The Density of Iterated Crossing Points and a Gap Result for Triangulations of Finite Point Sets

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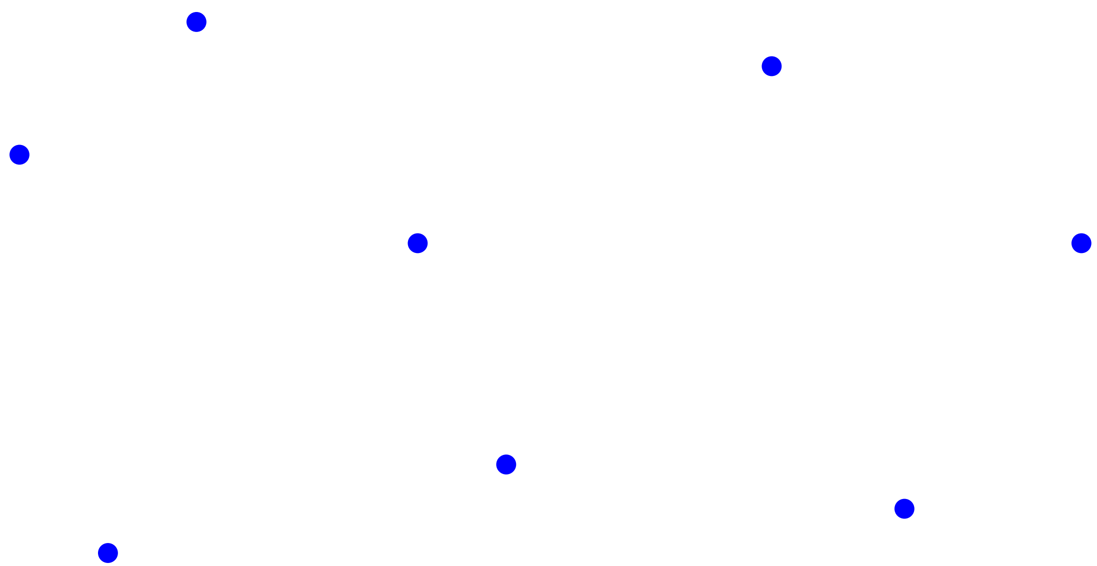
Germany

Iterated Intersections

- given: finite point set P in the plane



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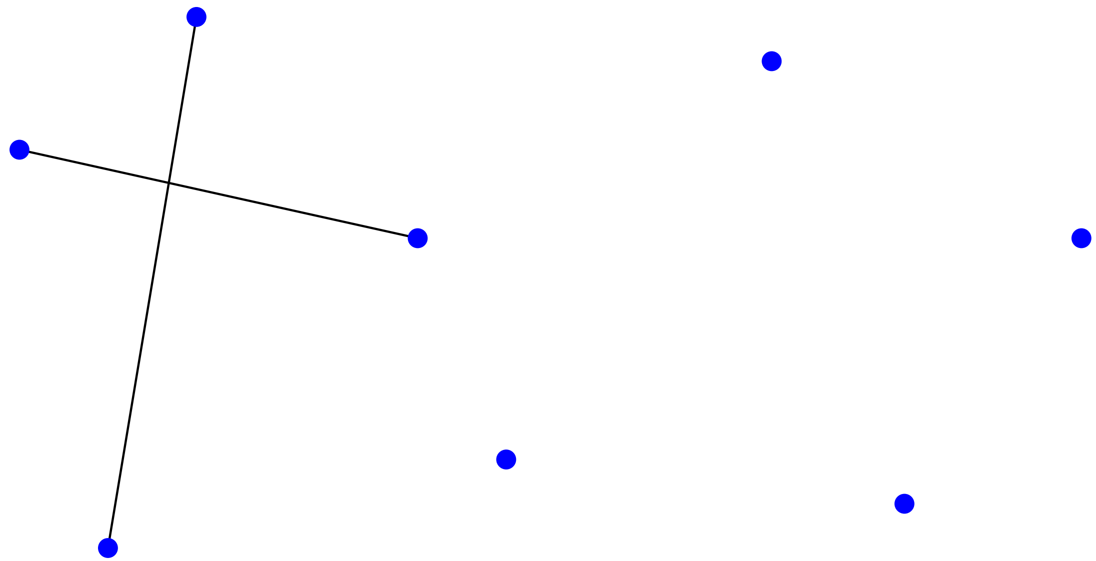

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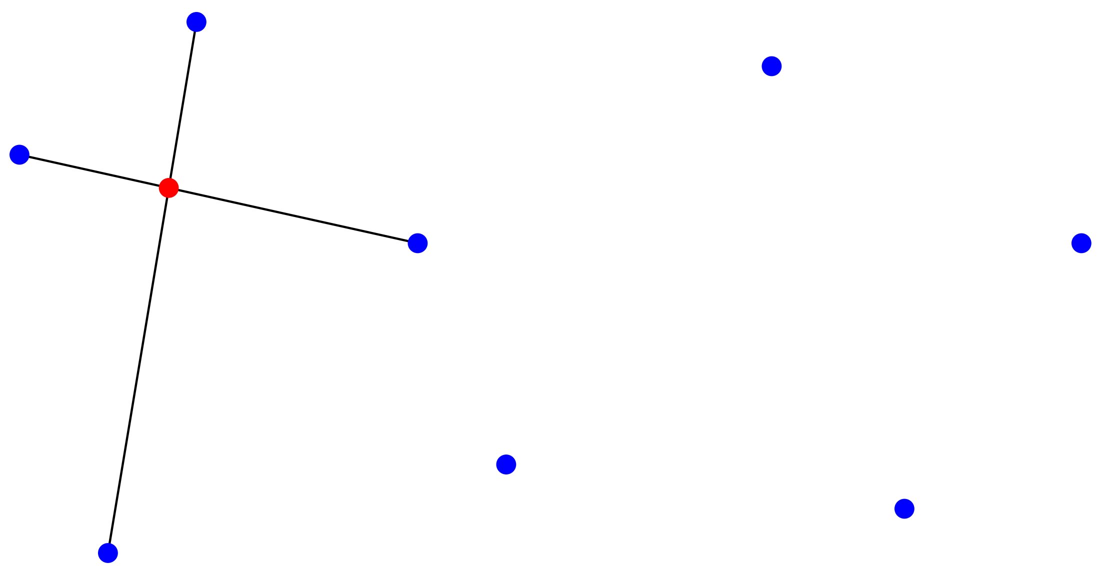

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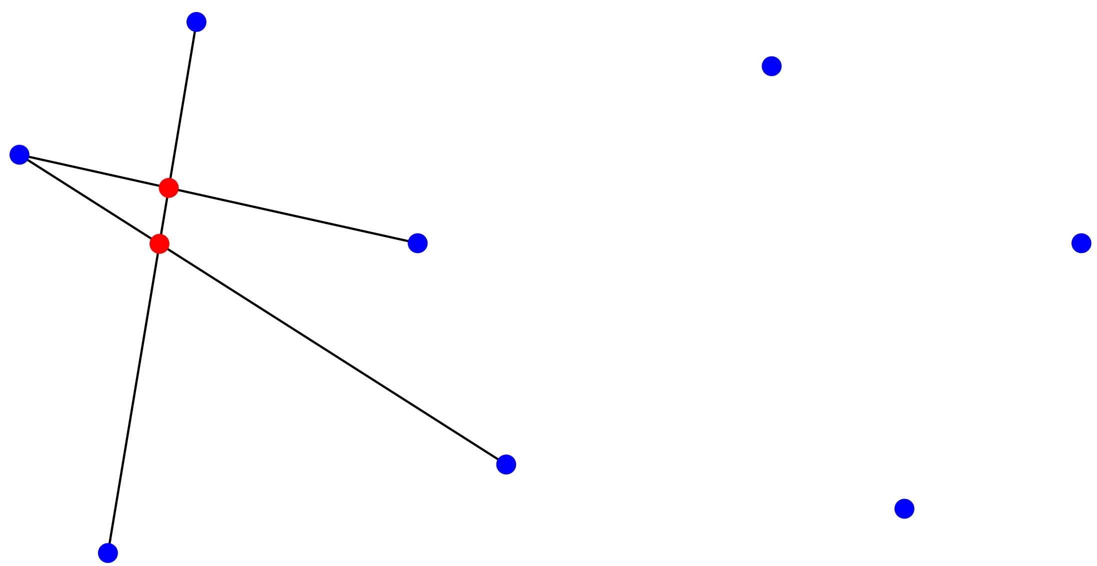

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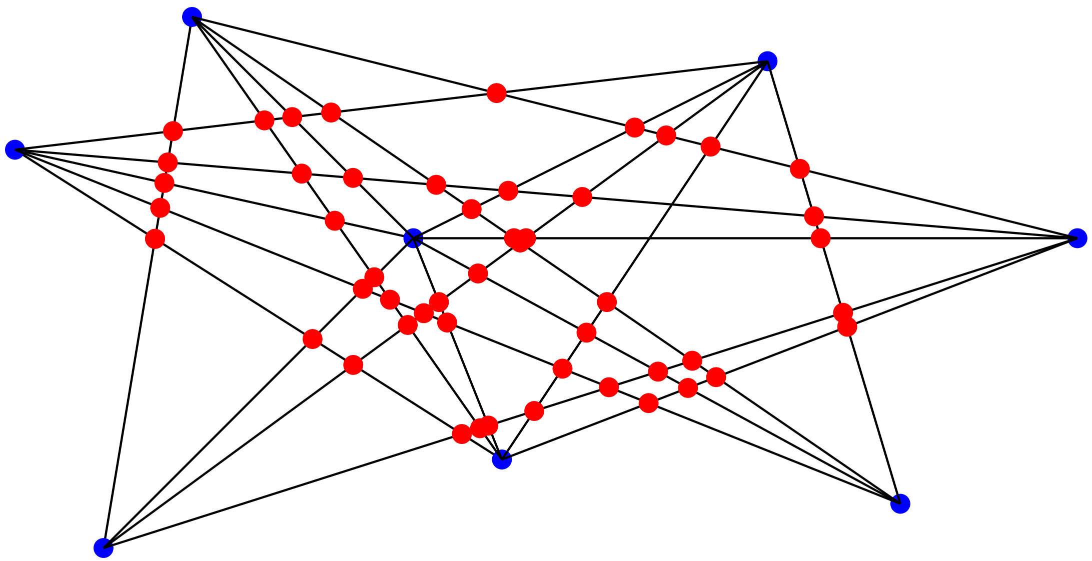

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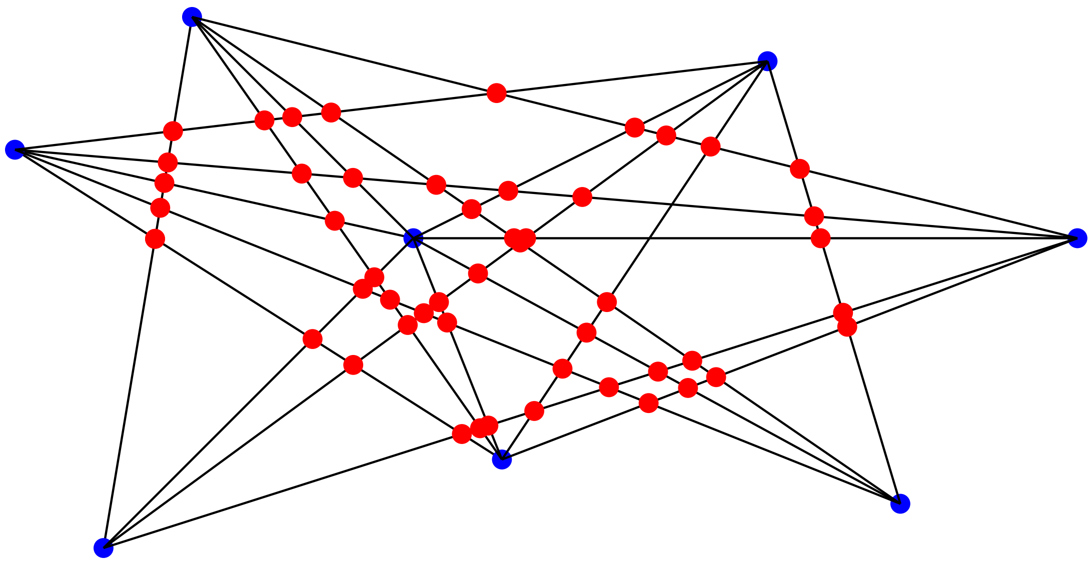

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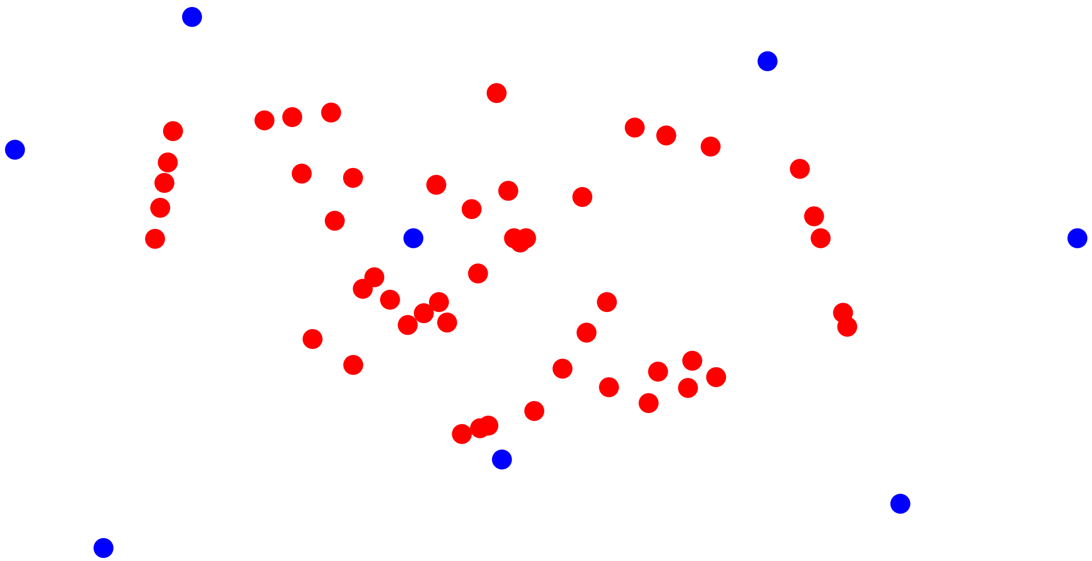

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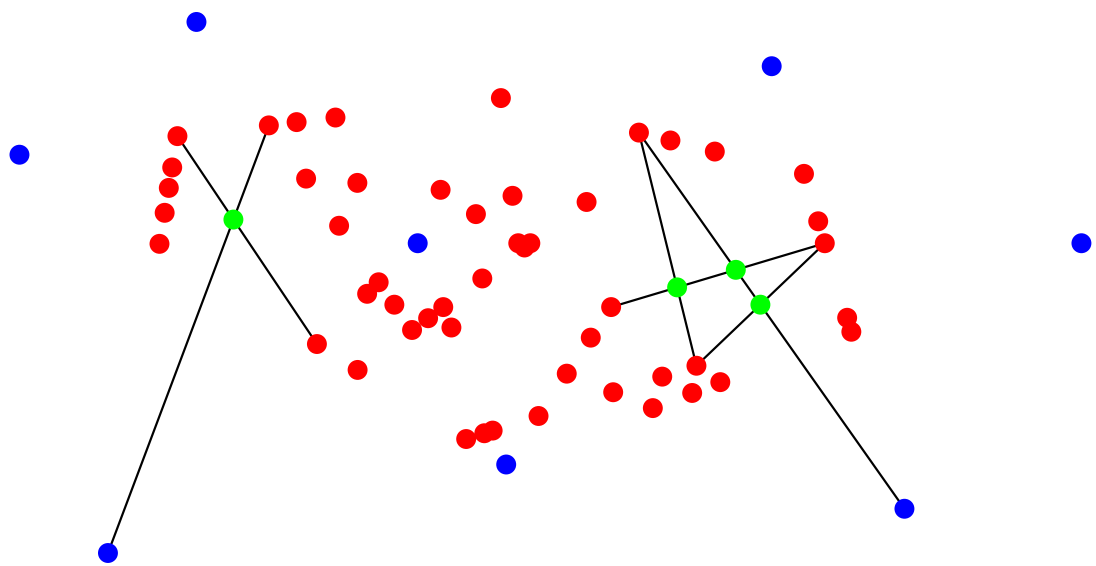

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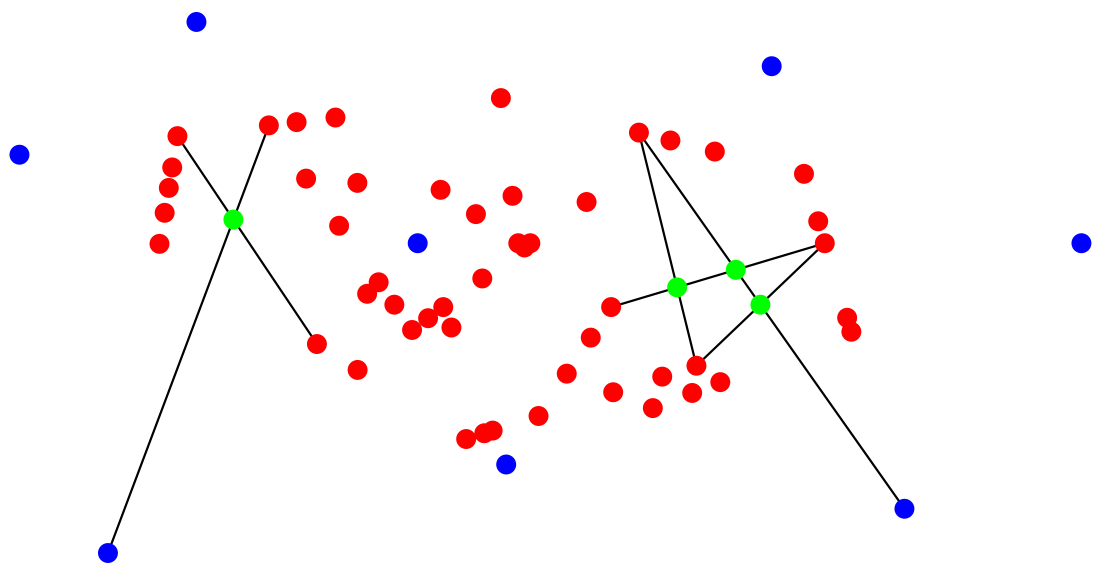

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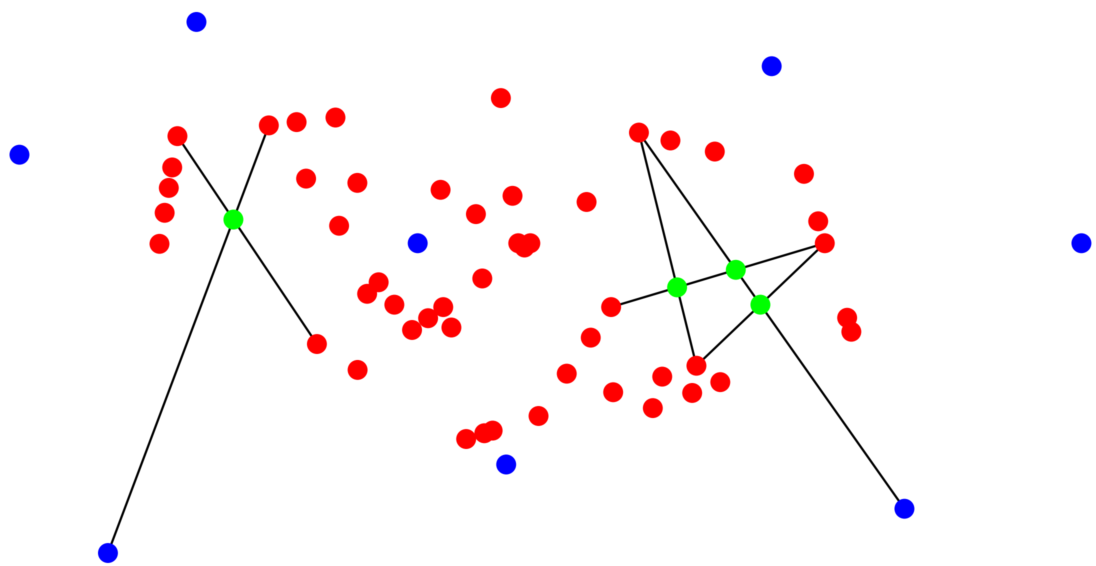
- form pairwise intersections of connecting line segments $\longrightarrow$ intersection set $P' = \{\bullet\text{'s}\} \cup \{\bullet\text{'s}\} \supseteq P$
- iterate process: $P' \longrightarrow P'' \longrightarrow P''' \longrightarrow \cdots \longrightarrow P^\infty$



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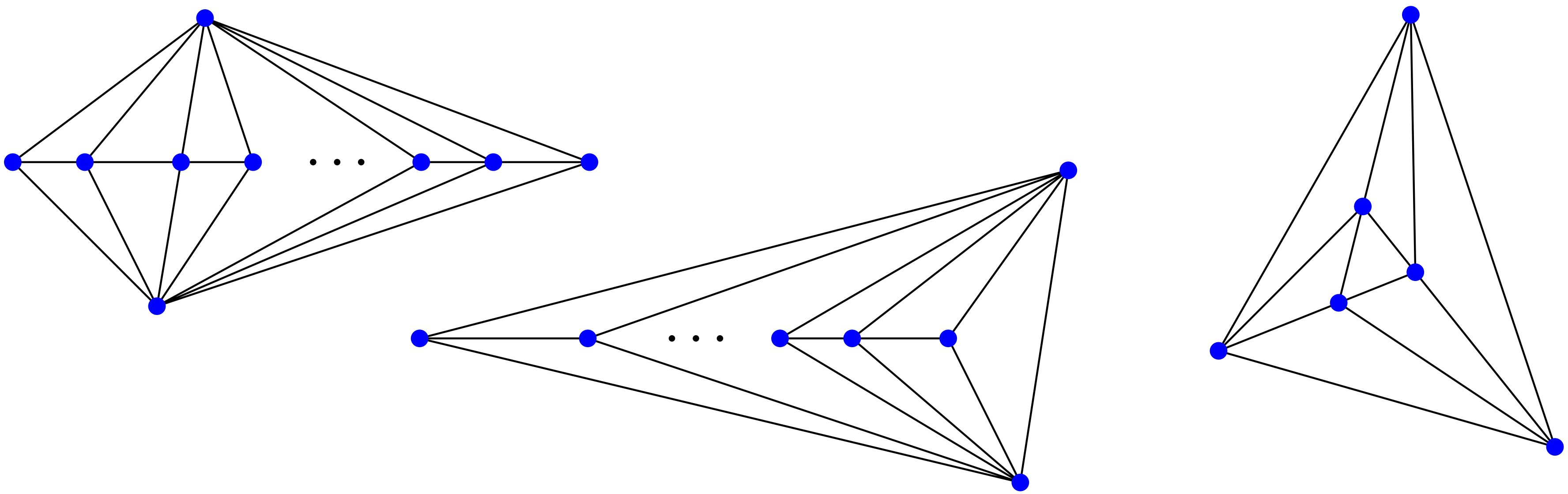
All exceptions are classified [Eppstein, *Geometry Junkyard*]:



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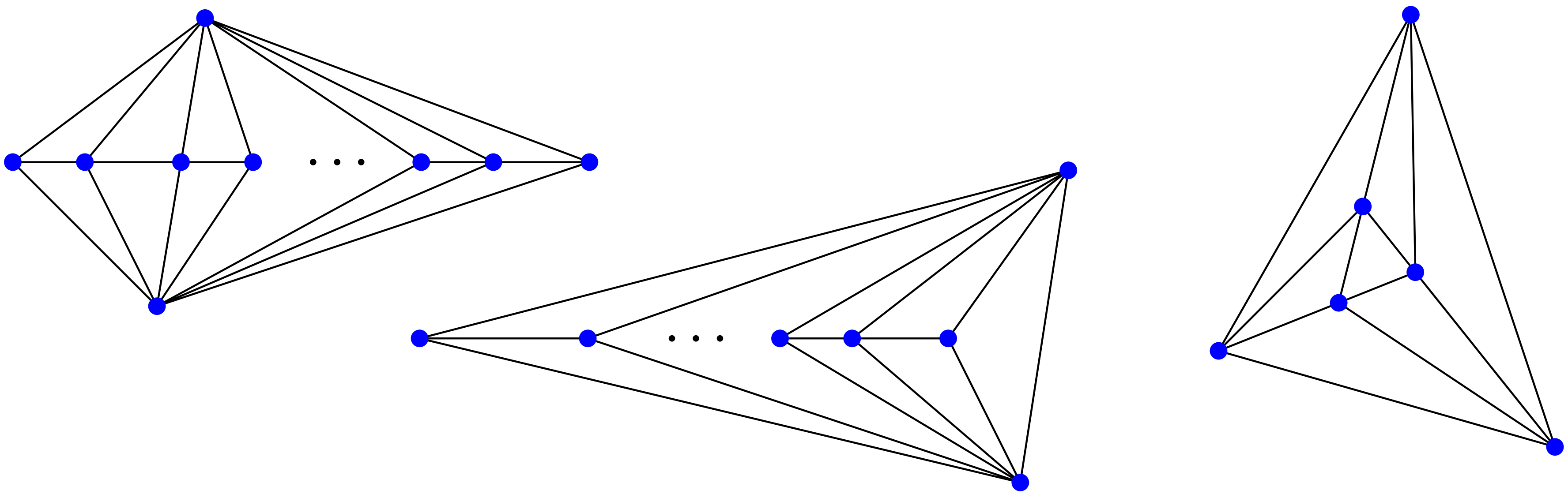
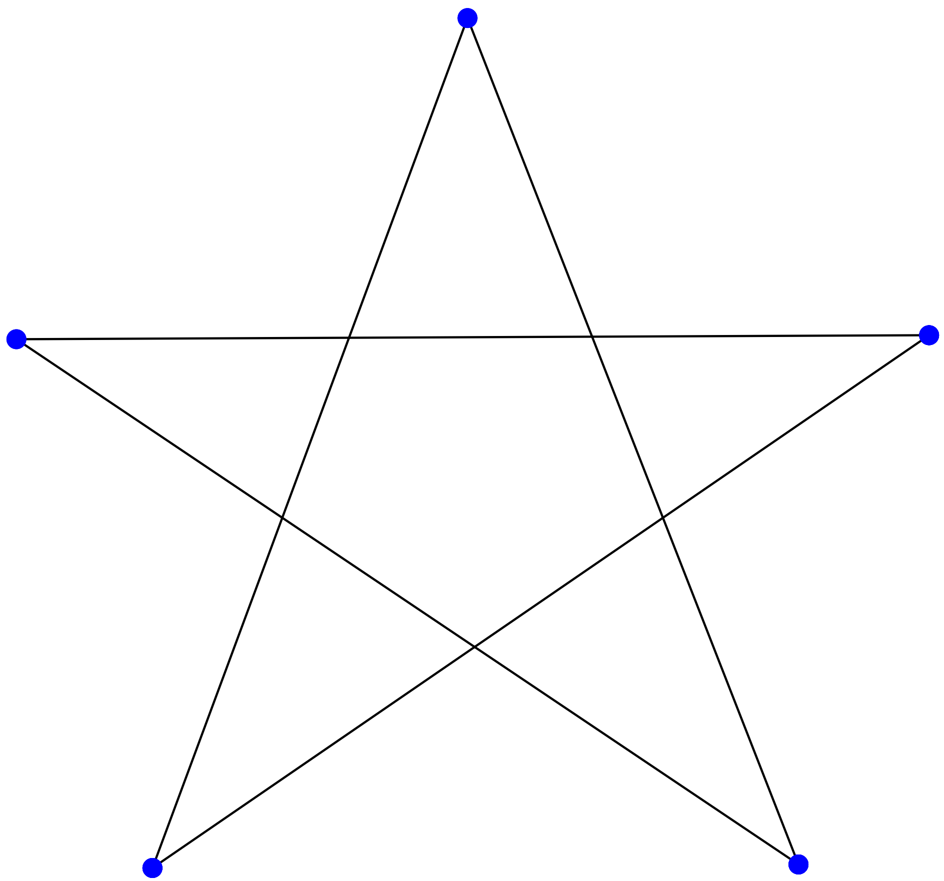


Theorem. For any other set P, the limit P^∞ is dense in some region of positive measure.

The Pentagon is Everywhere

For any non-stable P, the set P'''' contains 5 points in convex position.



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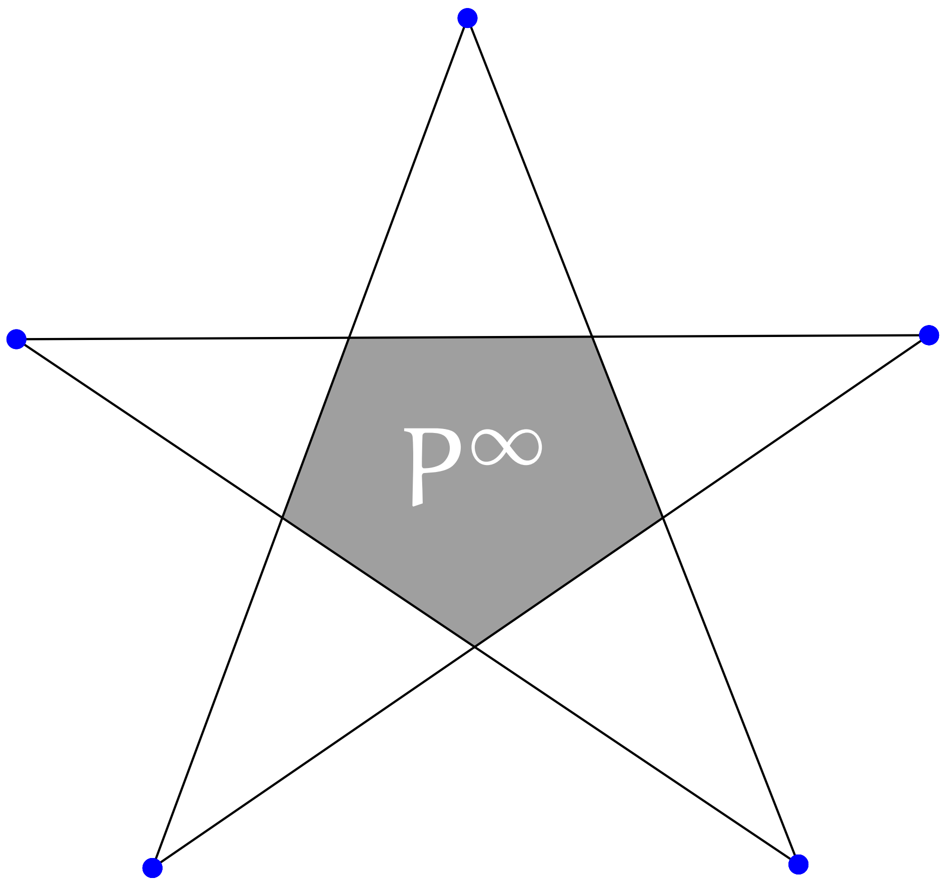

Allowing small Detours

Could we embed the initial point set in a finite network
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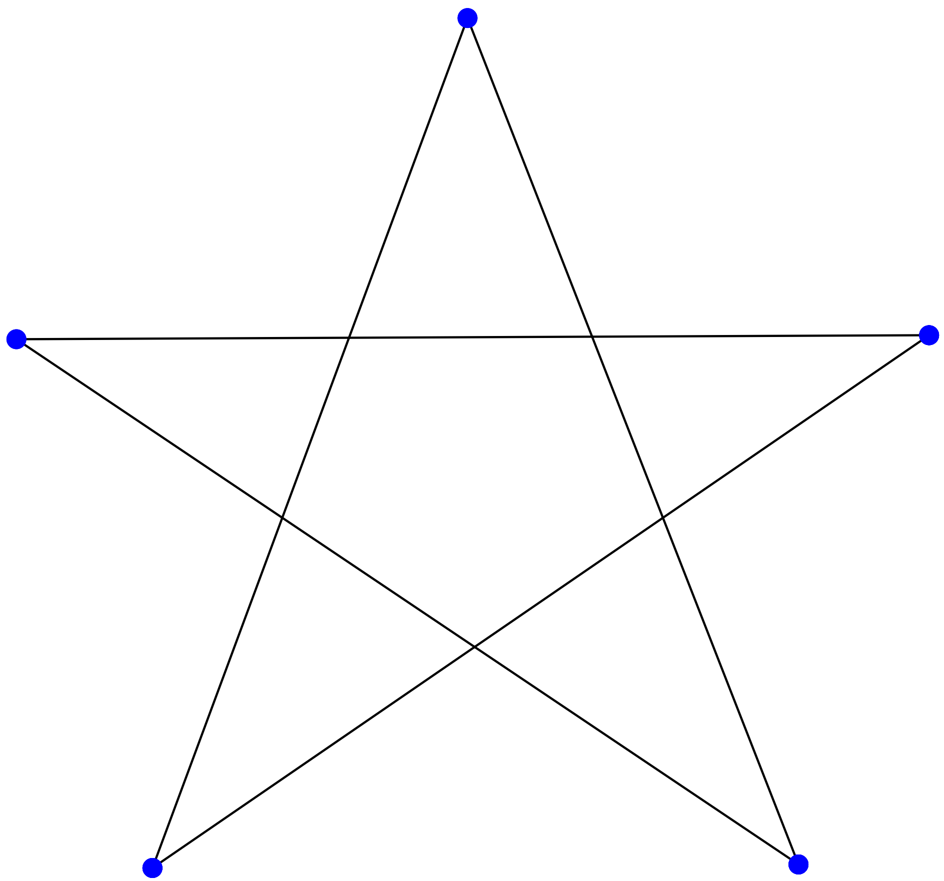

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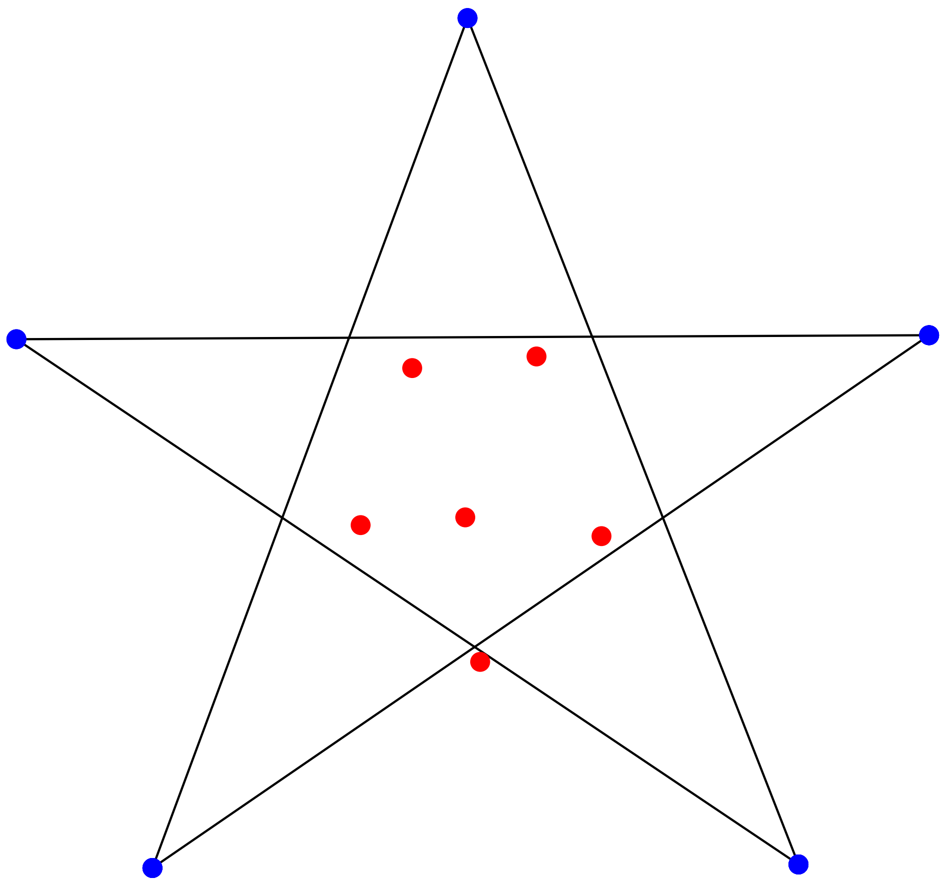

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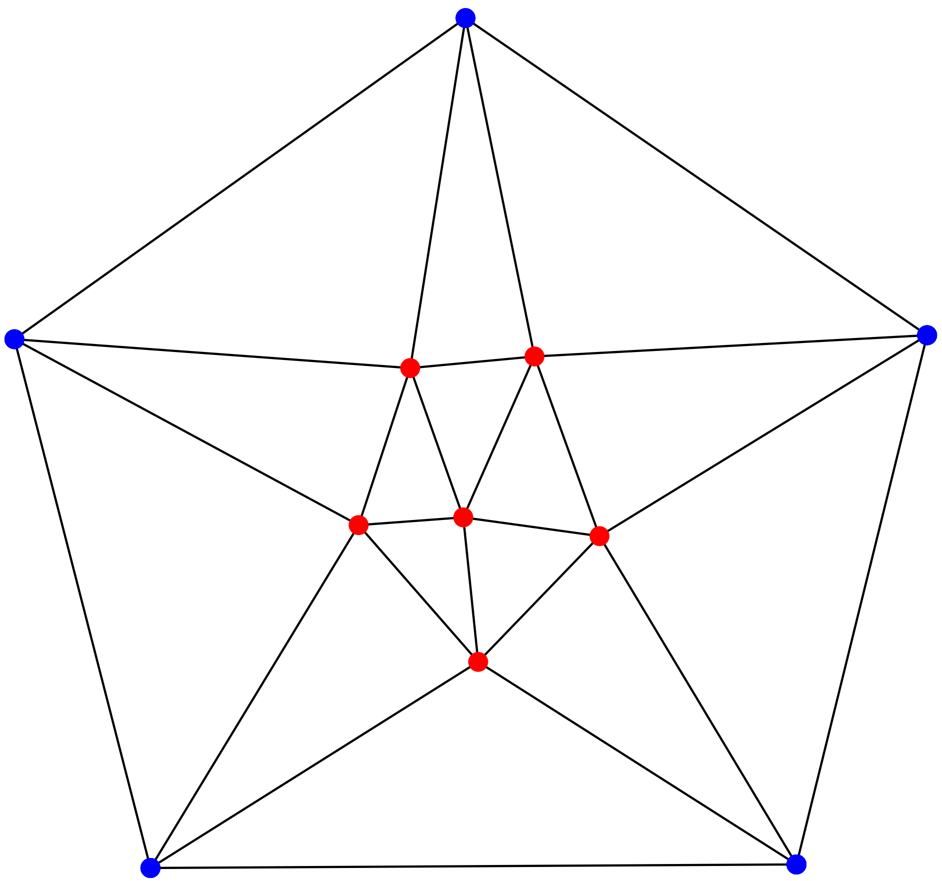

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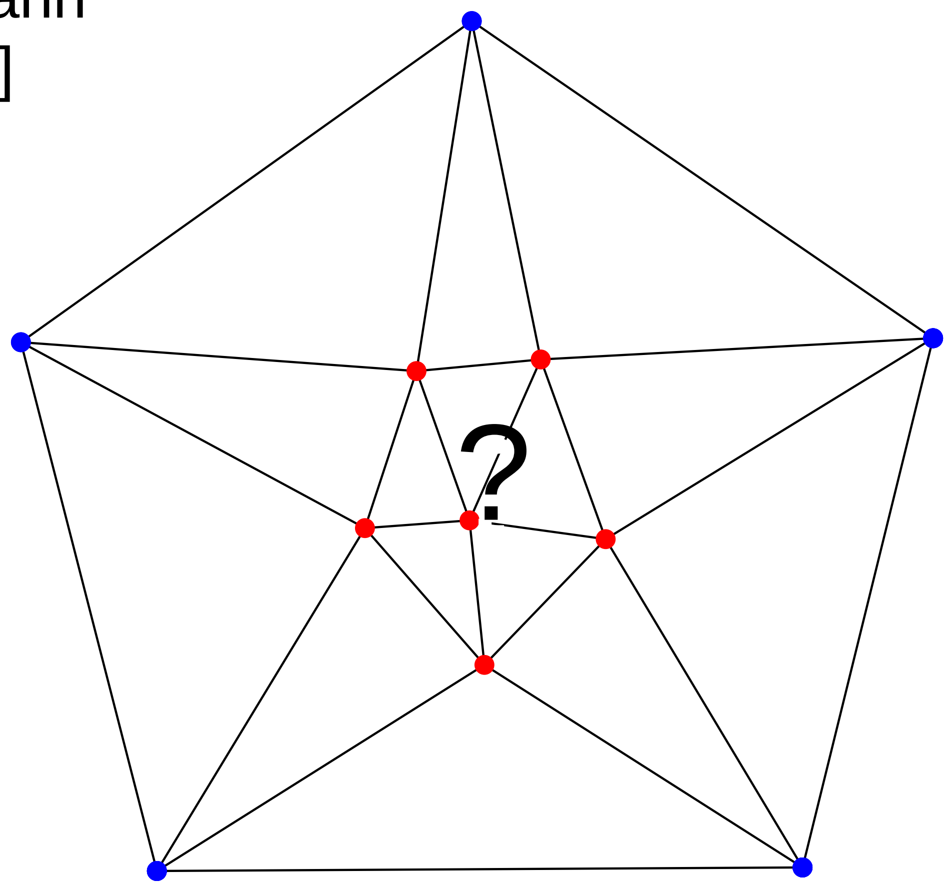
[Ebbers-Baumann et. al., 2005]



Dilation

$G = (V, E)$ plane graph (without crossings).

Define the *dilation* of G:

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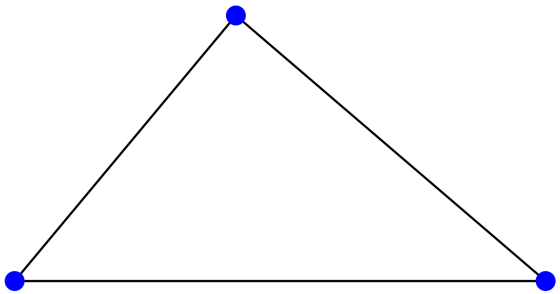
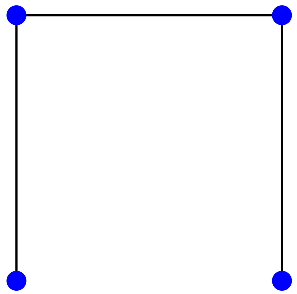


$\delta = 1$



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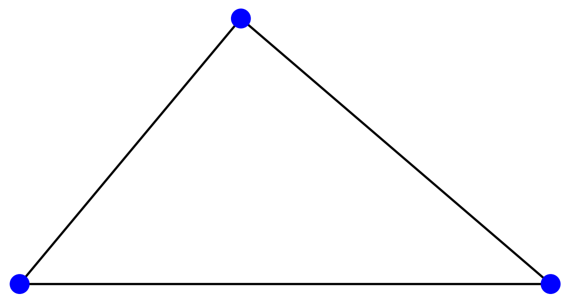

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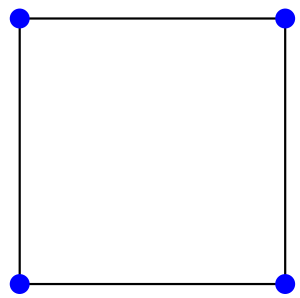

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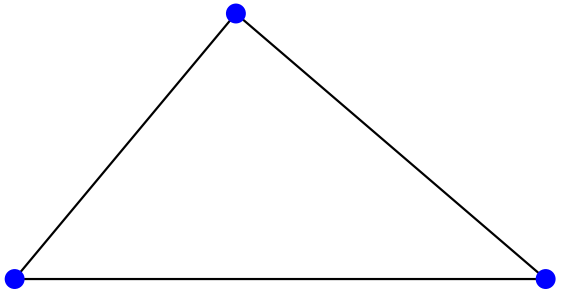

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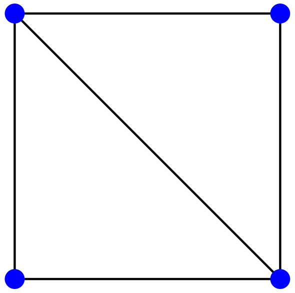

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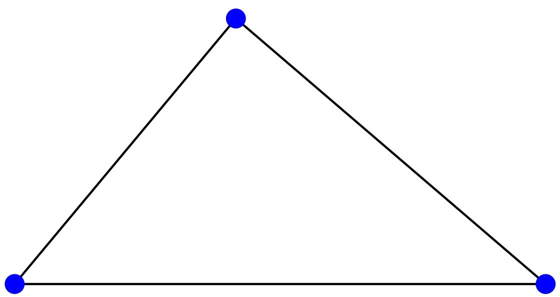

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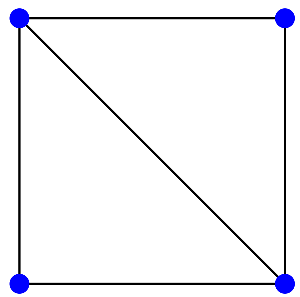

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Theorem. [Keil & Gutwin, 1992]
The dilation of any Dilaunay triangulation is bounded by 2.42.

Intersections and Dilation

iterated intersections		dilation-1 graphs
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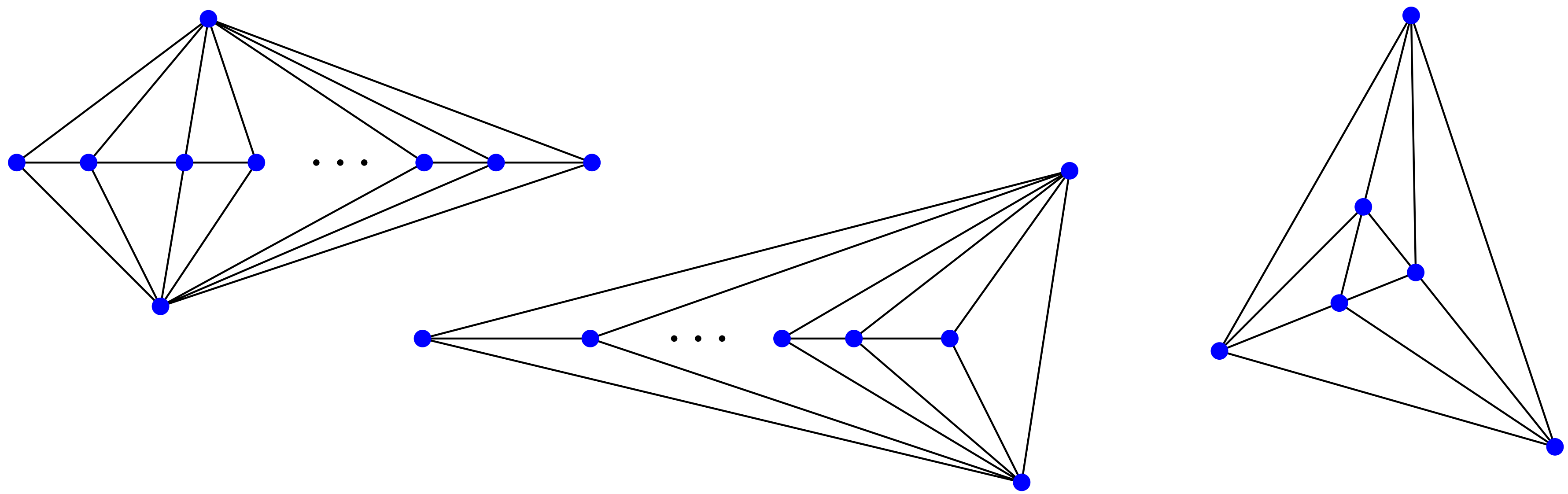

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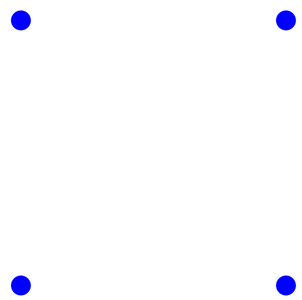

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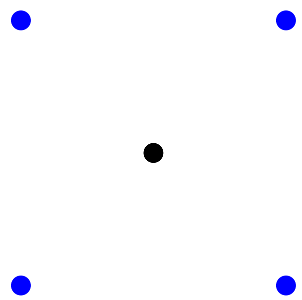

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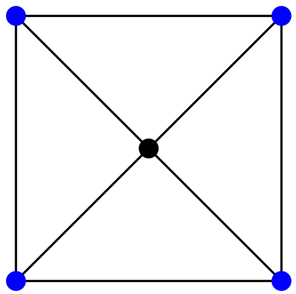

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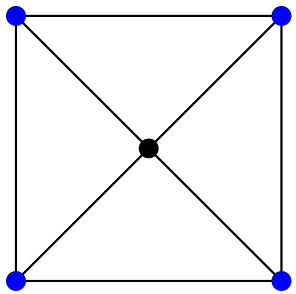

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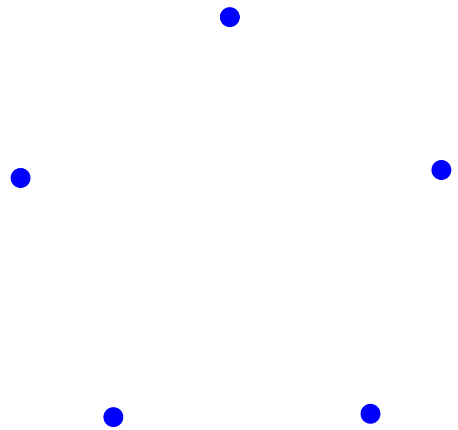

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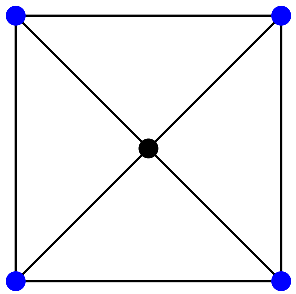

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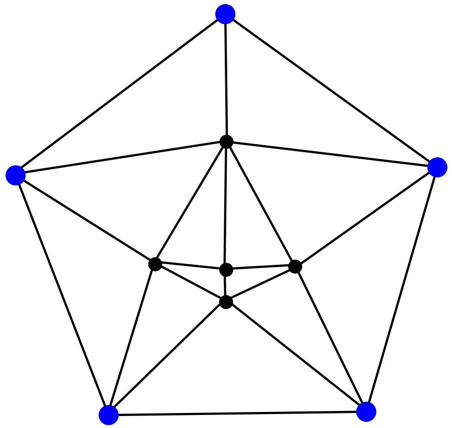

$\Delta < 1.02$ [Lorenz, 2004]

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Theorem. Every point set P with $\#P^{\infty} = \infty$ has dilation $\Delta(P) > 1$.

The dilation of any triangulation T that contains P as vertices
is bounded away from 1 by some γ_P. There's a *"gap!"*

Overview

Density Theorem.

If P^∞ is infinite then it is dense in some region.

$$\downarrow$$

Approximation Lemma. (*from exact intersections to dilation* > 1)

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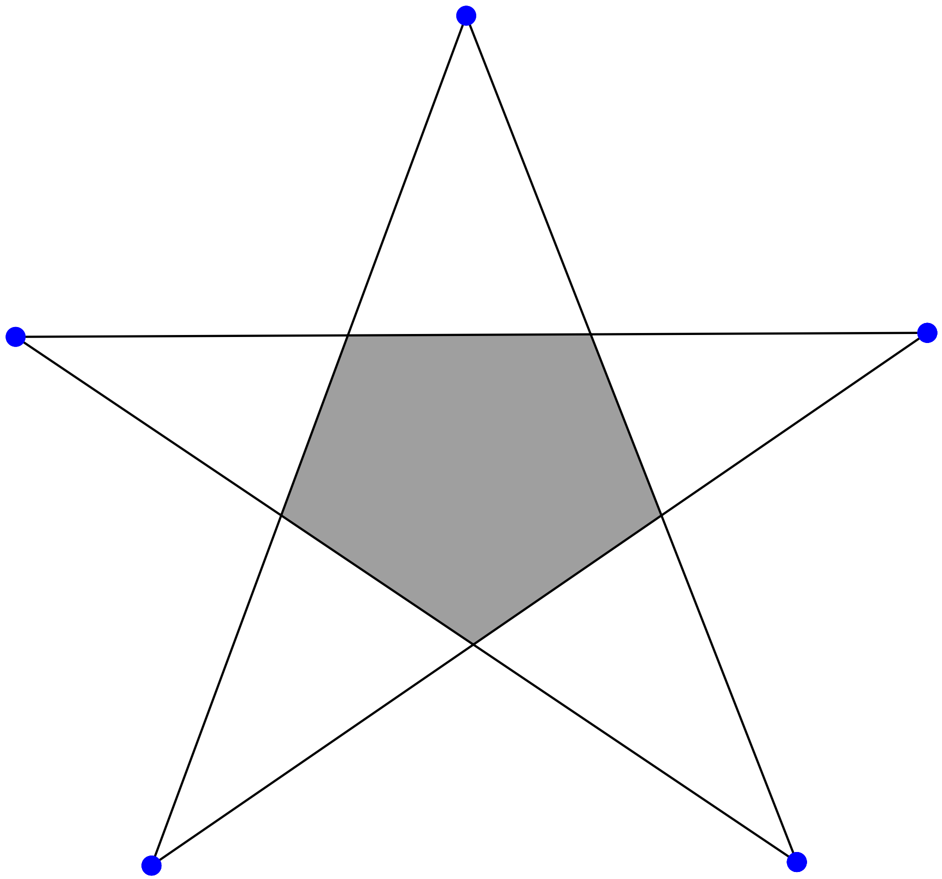

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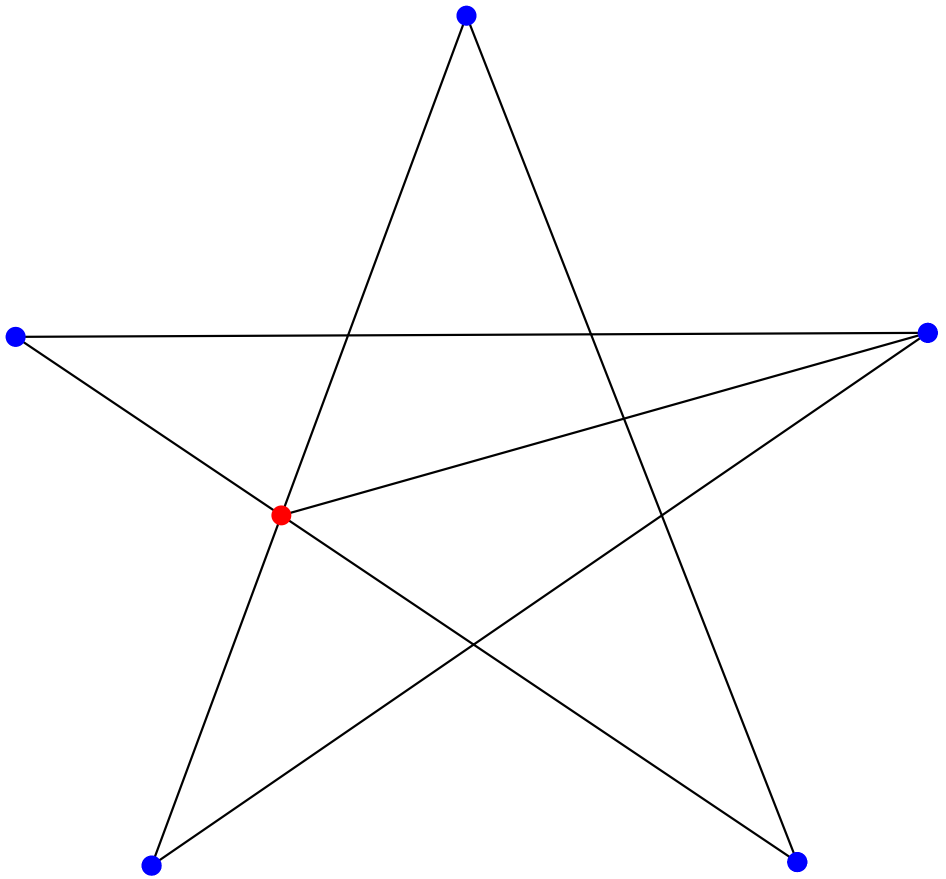

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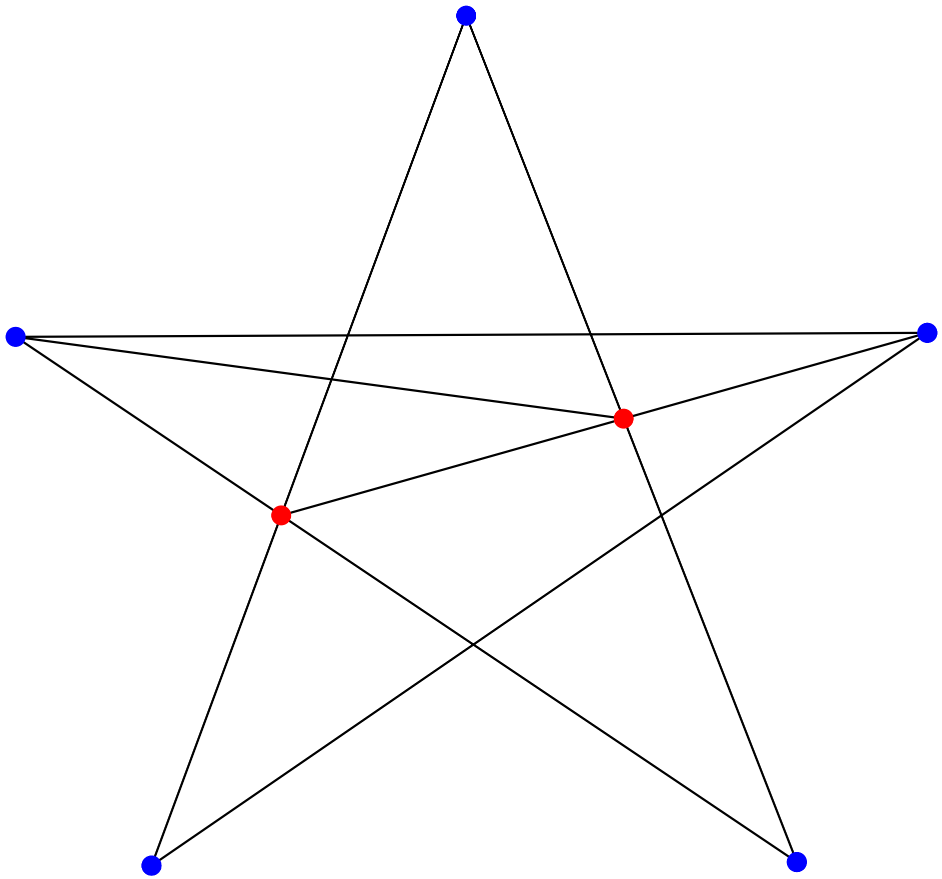

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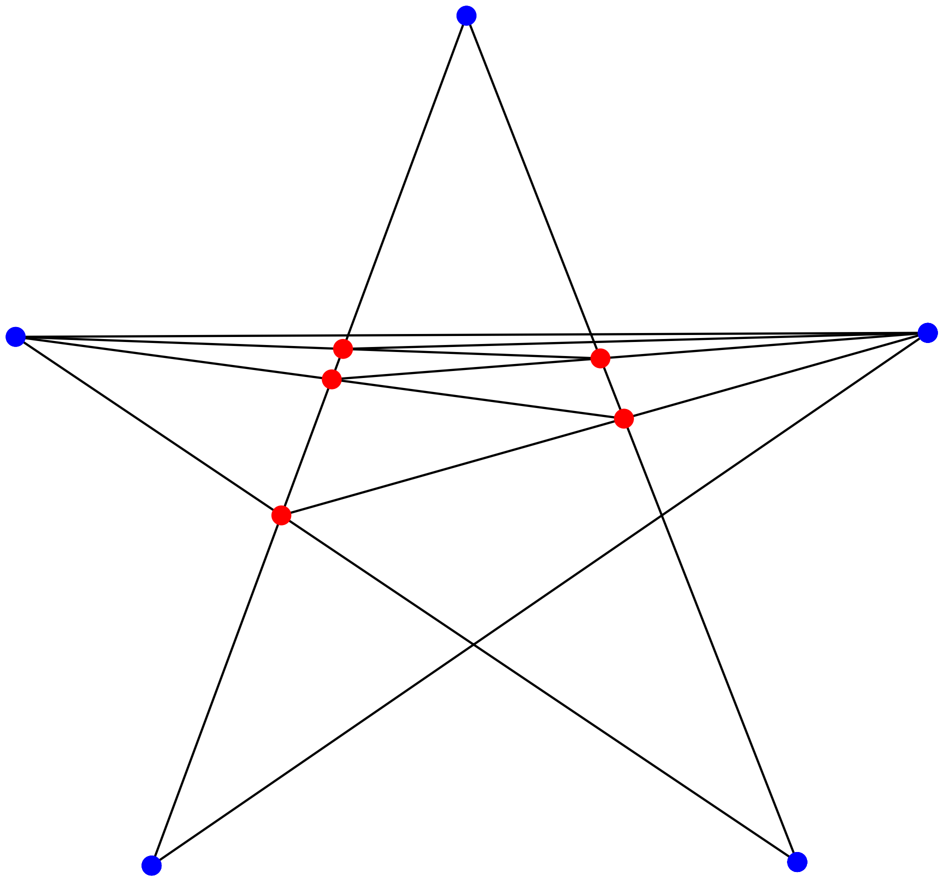

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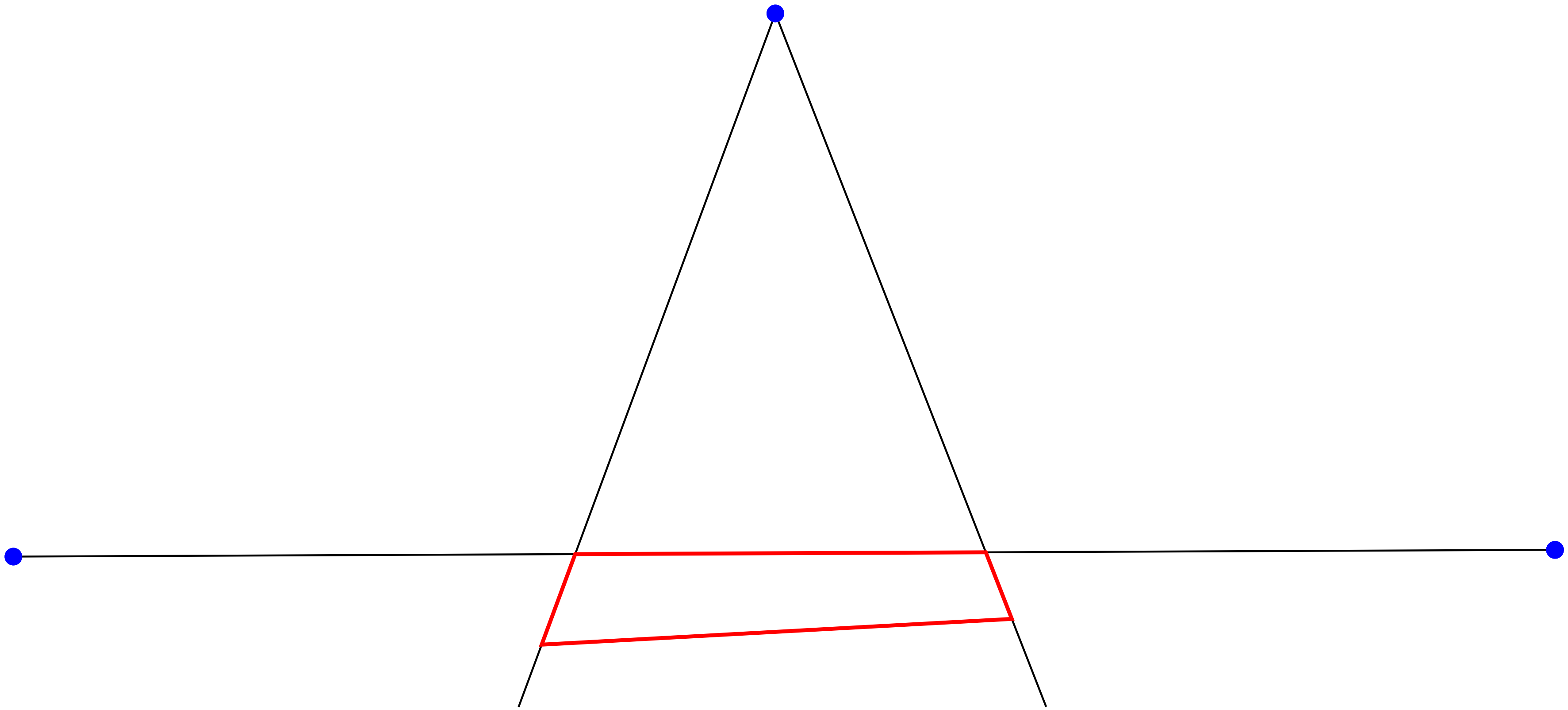

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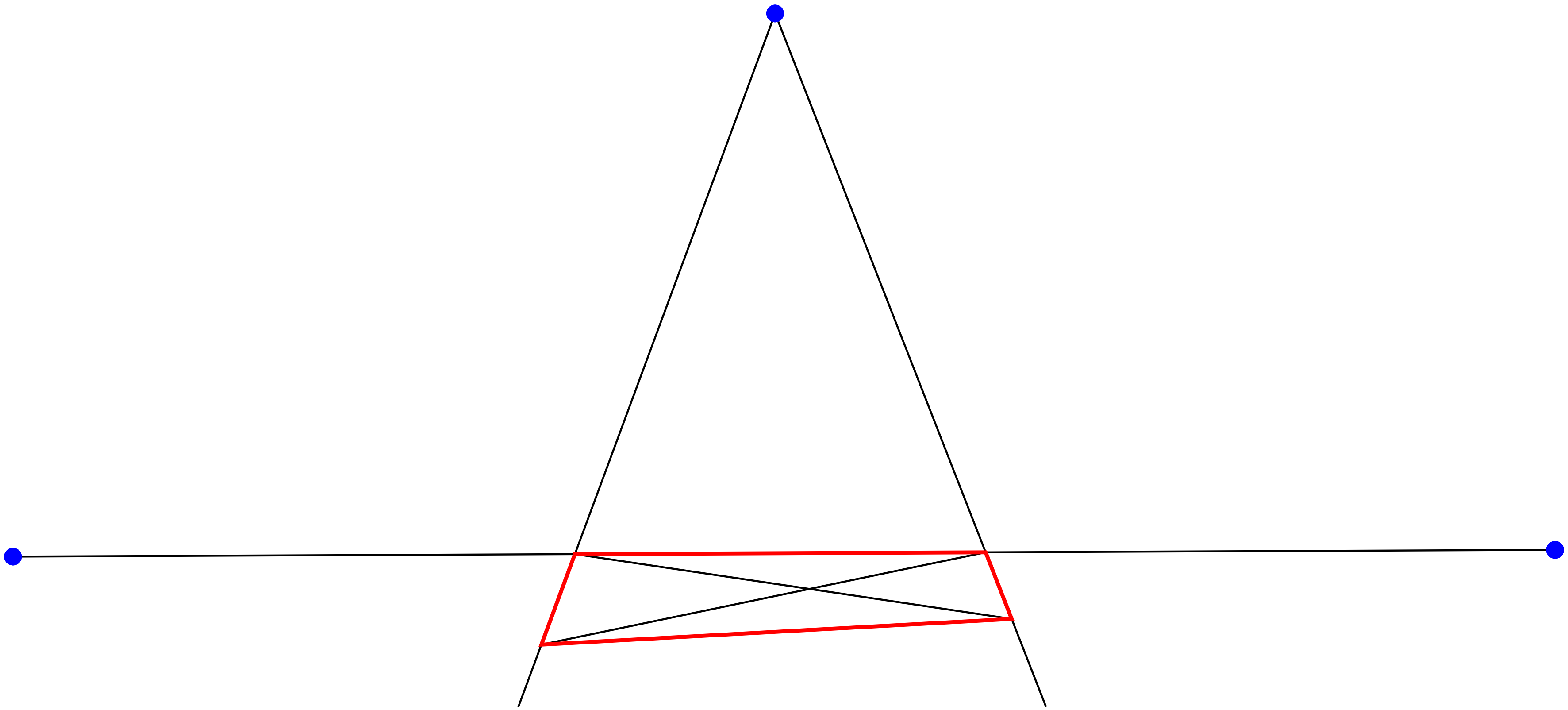

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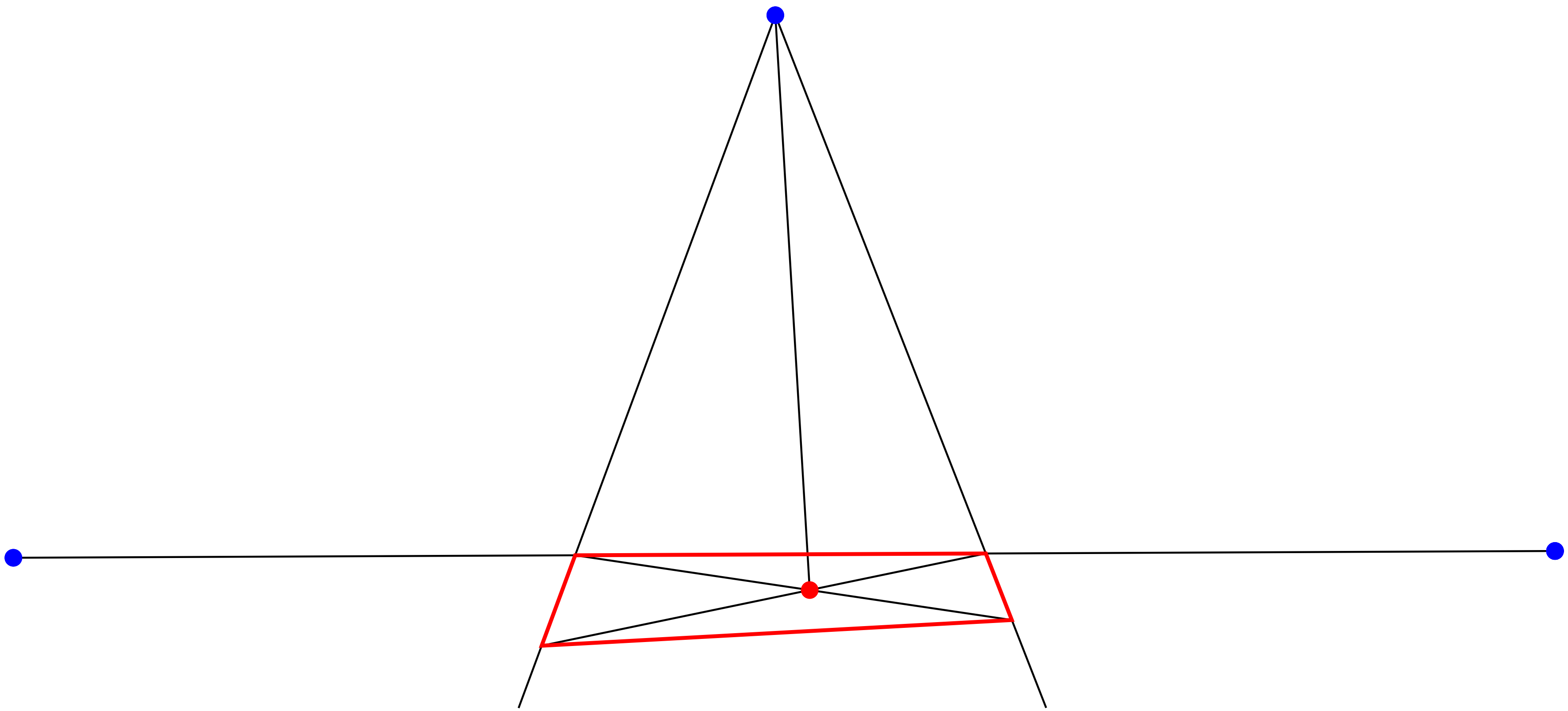

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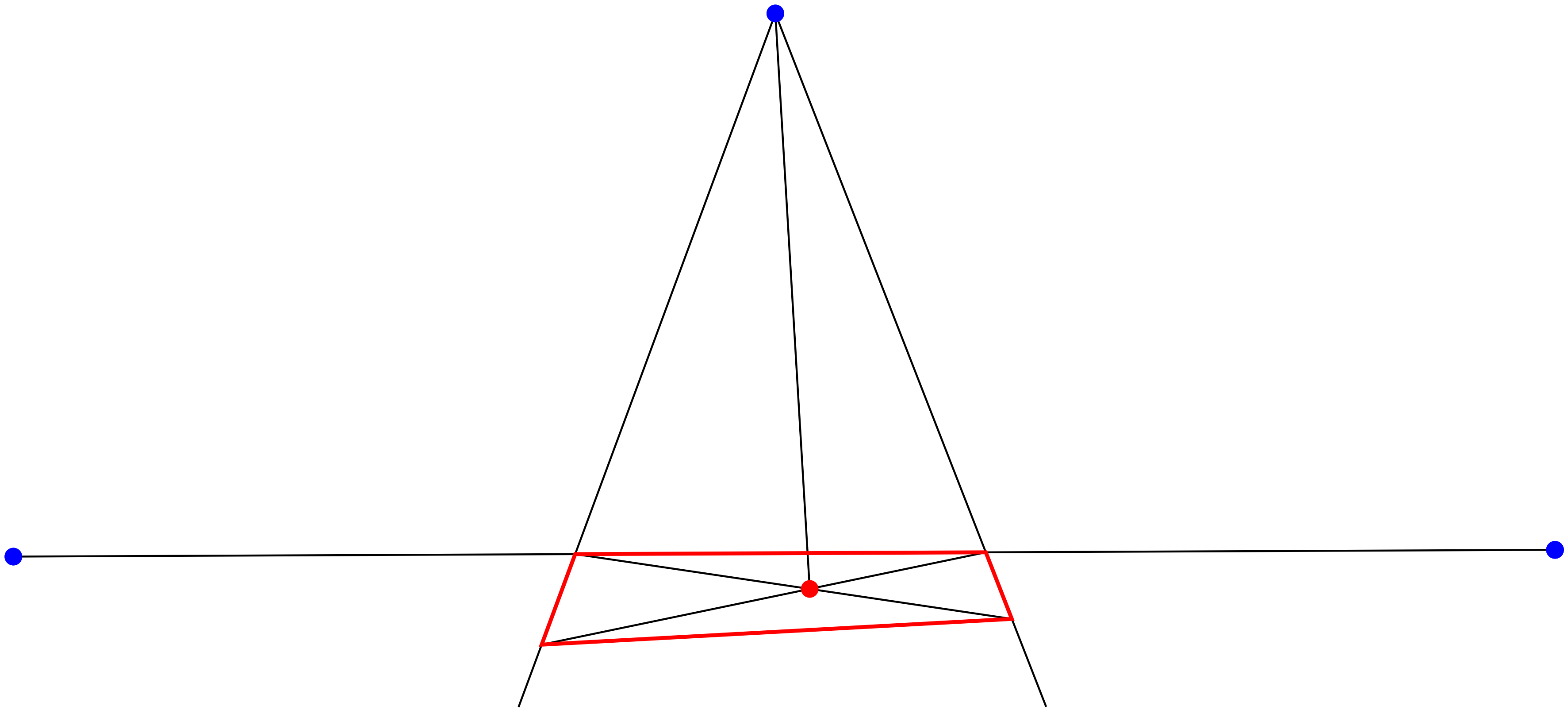

Cor. [Ismailescu & Radoičić, 2004]
If *whole lines* instead of segments, then the process on 4 pt's in non-convex position densely covers the whole plane.

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Then there exists $\delta > 1$ such that:
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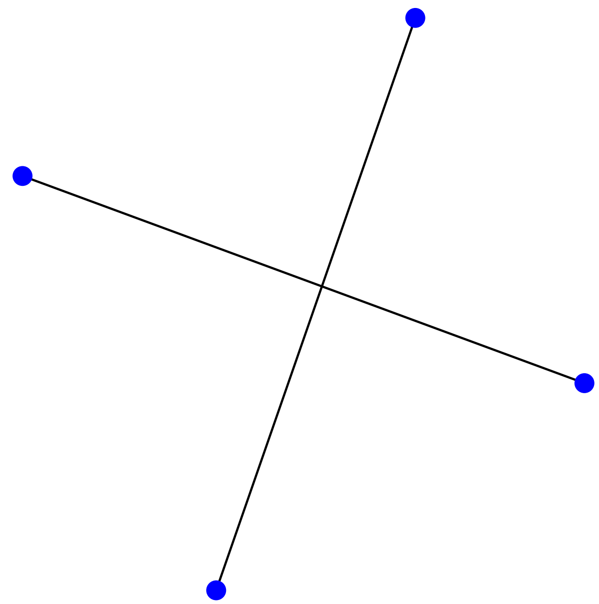

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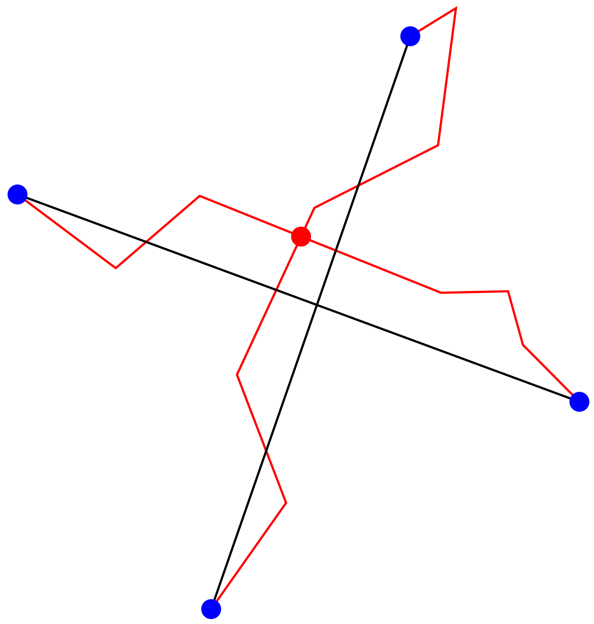

Approximating P^∞

Def. Point set Q is an ϵ-*cover* for point set P if the (closed) ϵ-ball around every point $p \in P$ contains a point $q \in Q$.
($\Leftrightarrow$ directed Hausdorff distance $\leq \epsilon$)

Lemma. Given finite point set P and parameters $k \in \mathbb{N}$ and $\epsilon > 0$.
Then there exists $\delta > 1$ such that:
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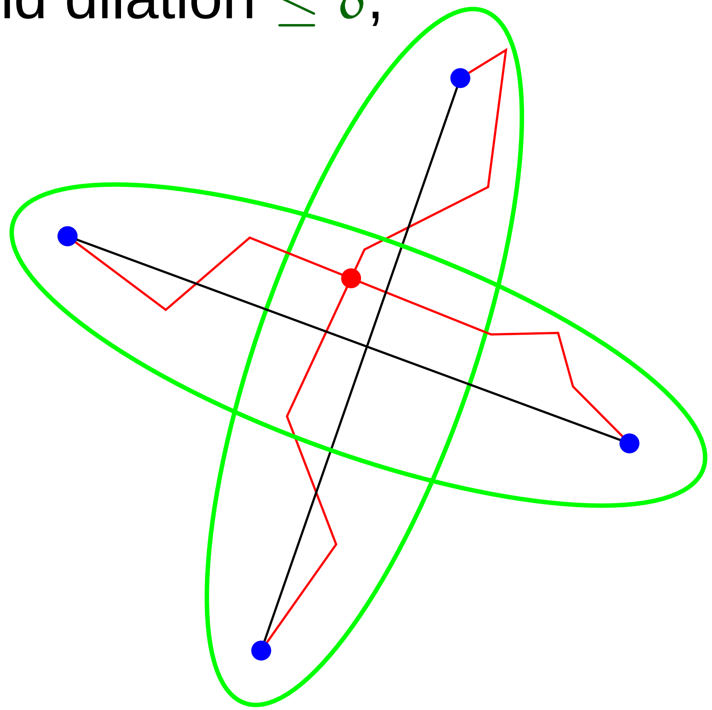

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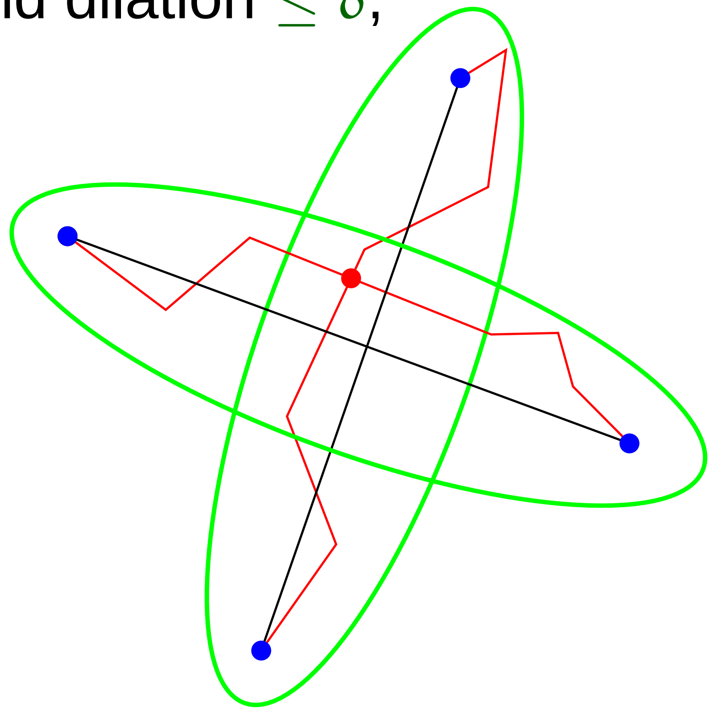
Pentagon Density Theorem
$+$ *ϵ-Cover Lemma*

$\Longrightarrow$ can force approximations of any set we want
(and in arbitrary quality)



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Overview

Density Theorem.

If P^∞ is infinite then it is dense in some region.

$$\downarrow$$

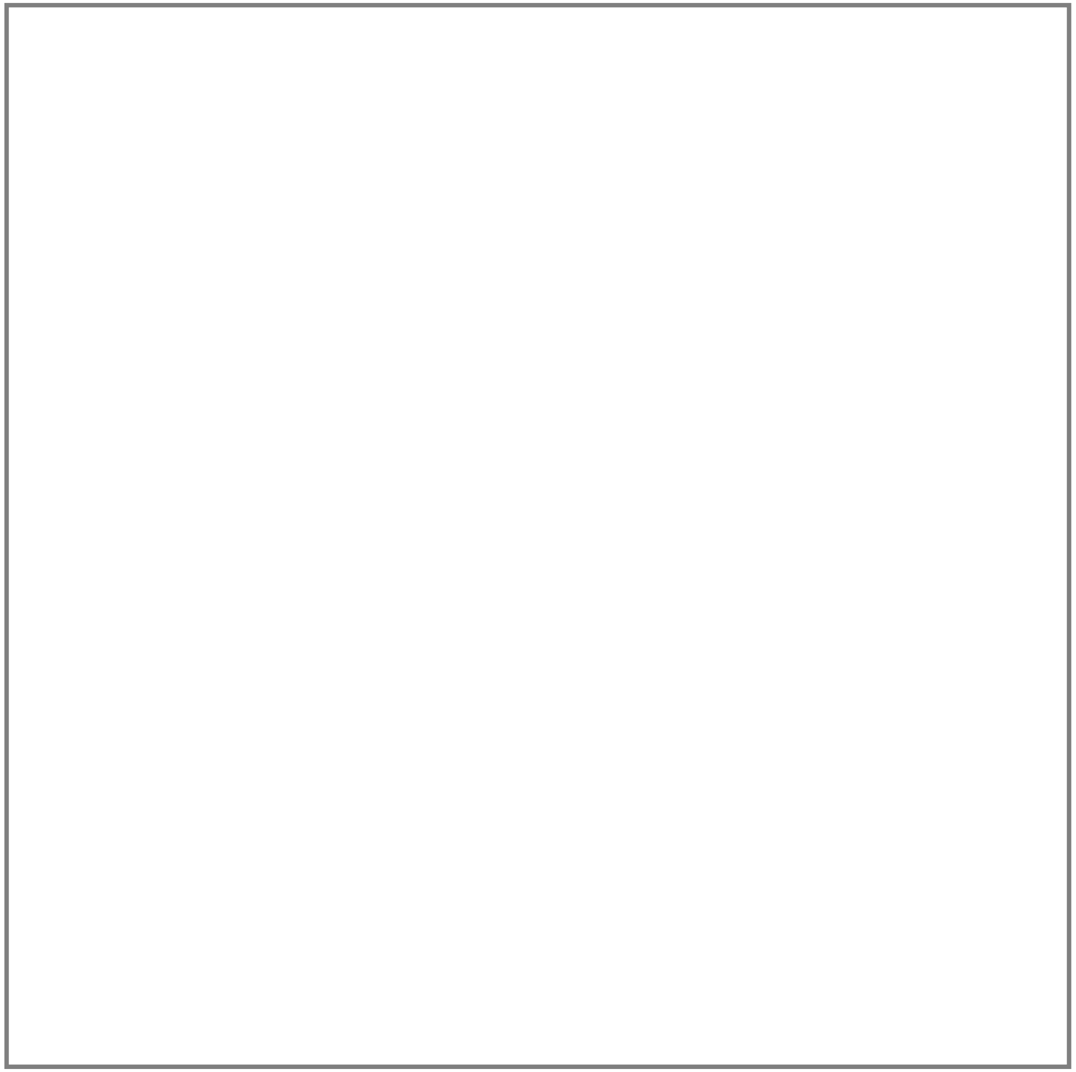
Approximation Lemma. (*from exact intersections to dilation* > 1)

$$\downarrow$$

Gap Theorem.

Every point set P with $\#P' = \infty$ has dilation $\Delta(P) > 1$.

A Self-Replicating Square



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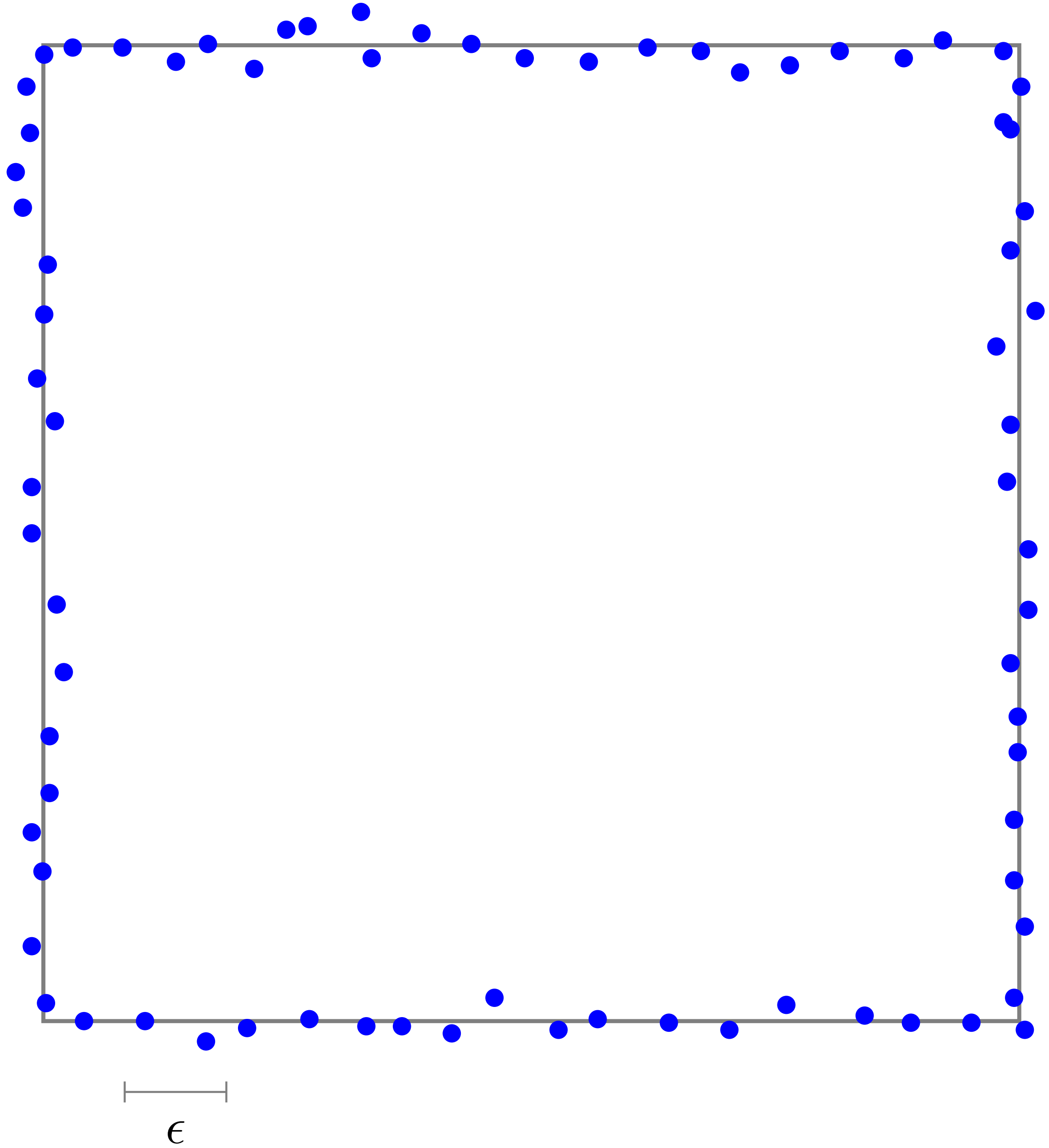

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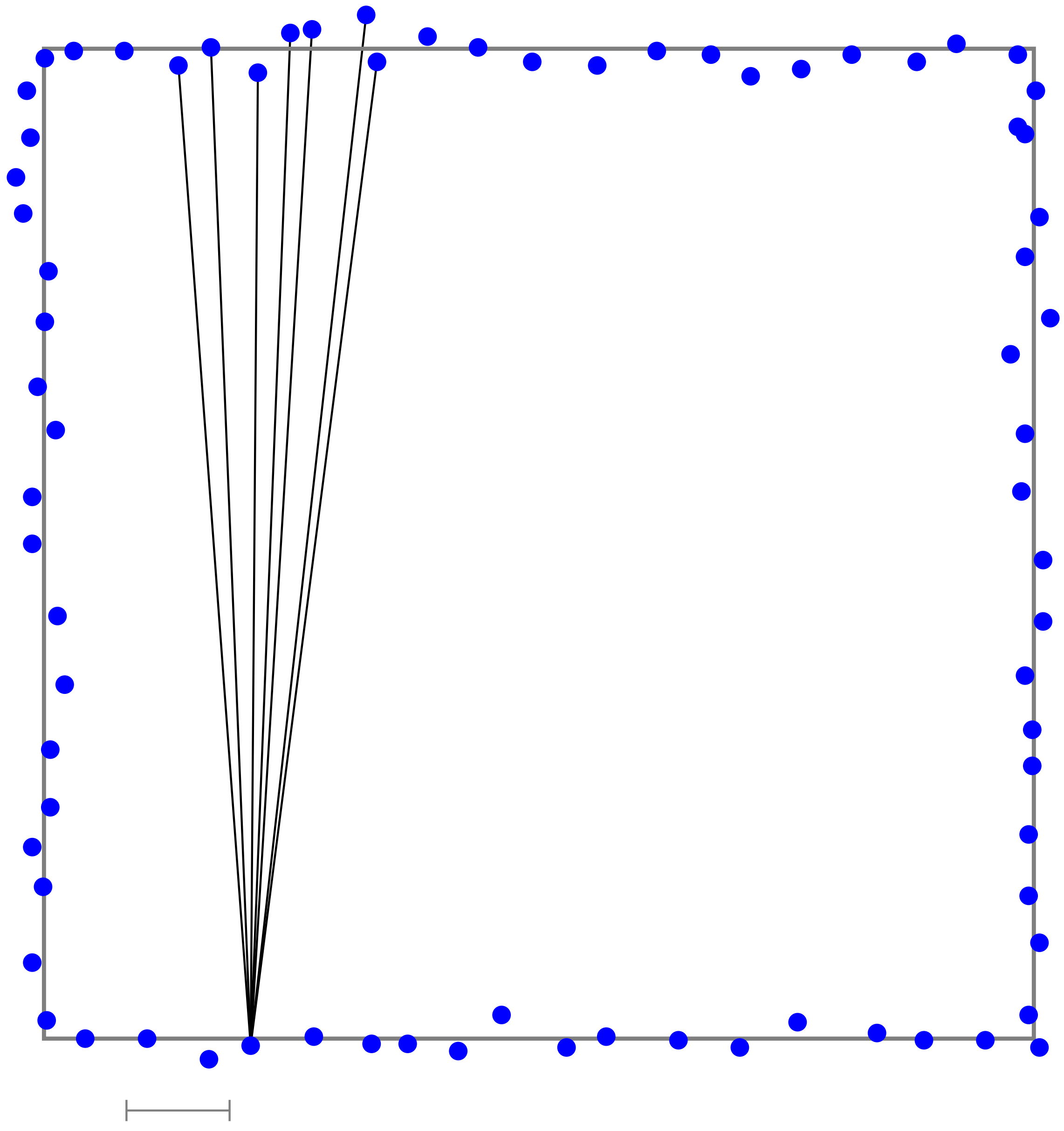

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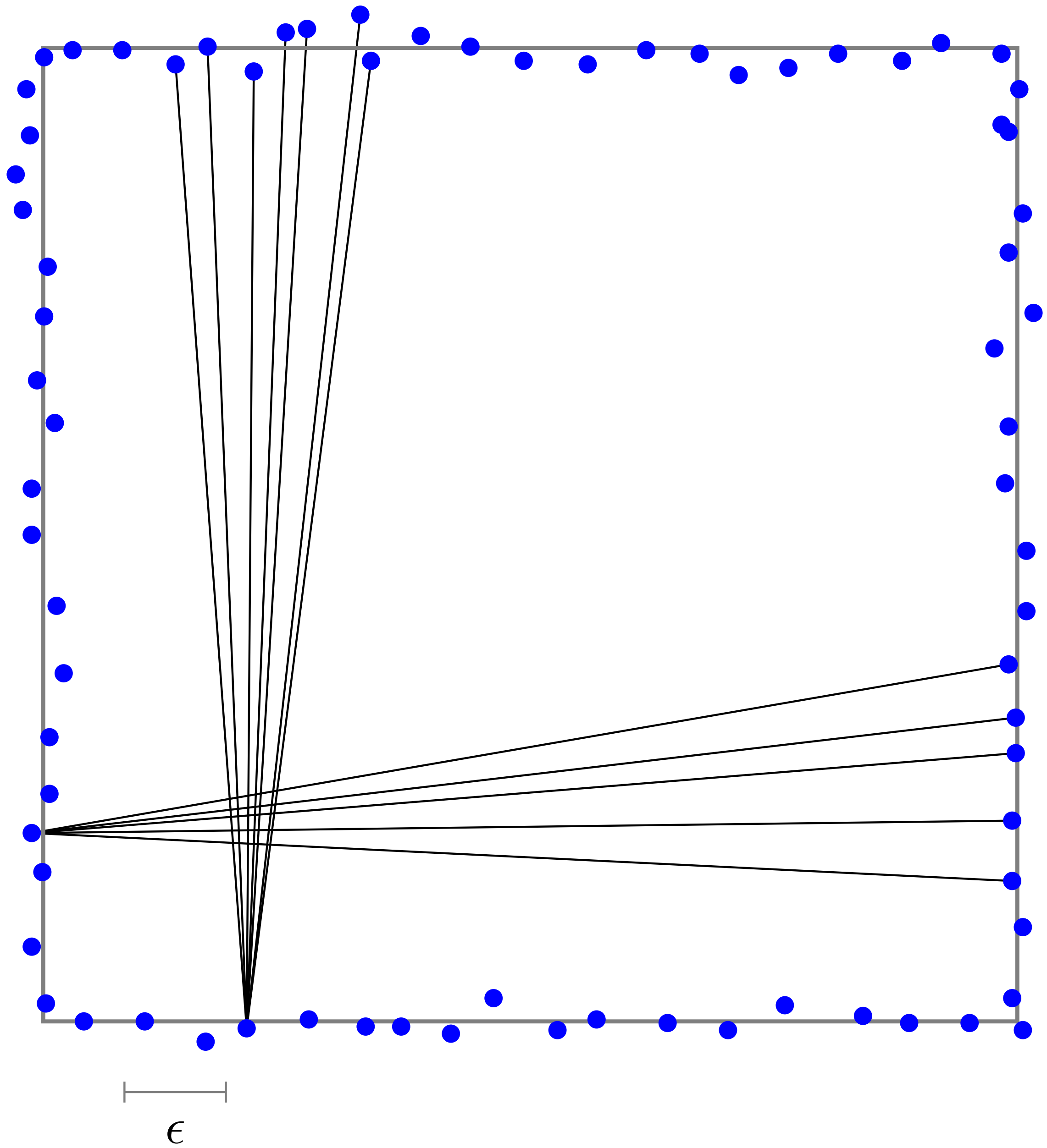

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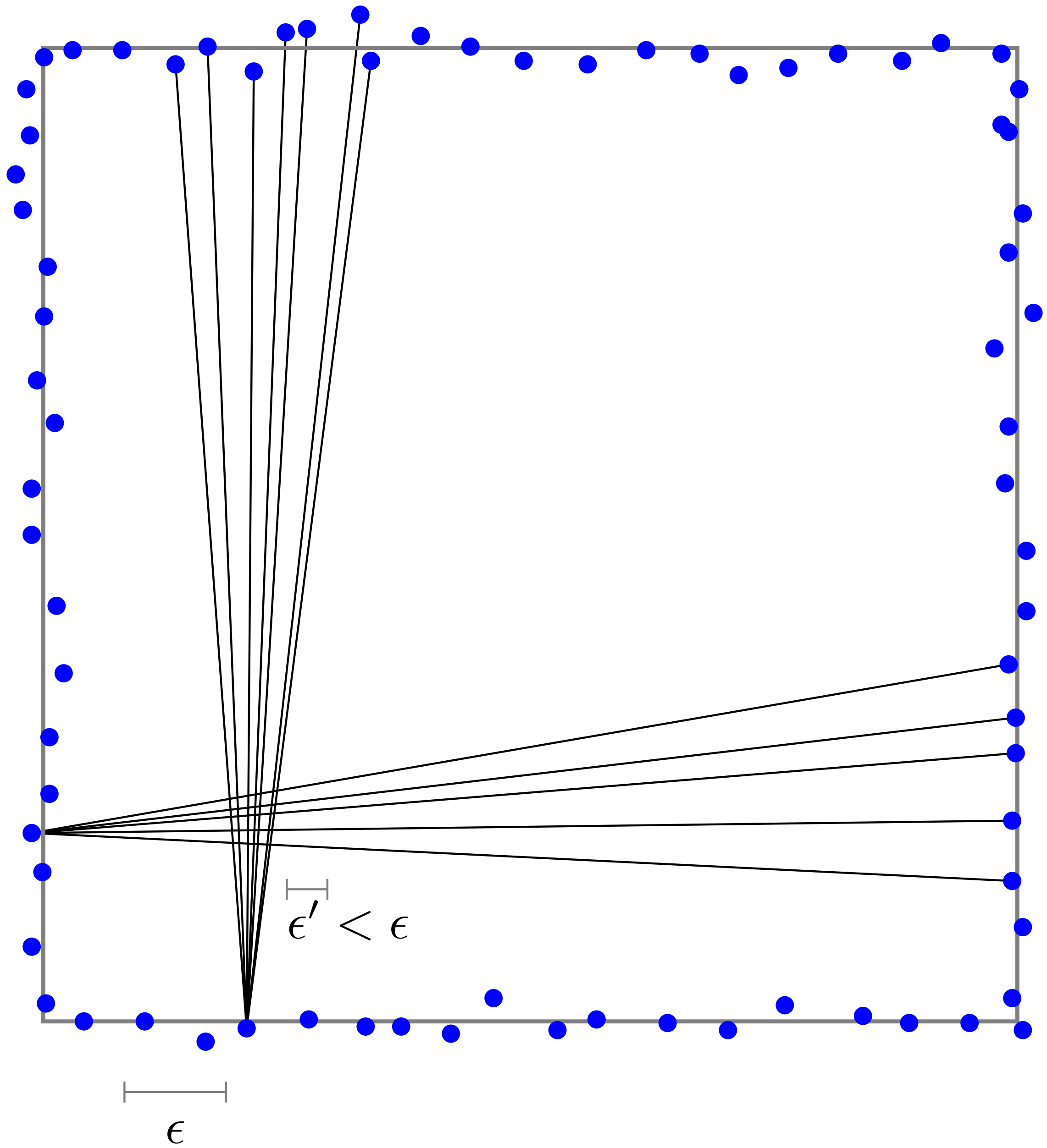

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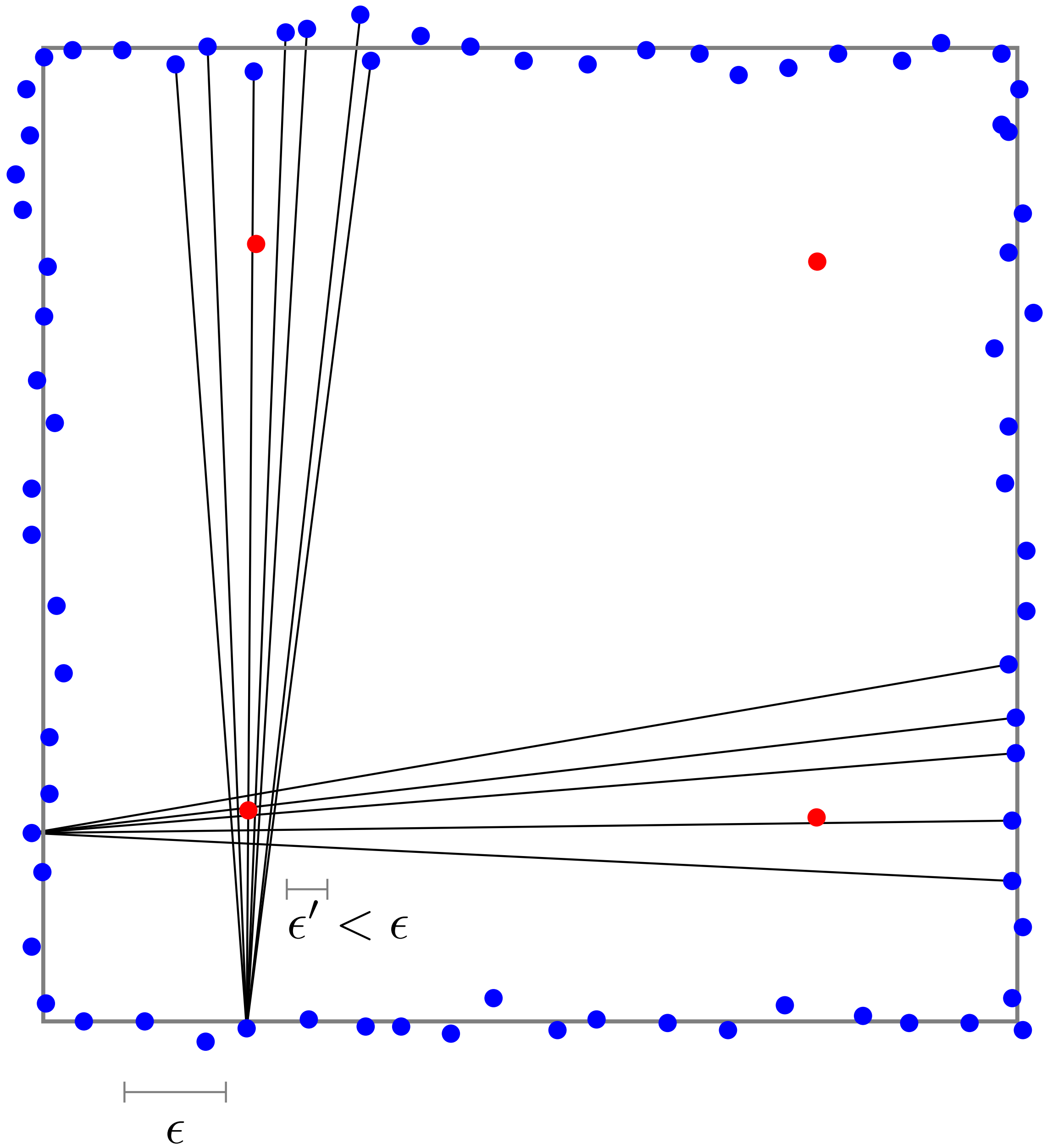

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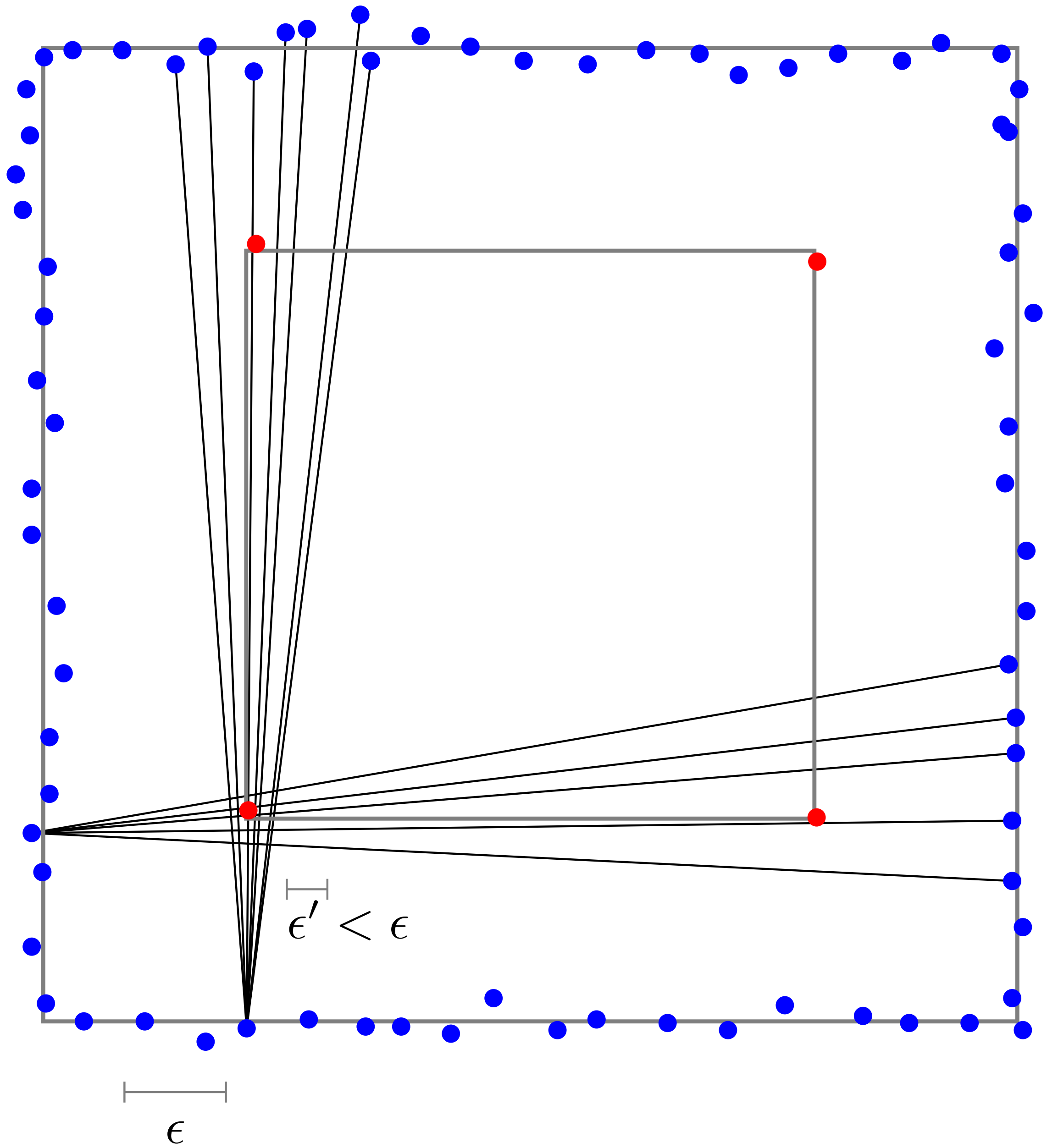

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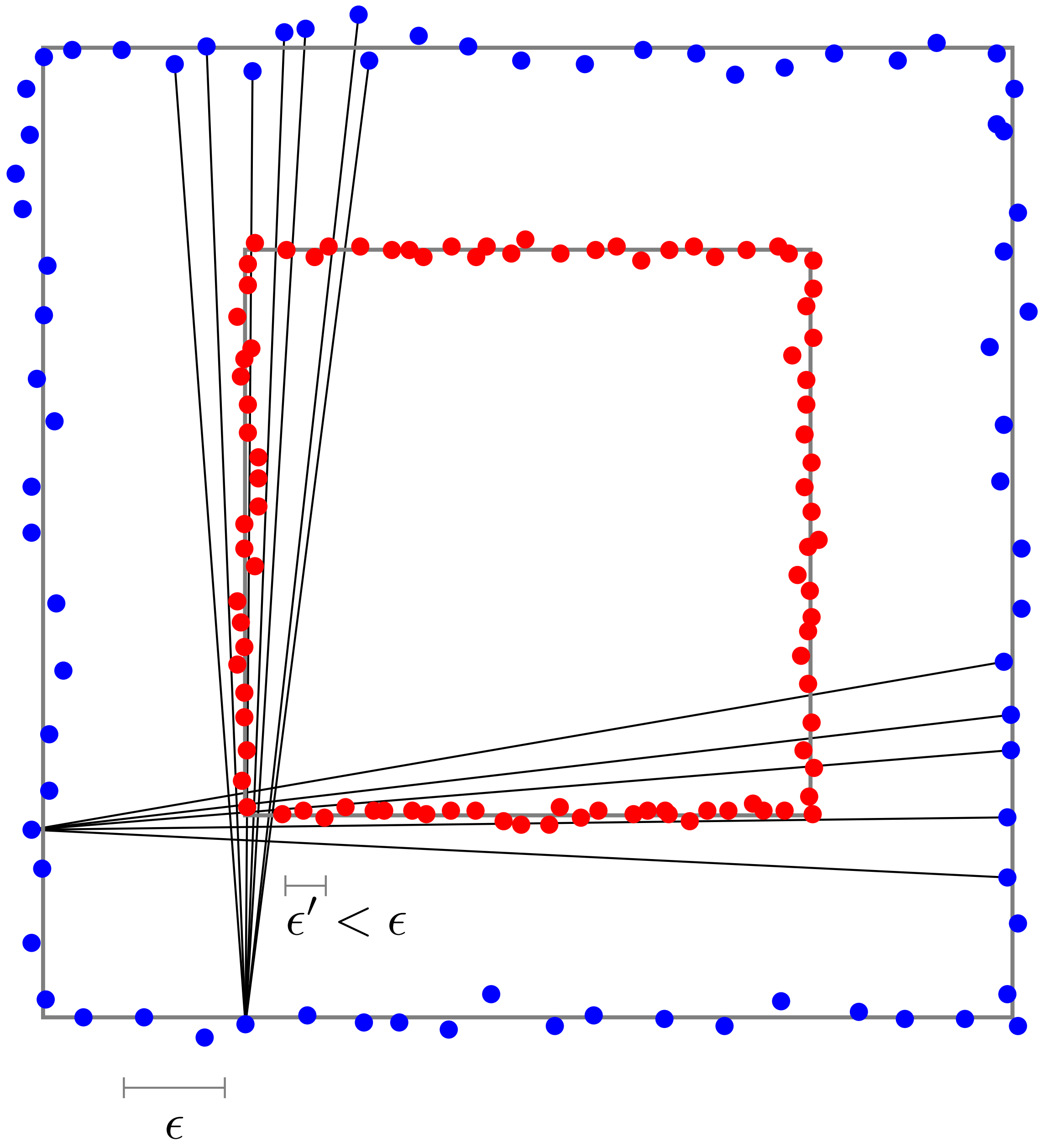

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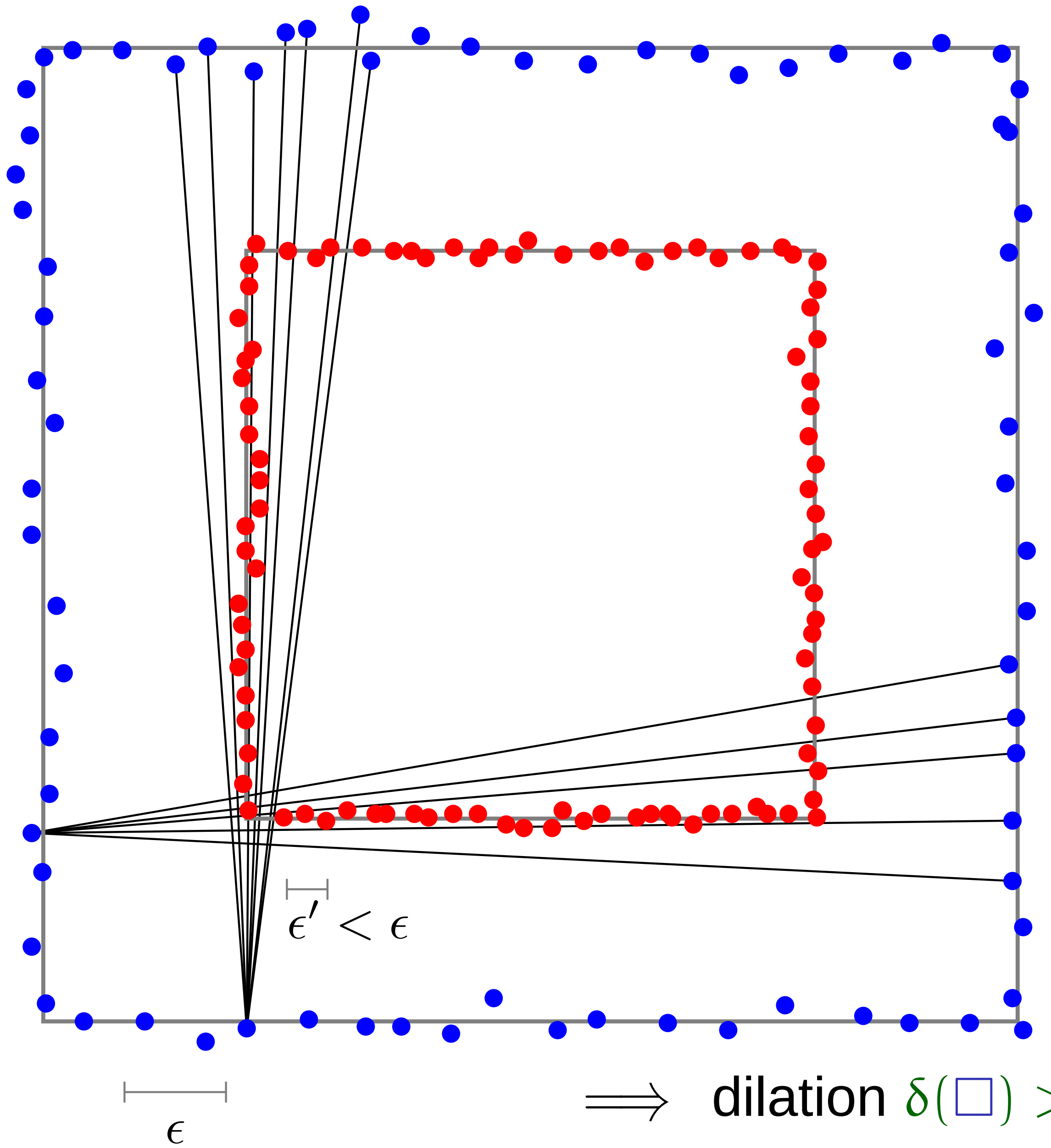

$\Longrightarrow$ dilation $\delta(\square) > 1.0000047$

Conclusion

Theorem. Every point set P with $\#P^\infty = \infty$ has dilation $\Delta(P) > 1$.

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Open Problems:

- What is the dilation of the regular pentagon?
- Is the infimum in $\Delta(P)$ always attained by some triangulation?