Computing Geometric Minimum-Dilation Graphs is NP-hard

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Geometric Dilation of a Plane Graph

Def. The *dilation* of a graph $G = (V, E)$ in the plane:

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Note: A complete graph has dilation 1.

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Task: Given point set $P \subset \mathbb{R}^2$,
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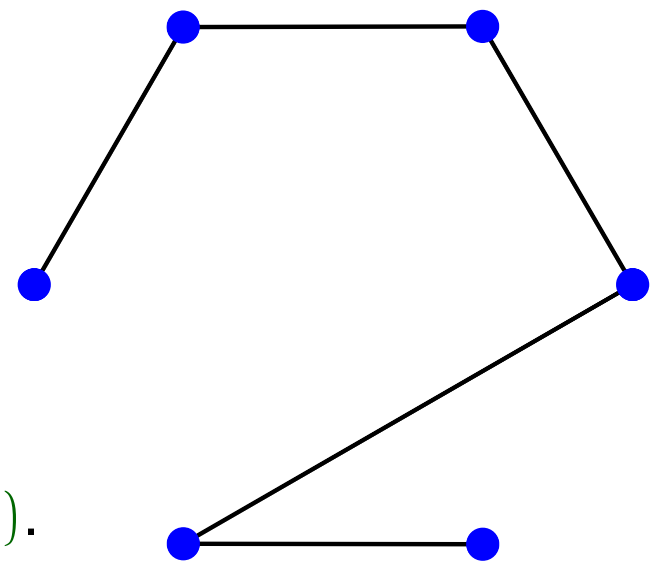

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For the vertices P of a regular n-gon,
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(NP-Completeness unclear because of sums of square roots . . .)

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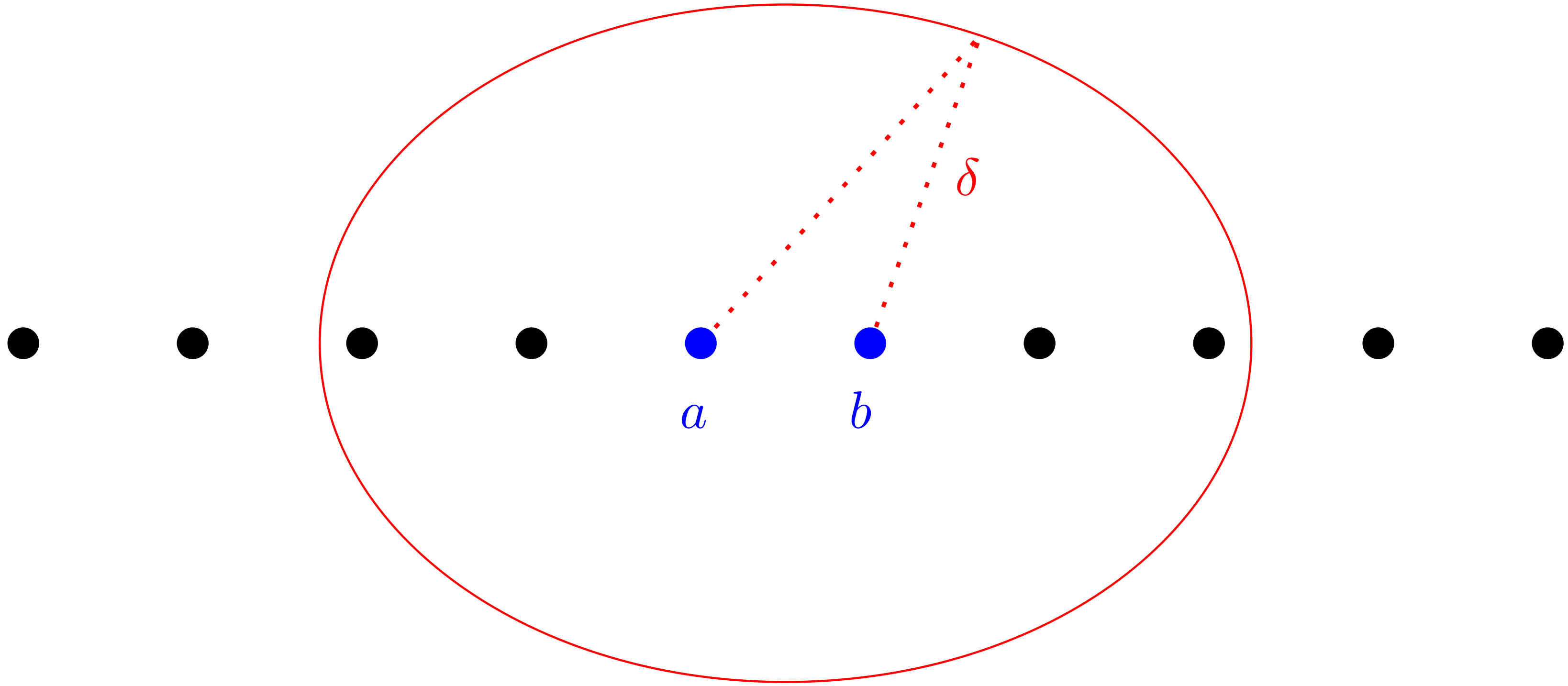
We consider restriction to geometric case with $G = K_n$.

Result

Theorem.

Given point set $P \subset \mathbb{R}^2$, $m \geq |P| - 1$, and a threshold $\delta > 1$,
it is NP-hard to decide whether there exists a (plane) graph $G = (P, E)$
with $|E| = m$ and $\delta(G) \leq \delta$.

Connections on a Line



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- A point set P and a dilation threshold δ given.
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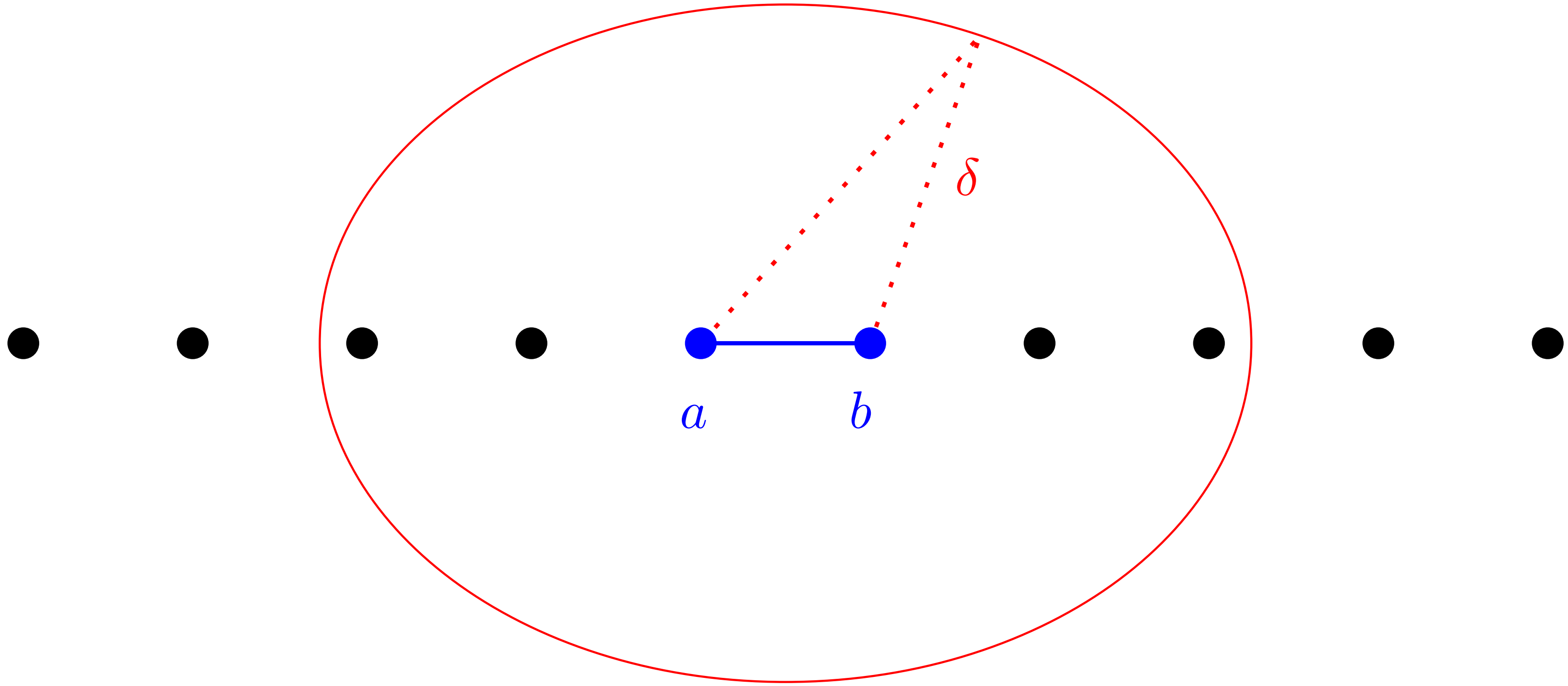

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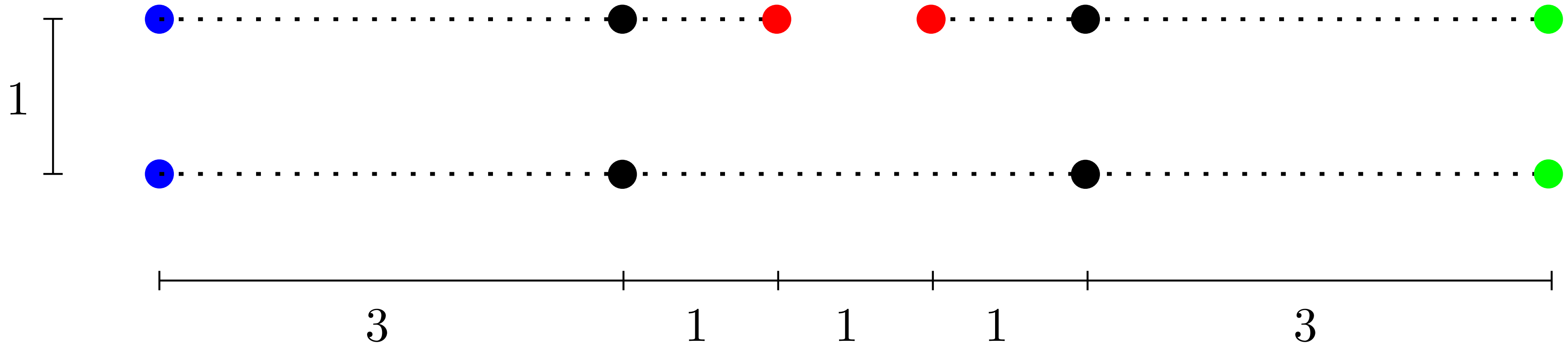
- *then* min-weight graph G (of dilation $\leq \delta$) contains edge ab.

A Tree on Two Rails



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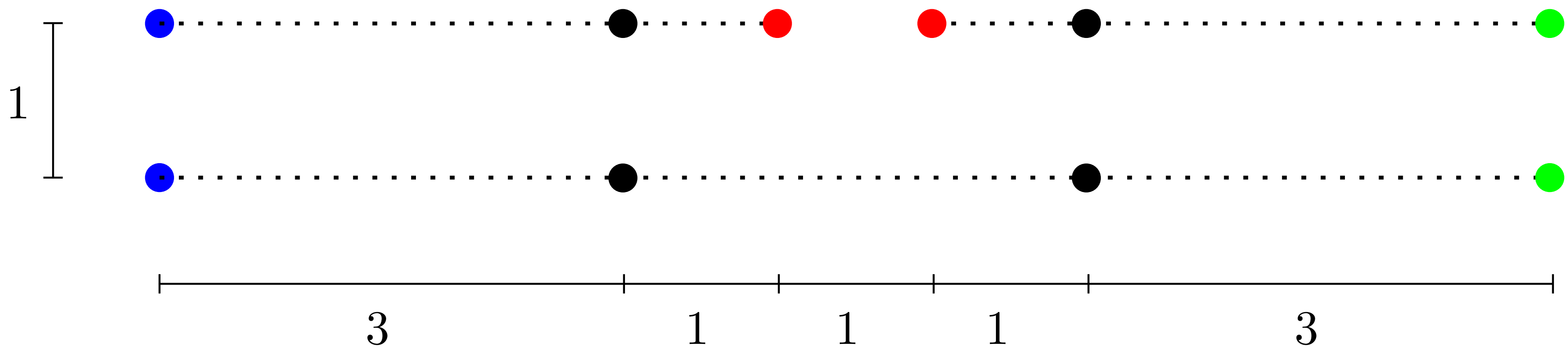

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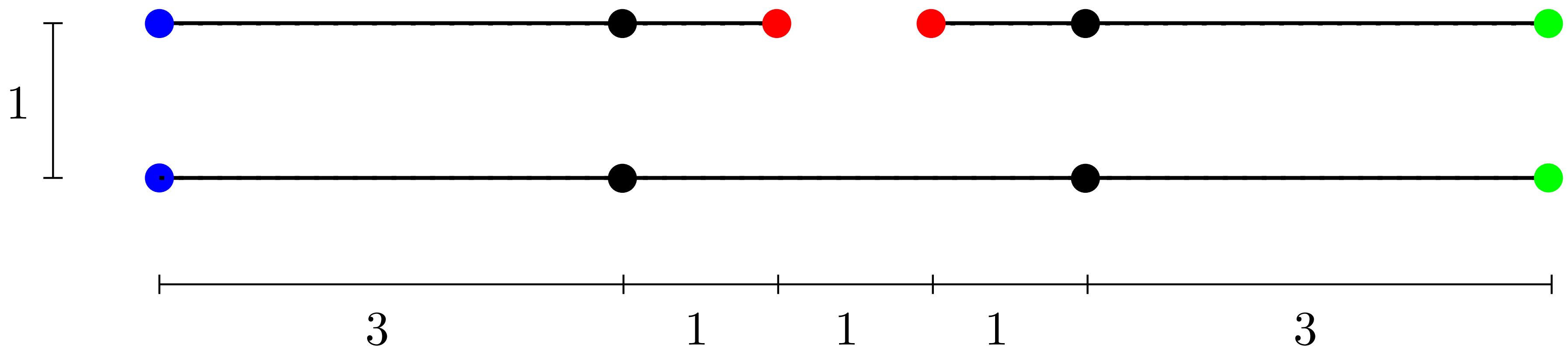

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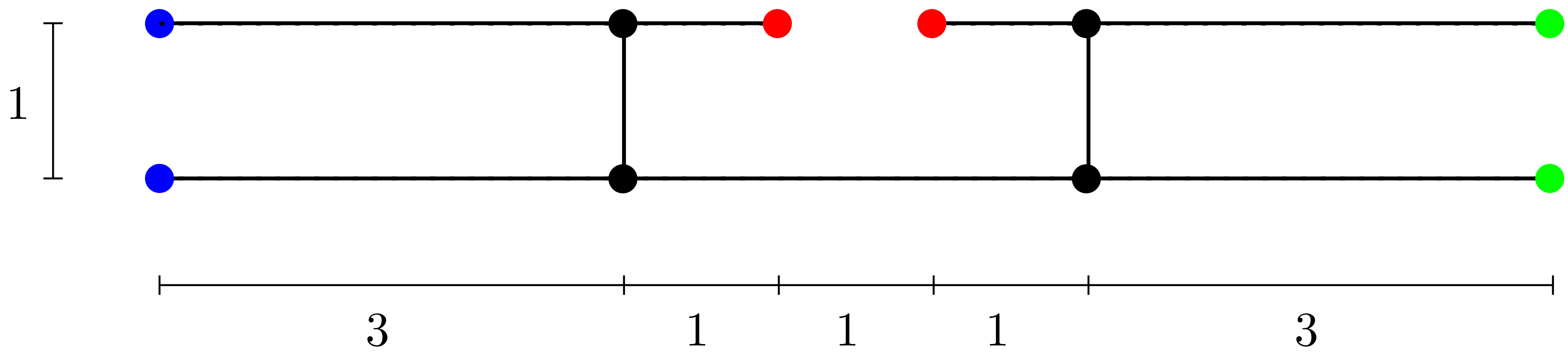

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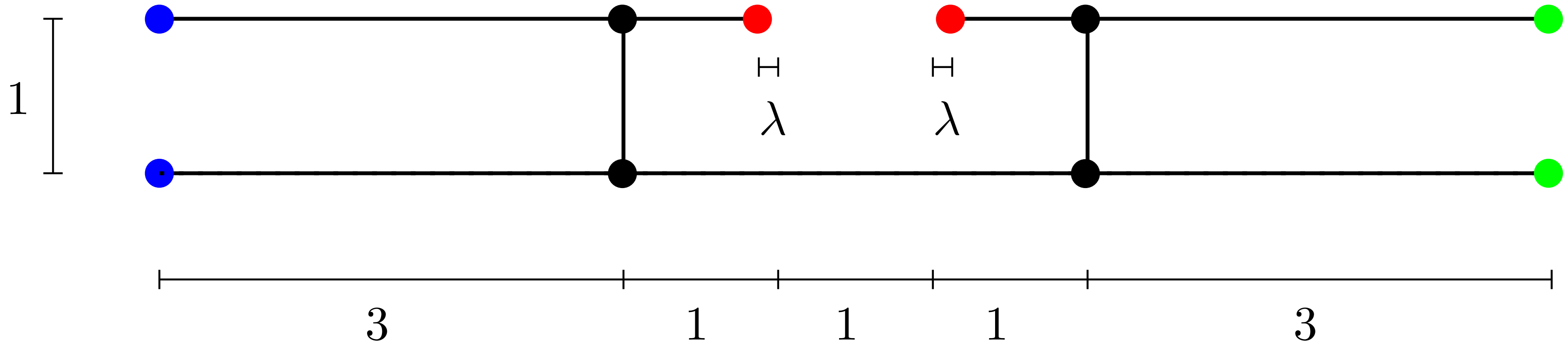

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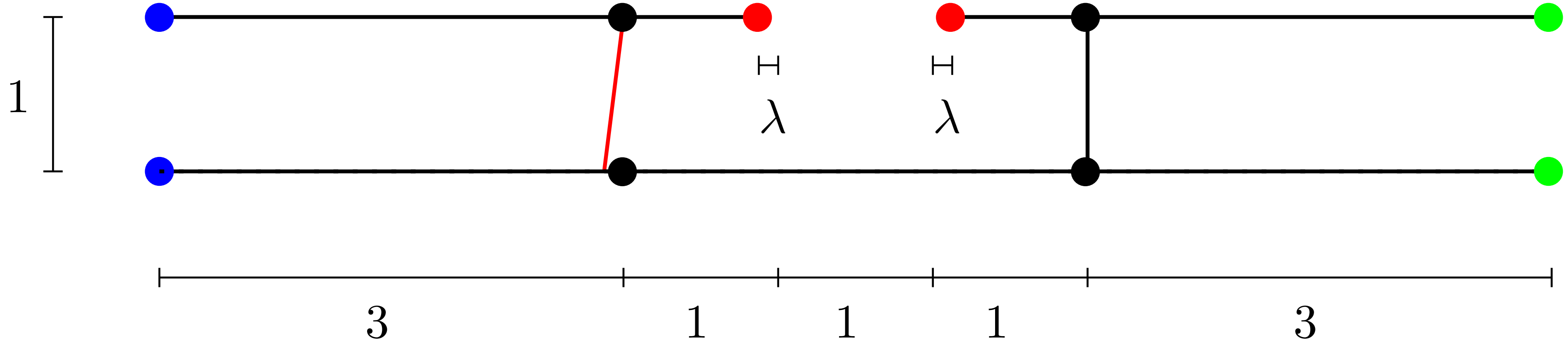

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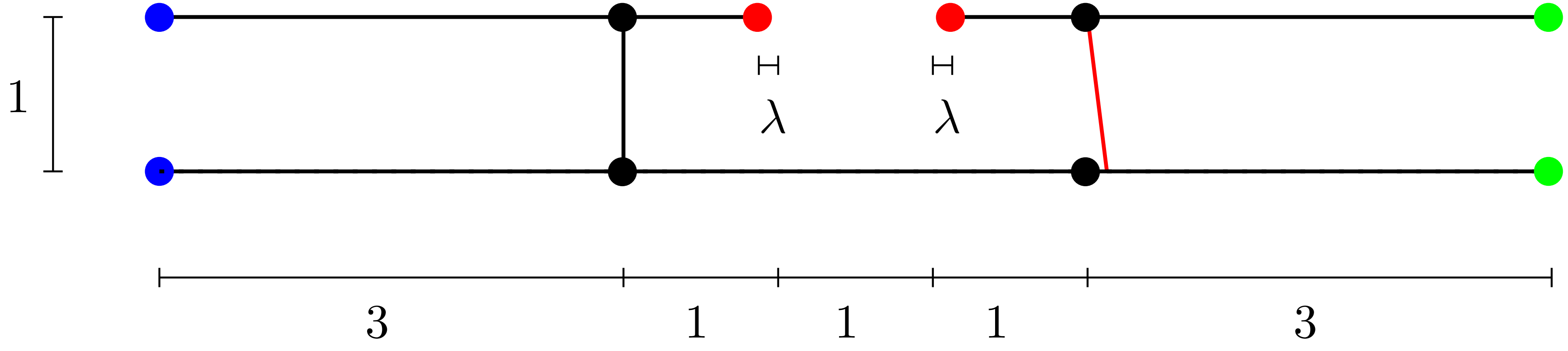

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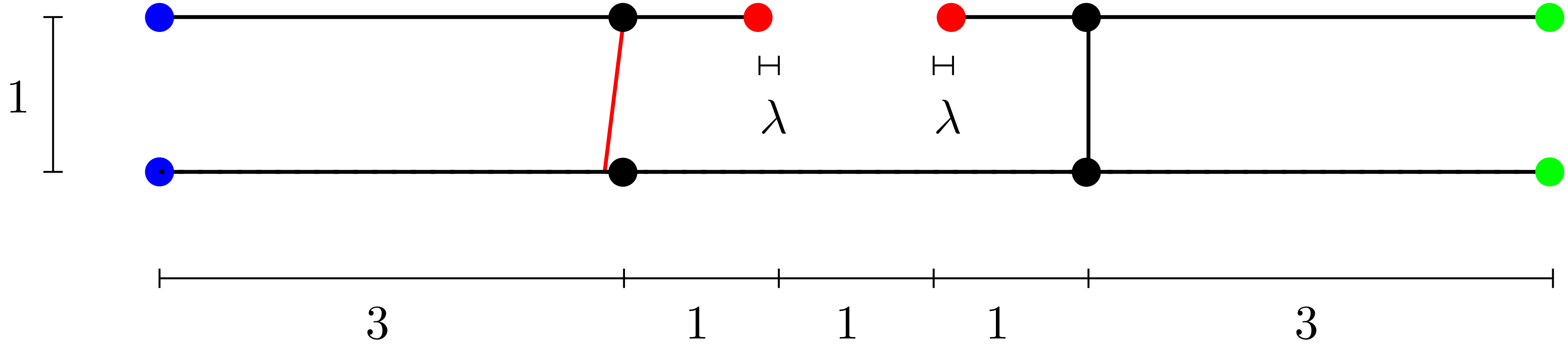

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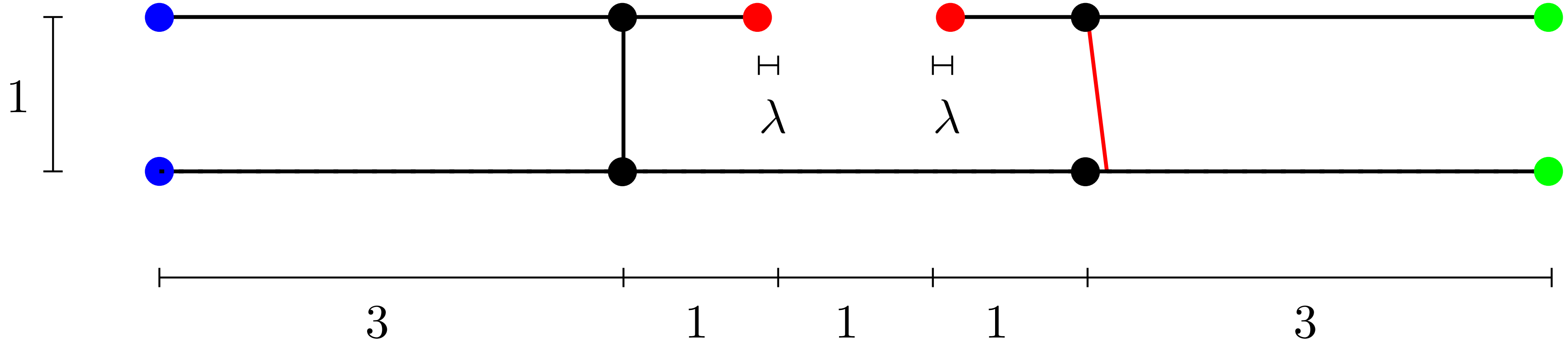

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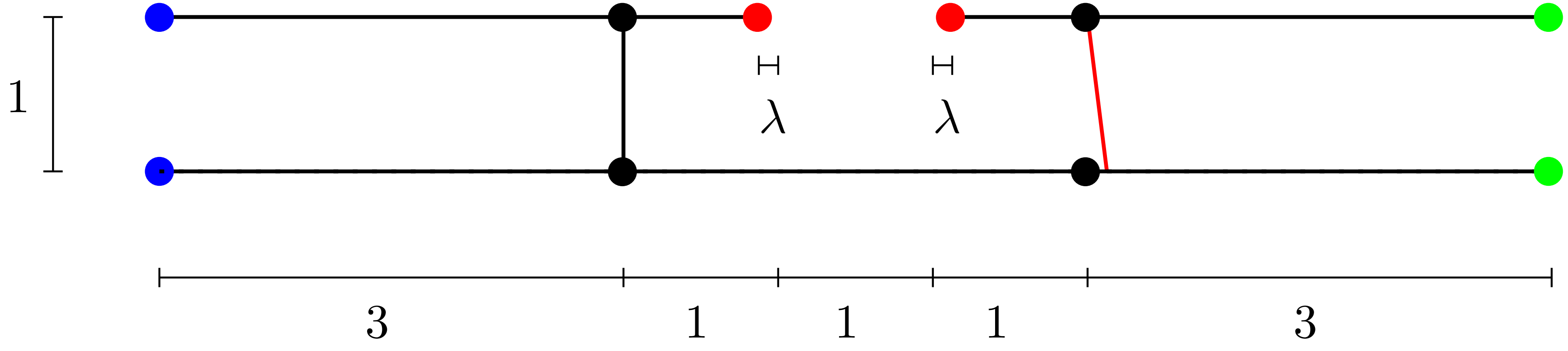

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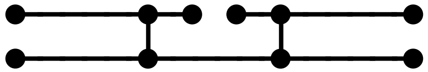
- . . . but only one of them!

Many Gadgets



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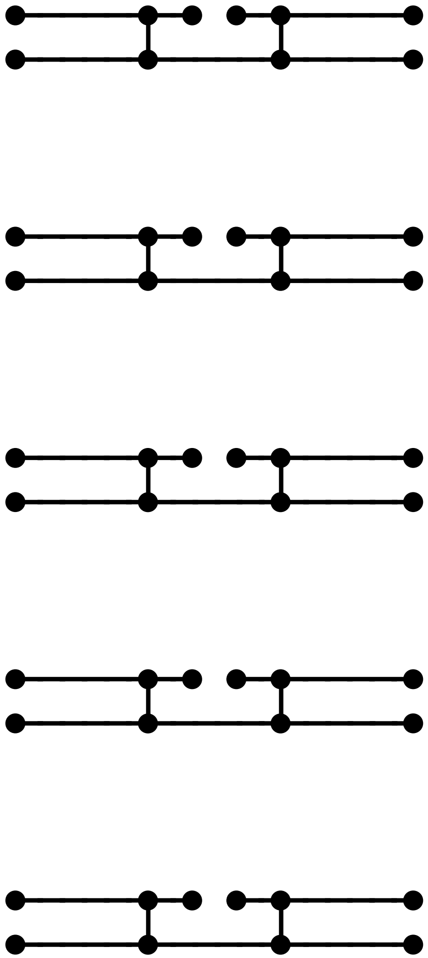

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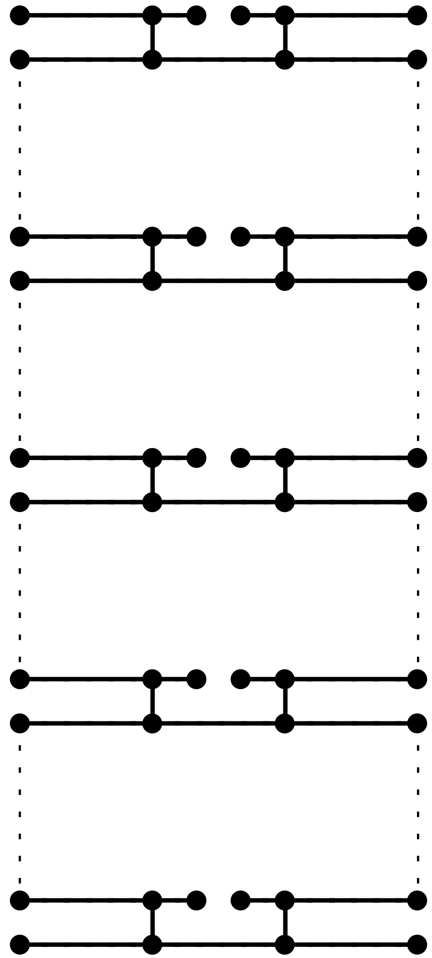

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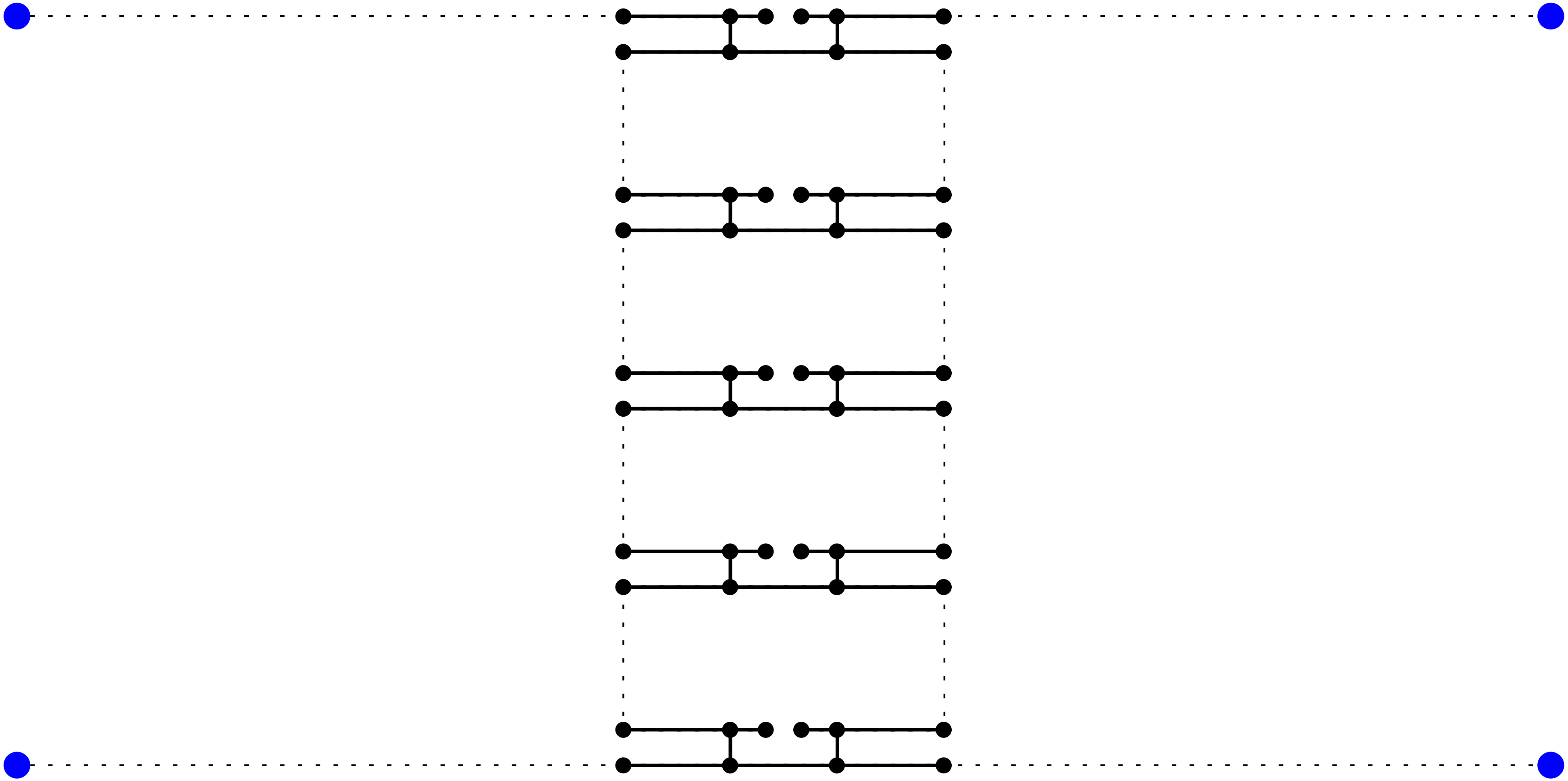

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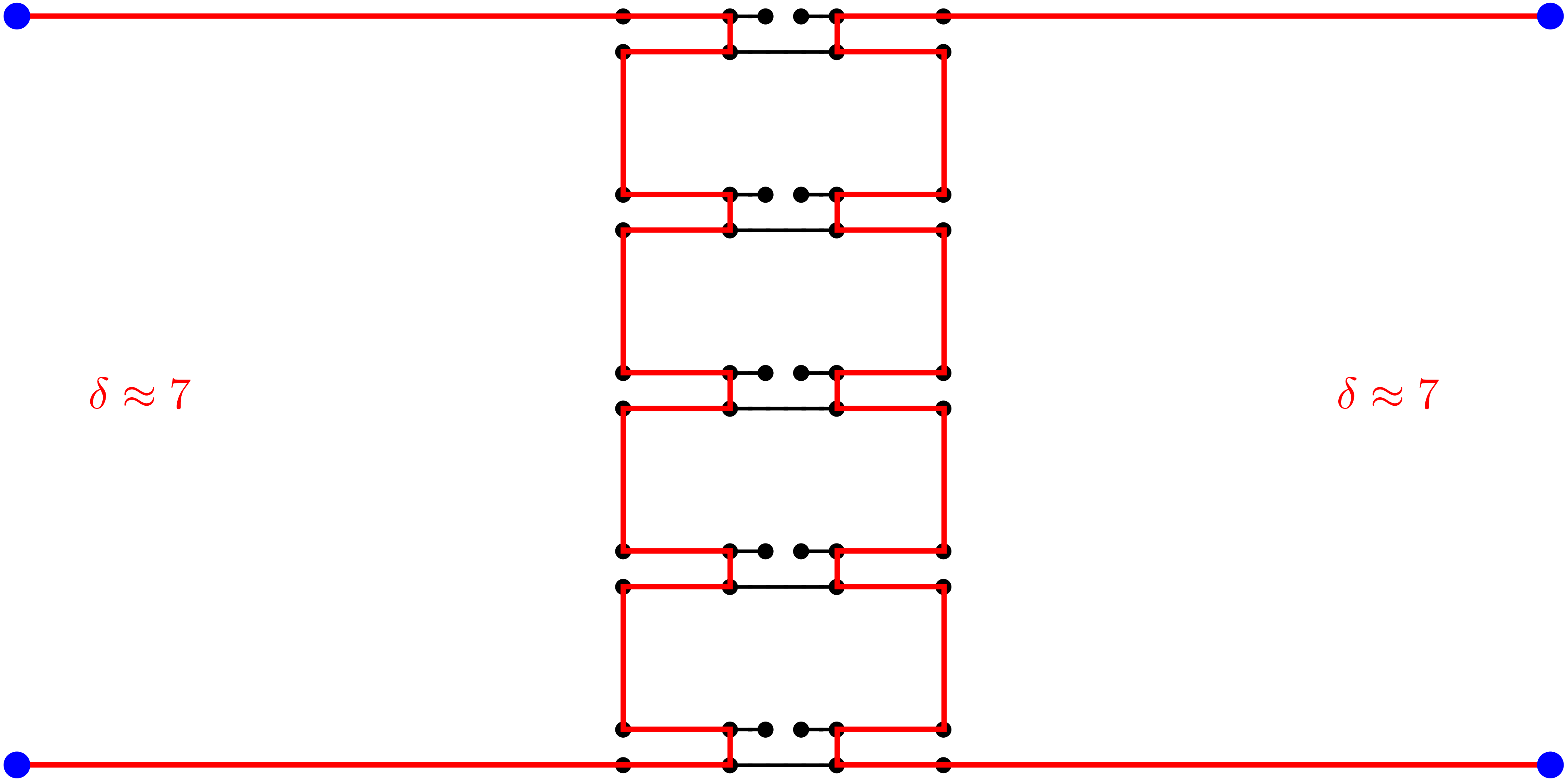

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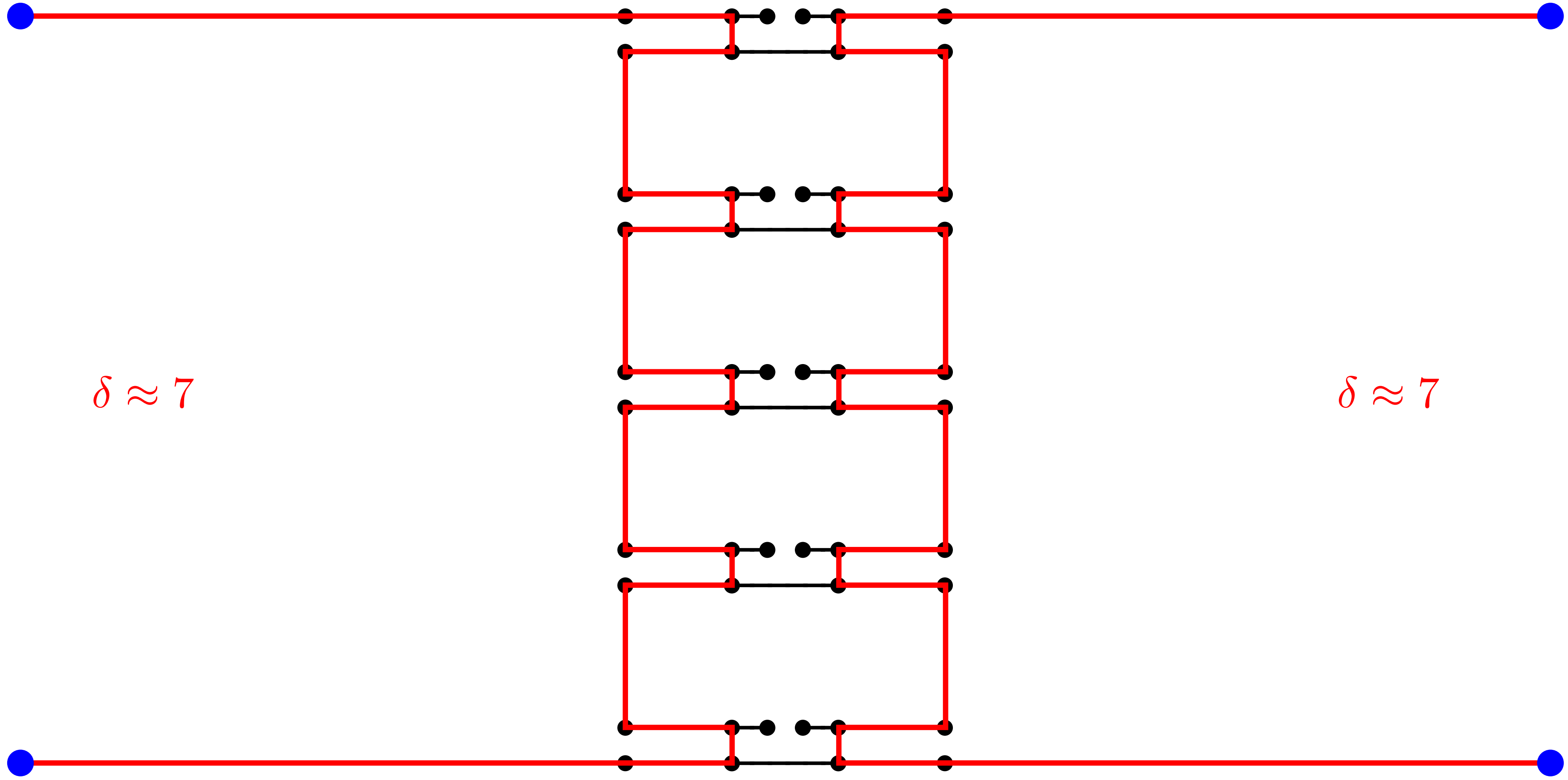

Total number of edges: $m = (|P| - 1) + (\#\text{gadgets} - 1)$

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Remark:
Even for large numbers, polynomial-size reduction easily possible.

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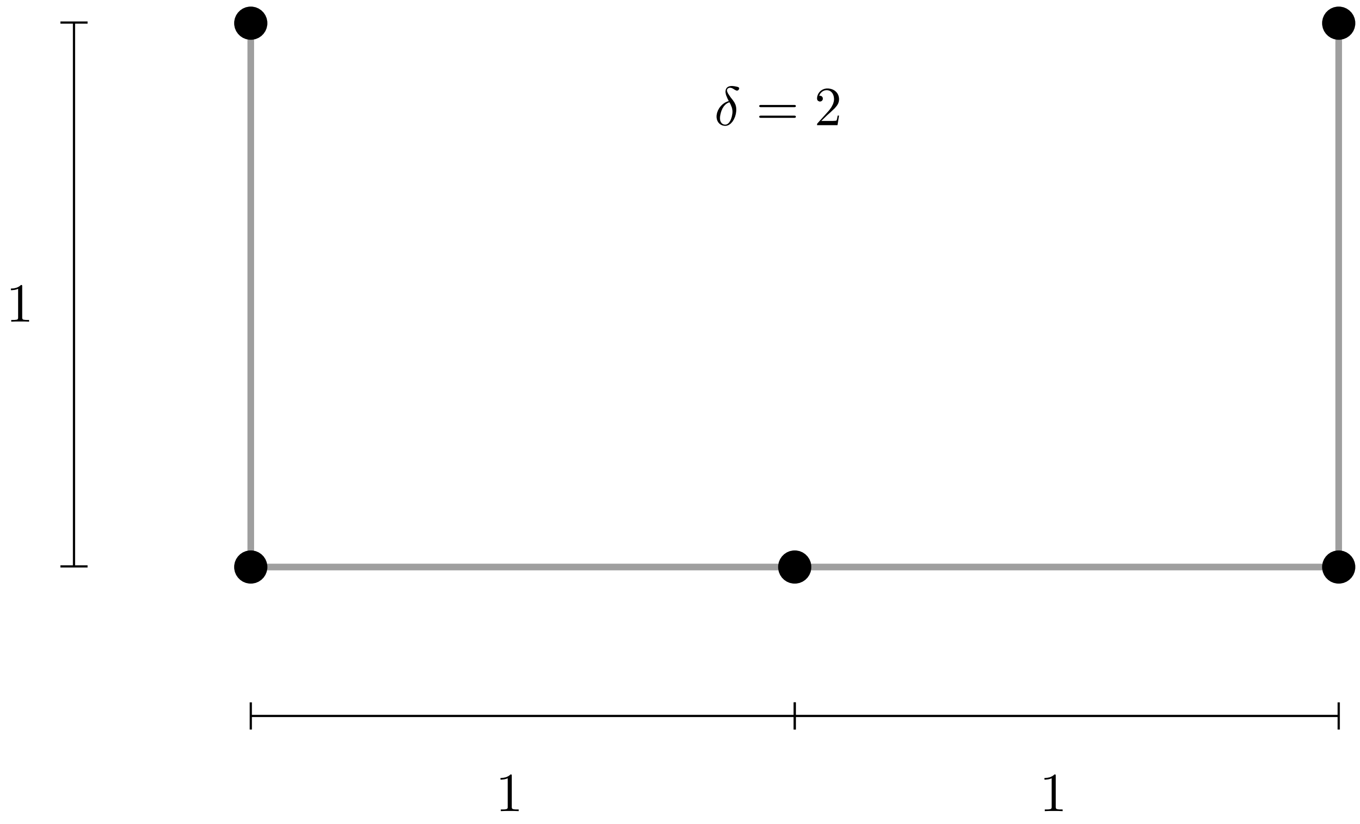
Cheong, Haverkort, Lee (unpub.): *Yes!*

Epilog — Crossings in a Min-Dilation Tree



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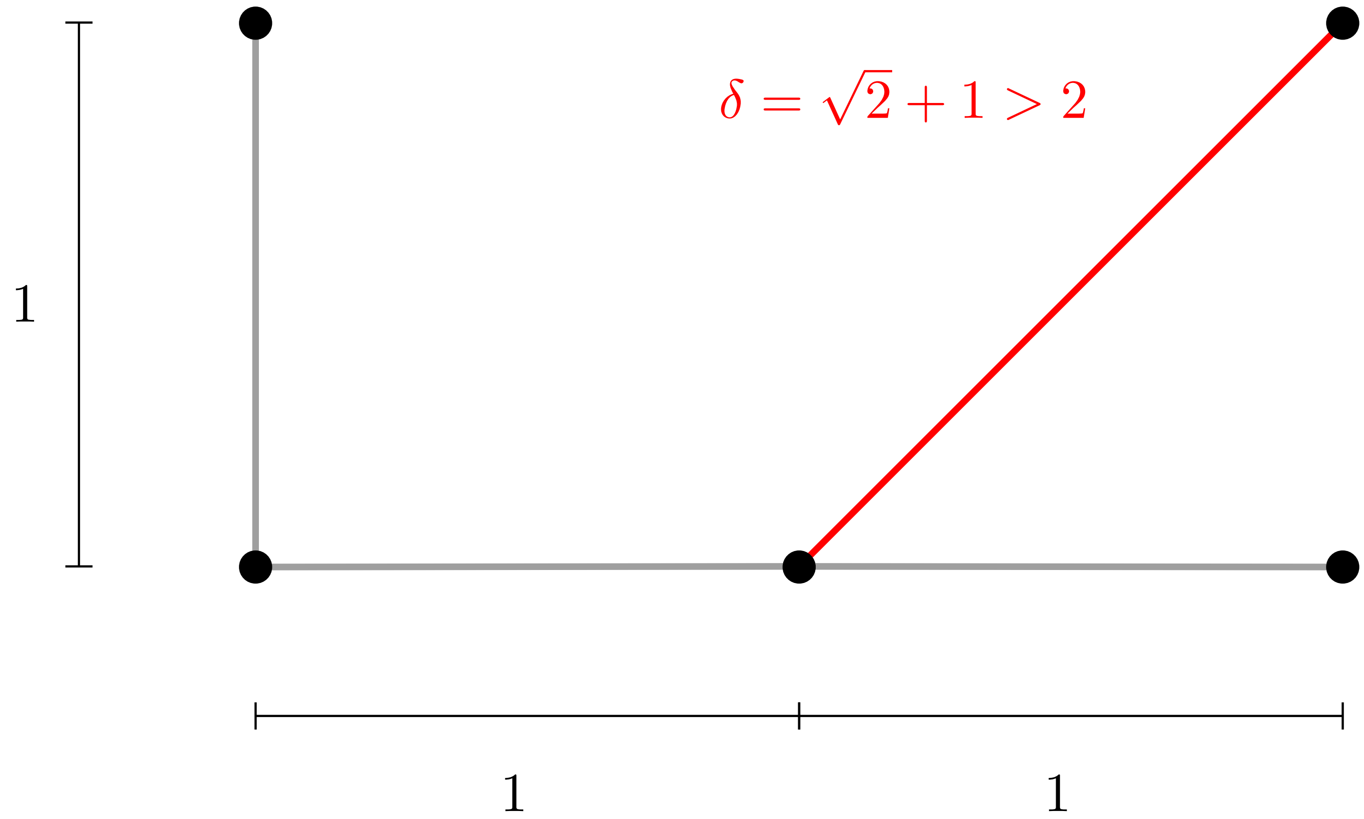

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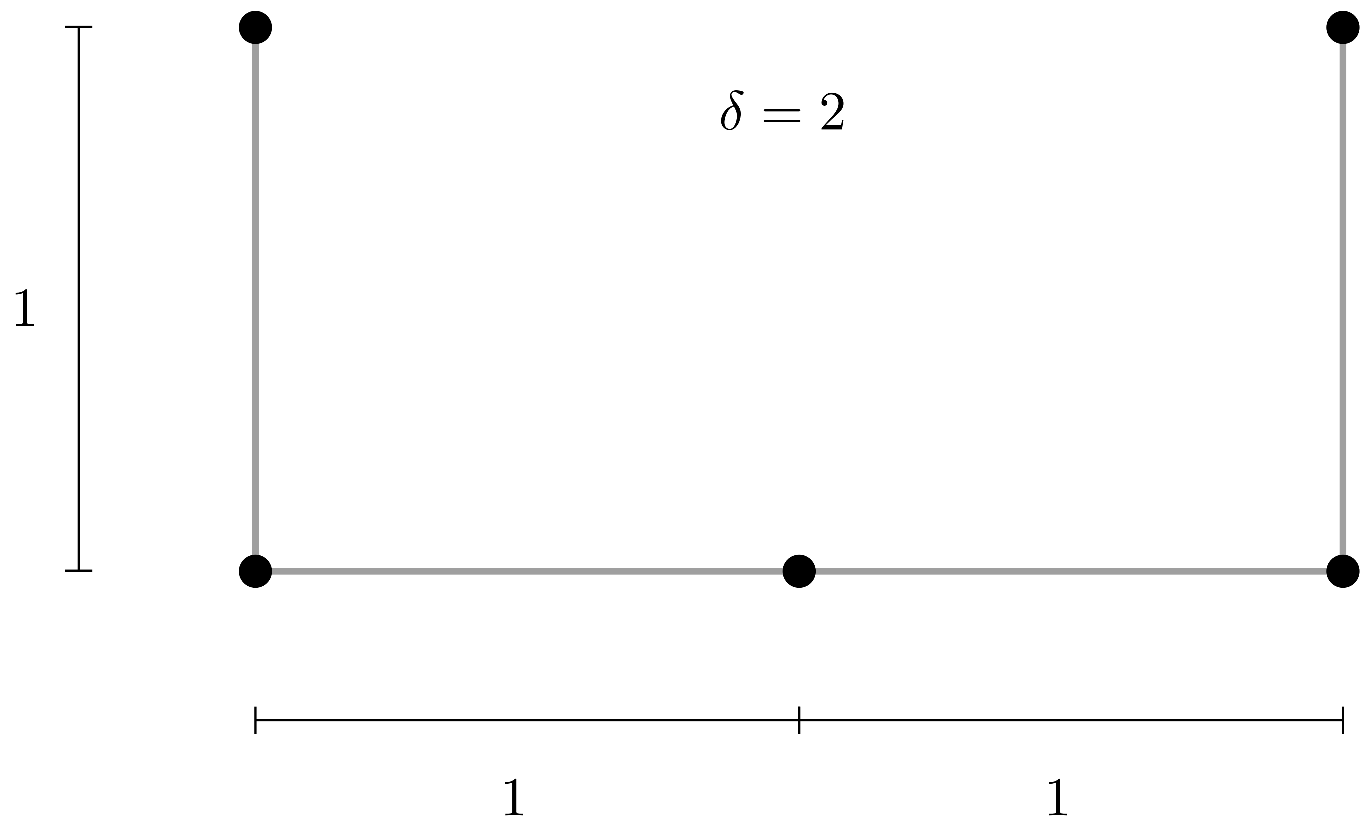

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