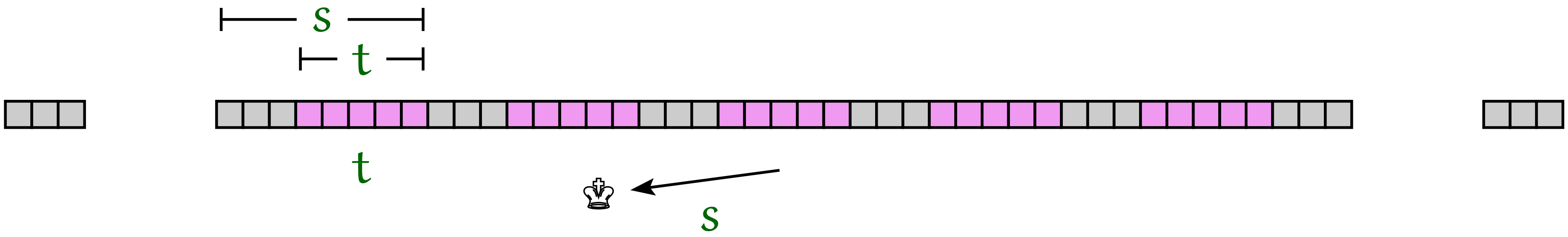

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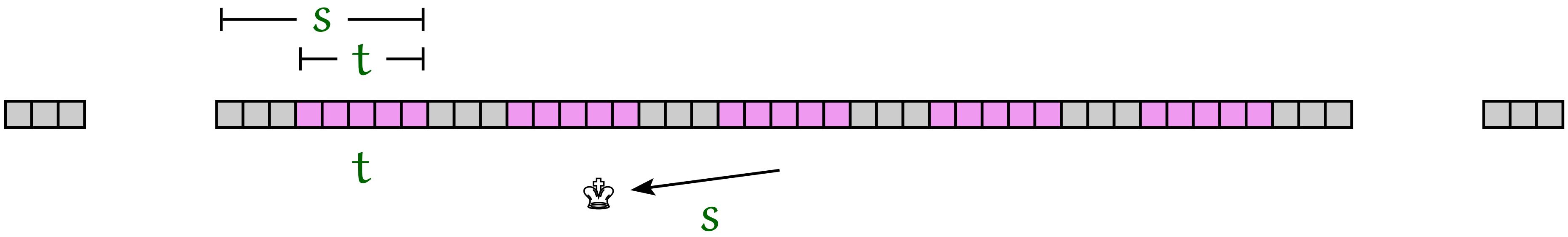

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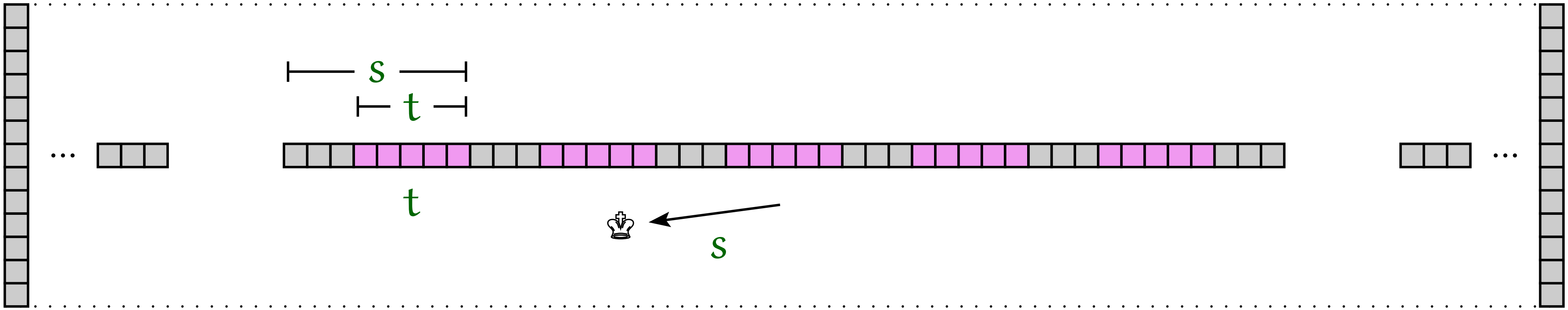

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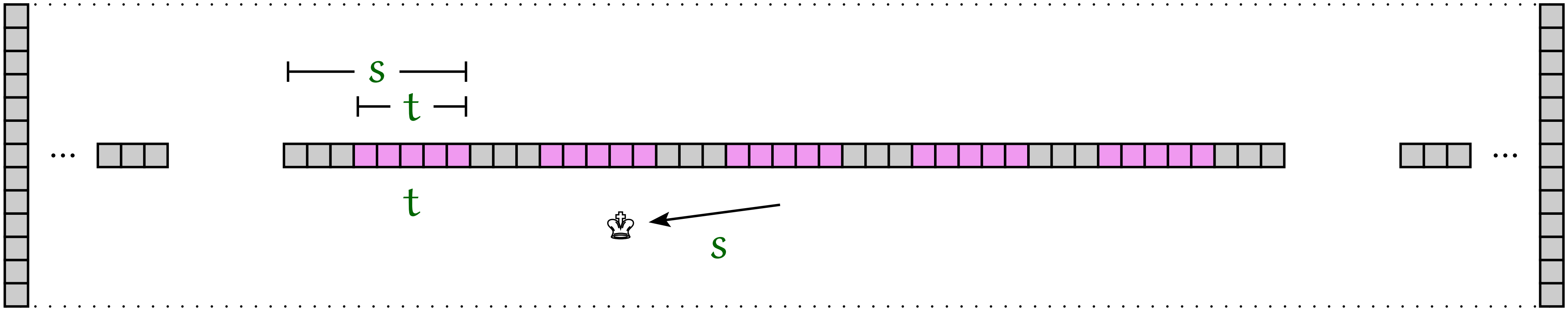

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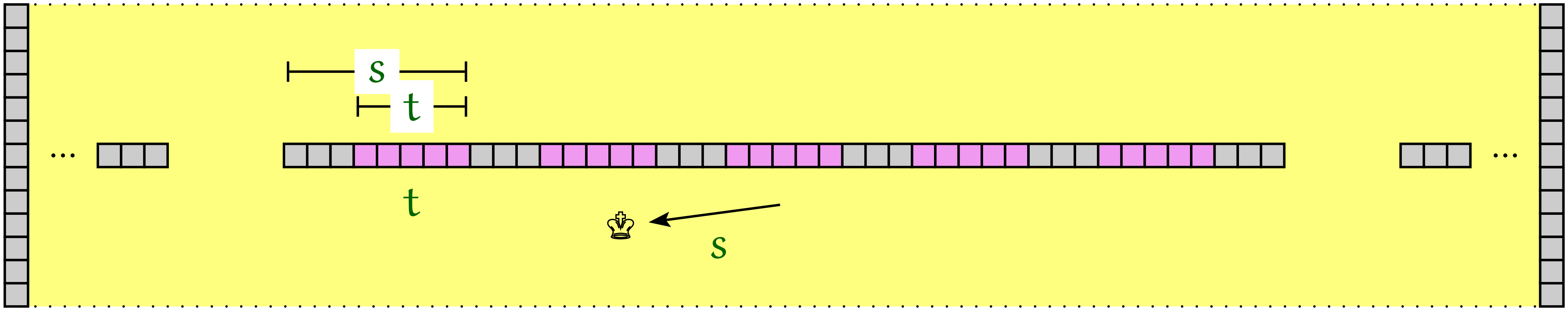

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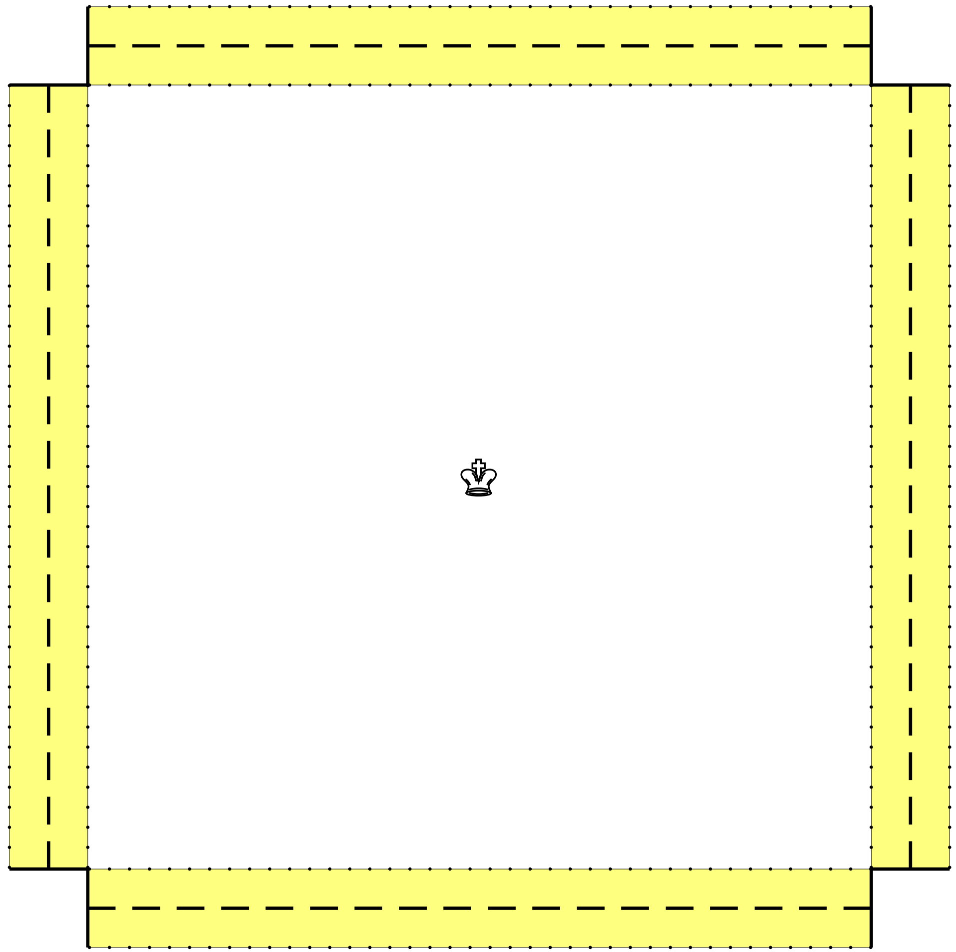
Building a Box

Encircle the King with a box of fences before he can reach the boundary.



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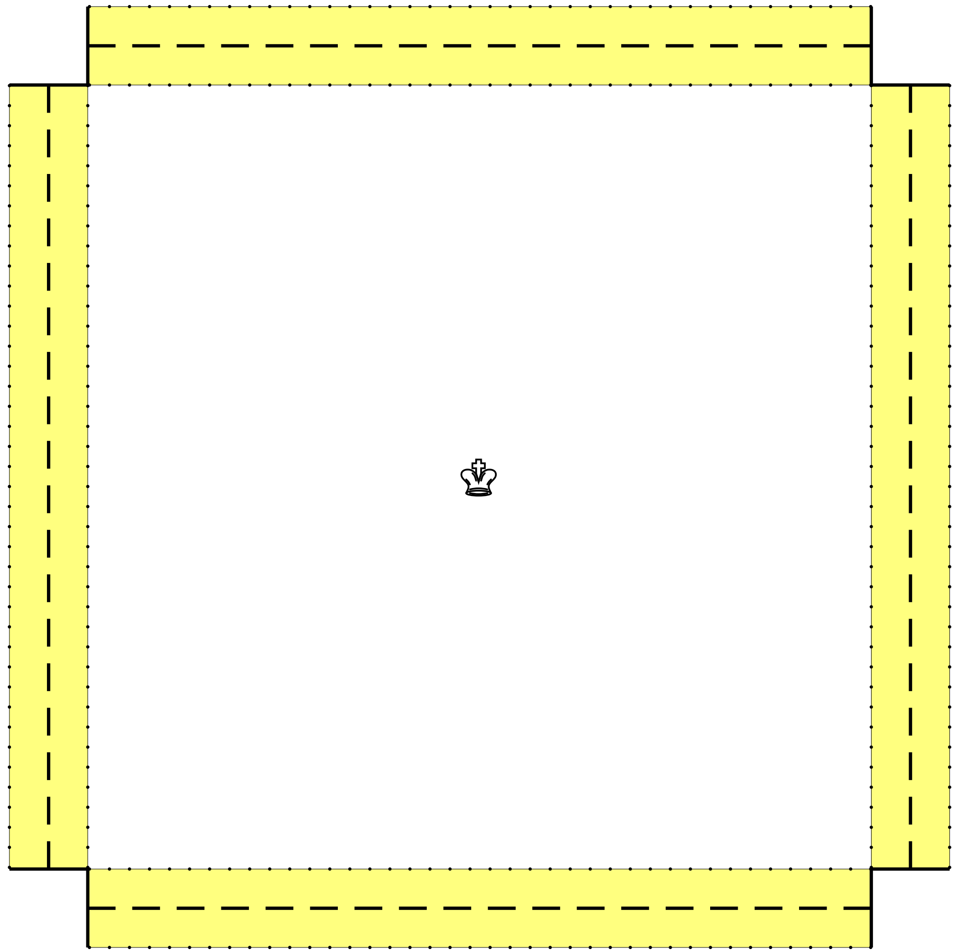

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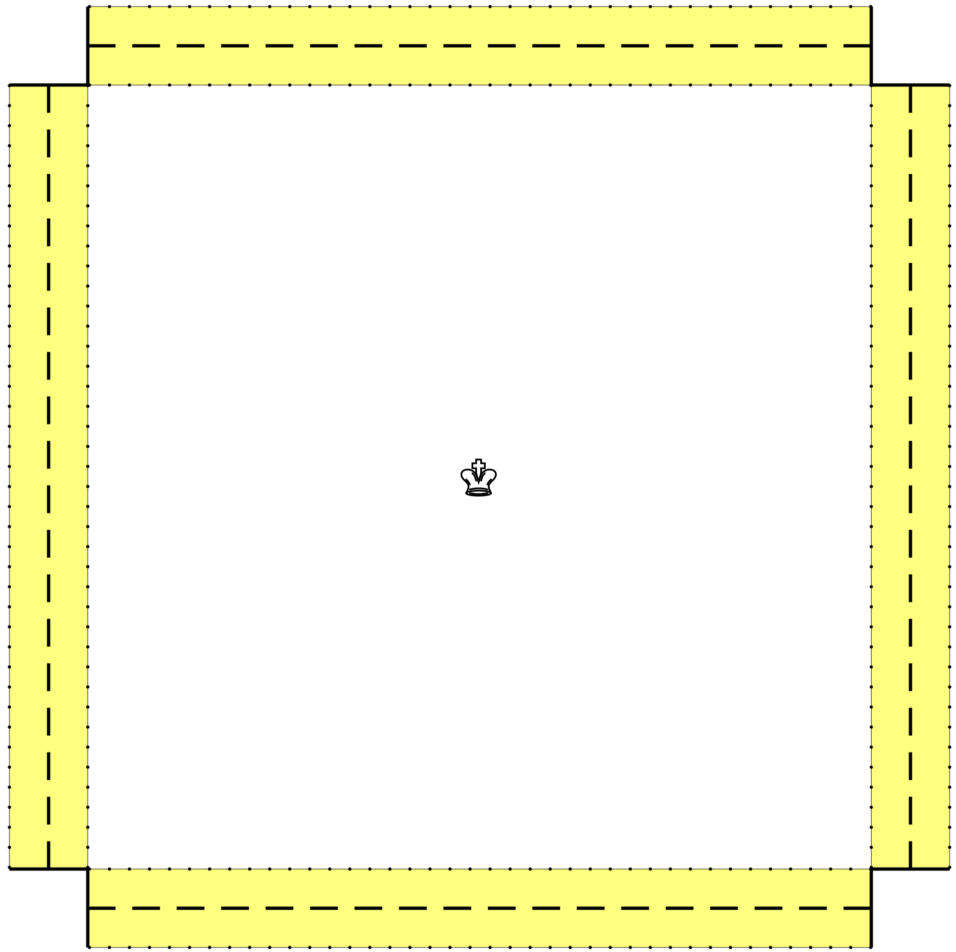
A first result:

For $\alpha < 9/8$ we get fences of density $< 1/9$, which the Devil builds

$$\frac{9}{9/8} = 8$$

times faster than the King runs.



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Against King speed $2-\varepsilon$, fence density $\frac{1}{2}-\varepsilon'$ is not enough.

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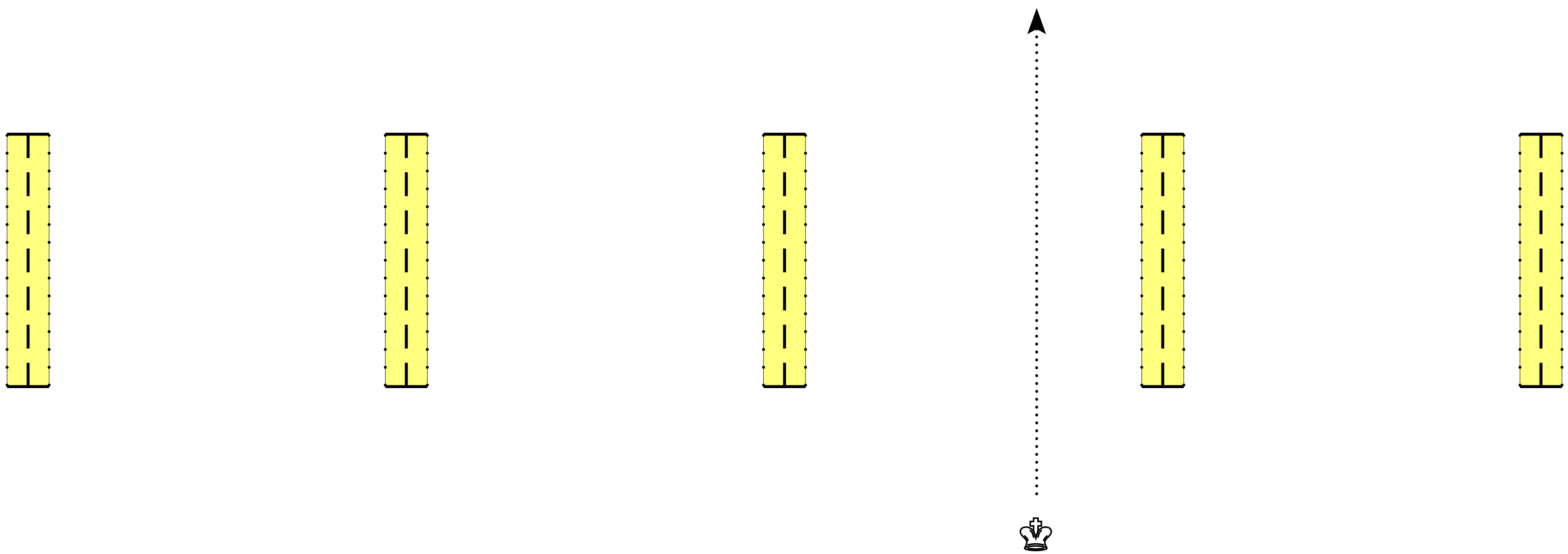
Solution: a fence of fences



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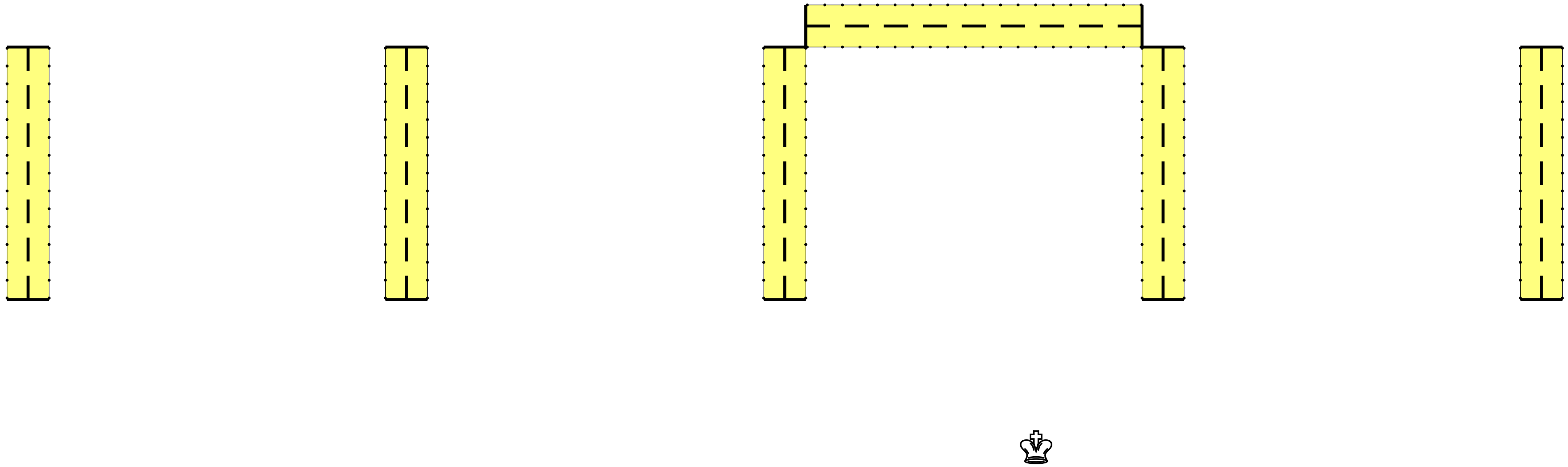

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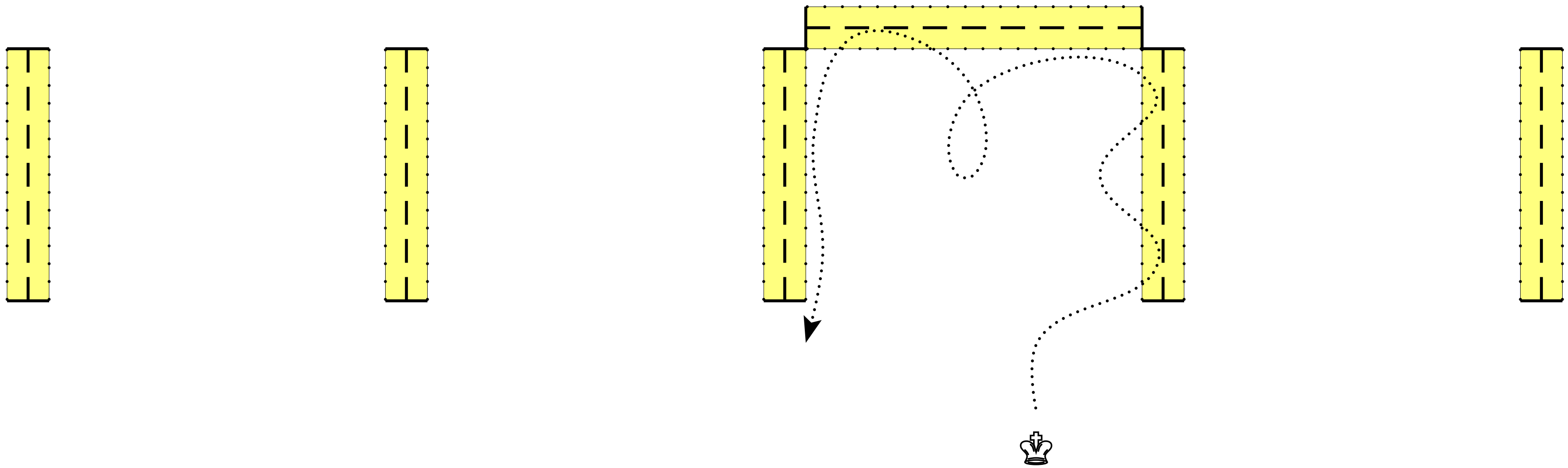

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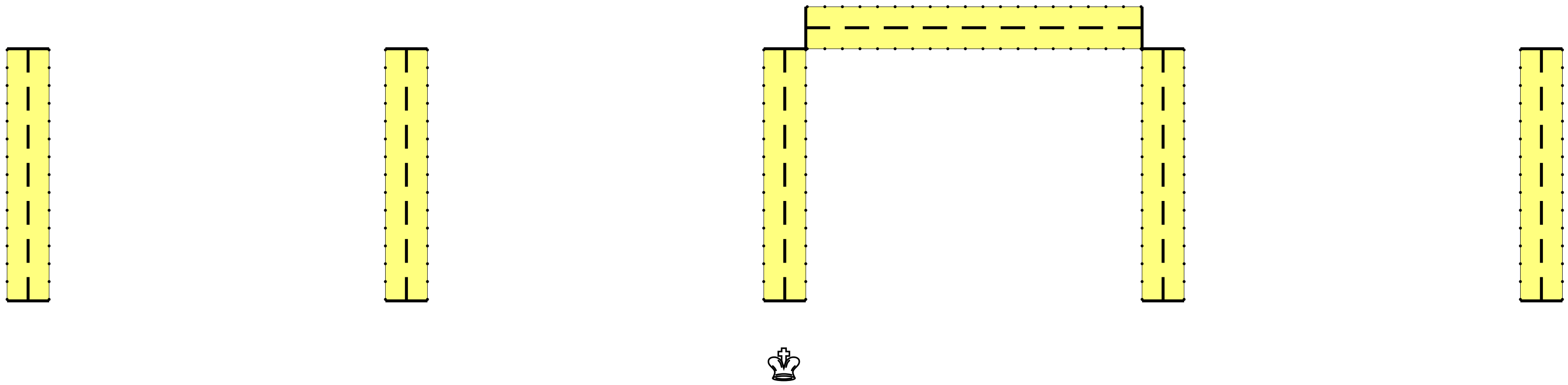

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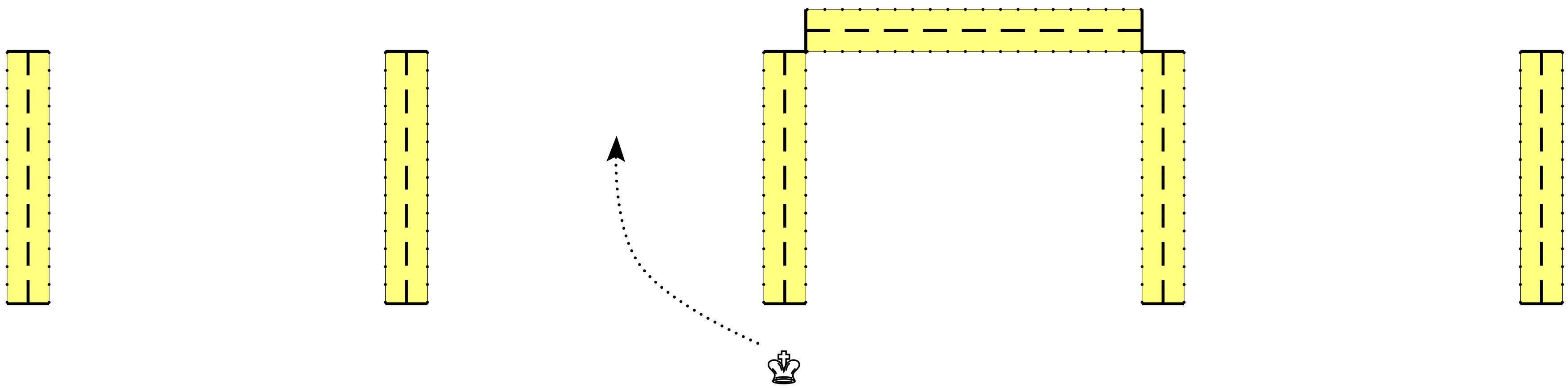

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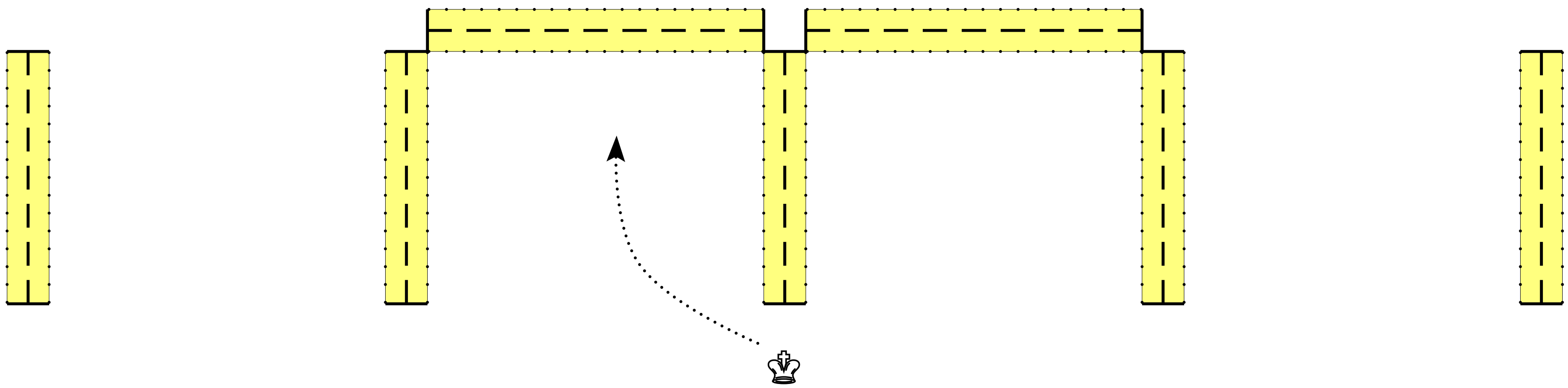

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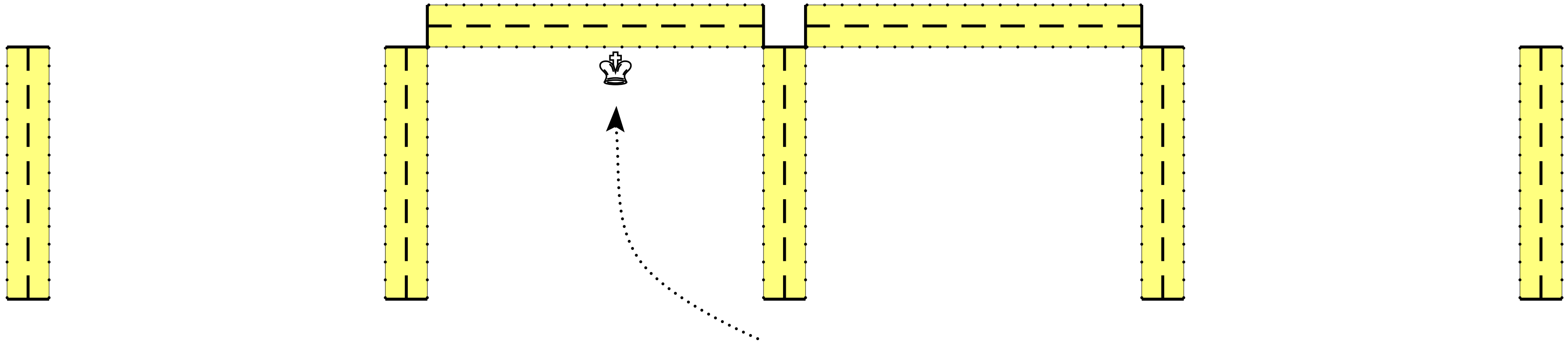

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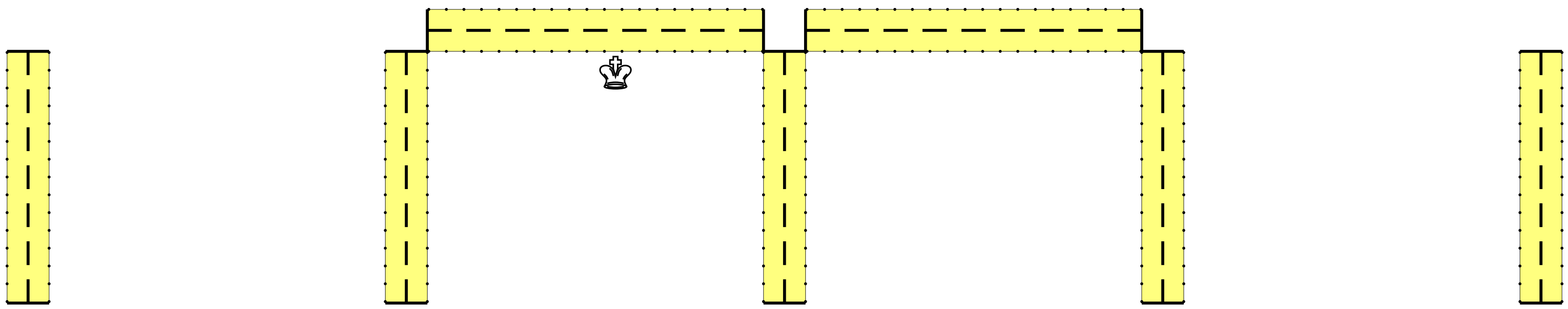

The slots are wider than they are deep, so the total density lies below that of the small fences. (works only for density $< 1/2$)

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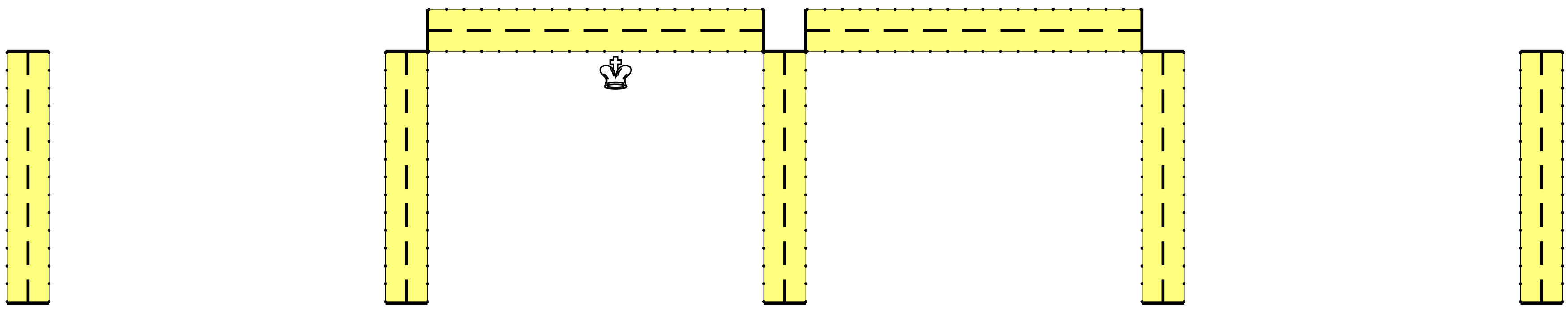
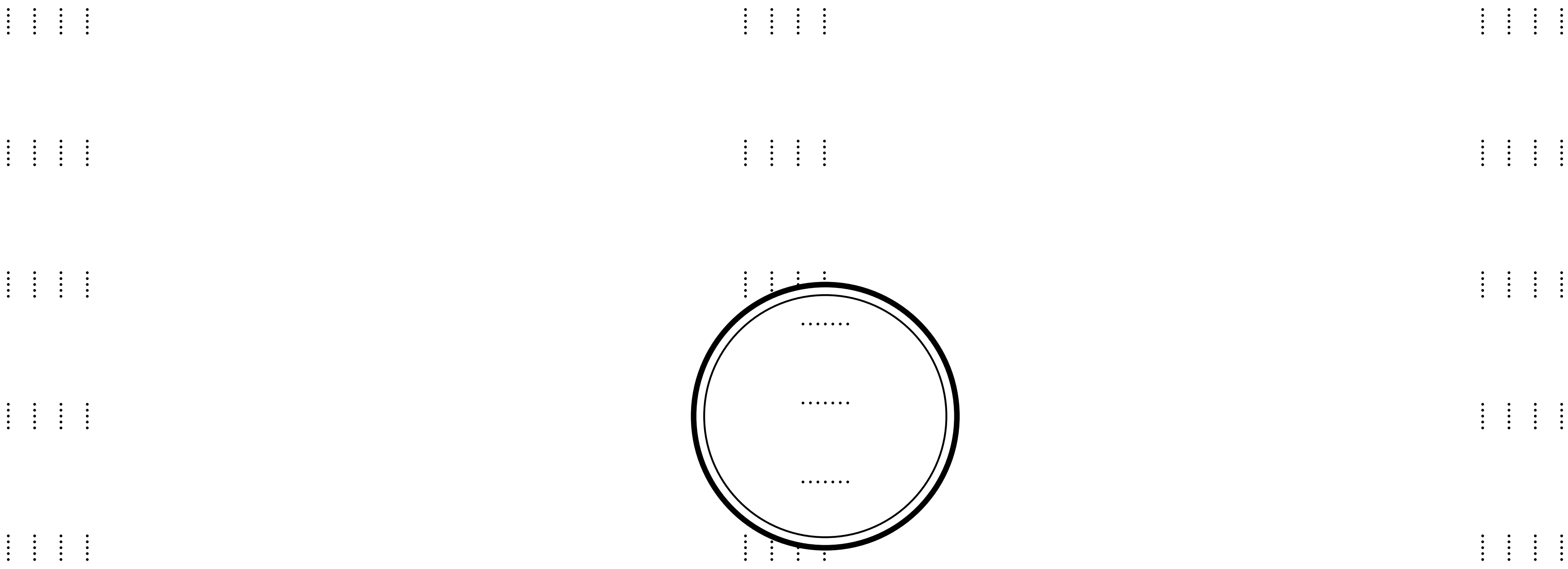
Need smaller densities.

Solution: a fence of fences



The slots are wider than they are deep, so the total density lies below that of the small fences. (works only for density $< 1/2$)

Iteration yields thinner and thinner and thinner and thinner fences ...



Conclusion

We introduced α-Kings (with any $\alpha \in \mathbb{R}^+$) to focus on speed as the essential parameter in the Angel Problem.

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Question Can he also catch the 2-King?