Weak Positional Games on Hypergraphs of Rank Three

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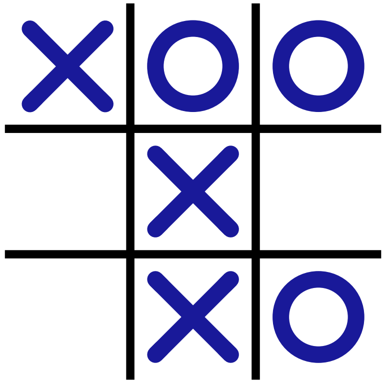
Tic-Tac-Toe



Two players alternatingly claim squares,
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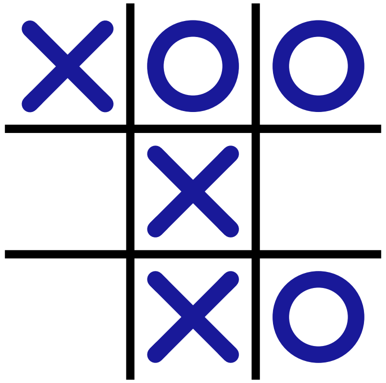

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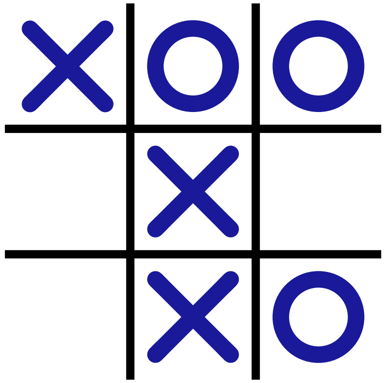

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two variants:

- strong positional game: both players trying to get an edge
  (draw possible but 2nd player never wins, by “strategy stealing”)
- weak positional game: 1st player (*Maker*) tries to get an edge while 2nd player (*Breaker*) tries to prevent this
  (no draw, by definition)

Tic-Tac-Toe

strong-game 1st-player win	$\Rightarrow$	weak-game Maker win
strong-game draw	$\Leftarrow$	weak-game Breaker win

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Weak Games — Previous / Classical Results

- local criterion [Hales & Jewett, '63]
  n-uniform hypergraph:
  max deg $\leq n/2 \Rightarrow$ Breaker win

- global criterion [Erdős & Selfridge, '73]
  n-uniform hypergraph $H = (V, E)$:
  $|E| < 2^{n-1} \Rightarrow$ Breaker win

- Ramsey criterion [Beck]
  $\chi(H) \geq 3$ (chromatic number) $\Rightarrow$ Maker win

Computational Complexity

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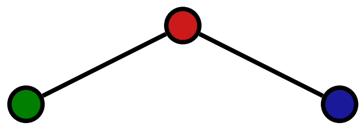

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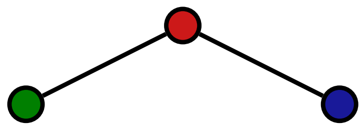

We set out to solve rank-3 hypergraphs . . .

(efficient classification and thus, optimal play)

Main Result

Theorem. We can decide in polynomial time, who wins the weak game on a given hypergraph of rank 3.

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Def. A hypergraph is called almost-disjoint if any two edges share at most one vertex.

This is not an unnatural property.
(satisfied, e.g., by arbitrary-dimensional Tic-Tac-Toe and often considered in the context of hypergraph coloring.)

It does not define away the problem.

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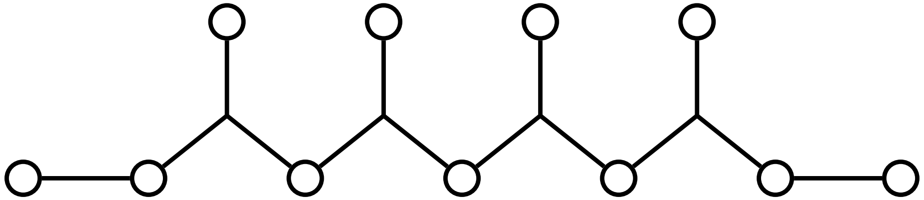
Def. Call a hypergraph a winner if Maker (playing first) can win on it.

Playing Along Paths



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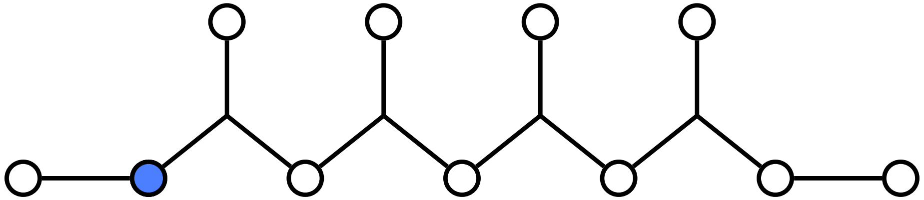

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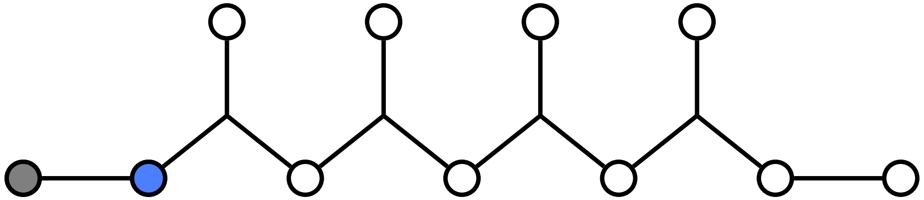

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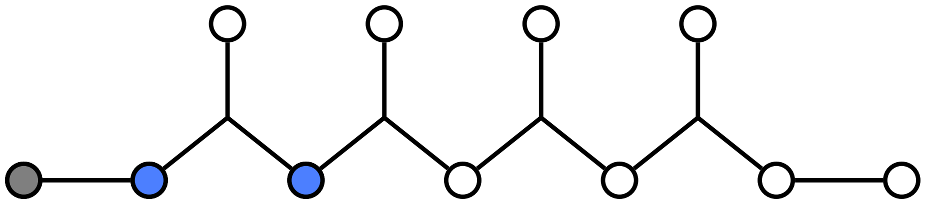

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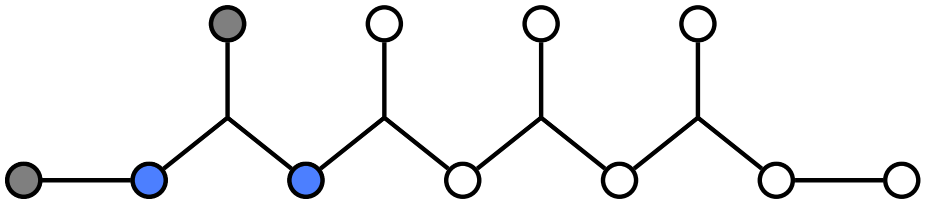

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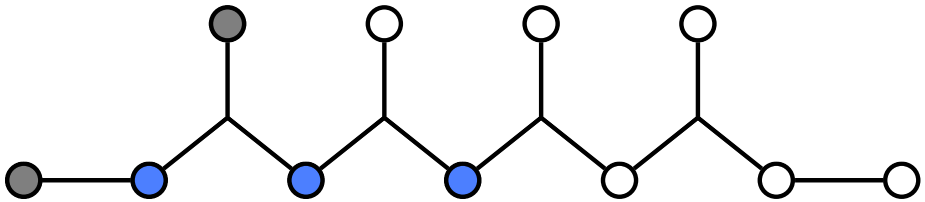

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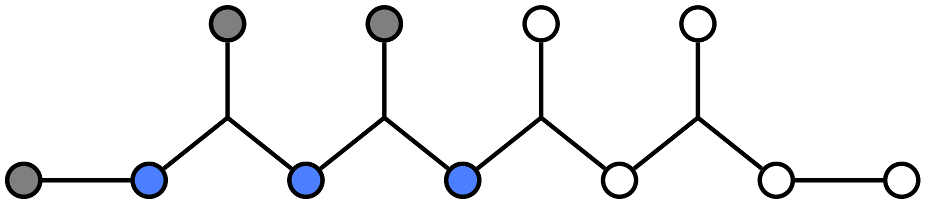

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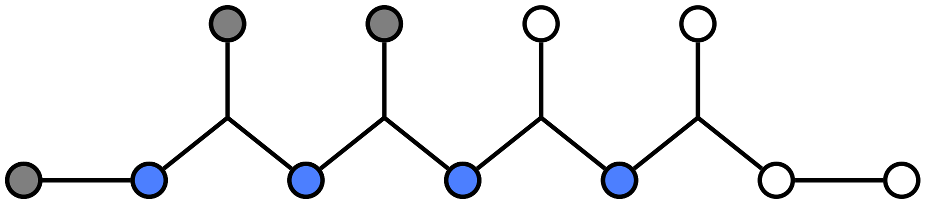

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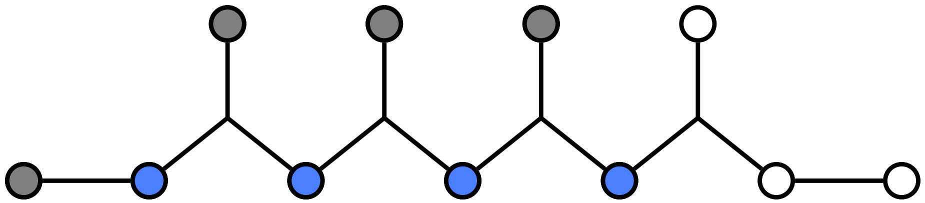

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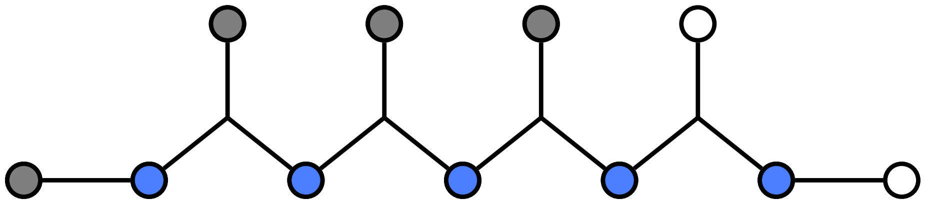

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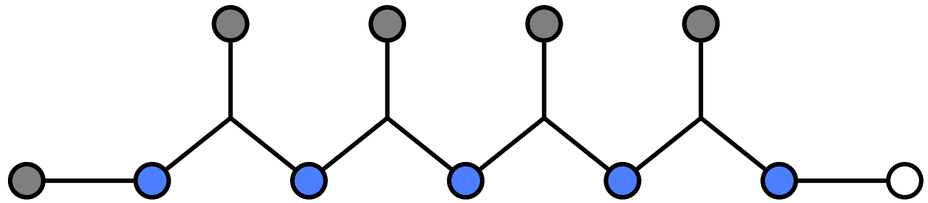

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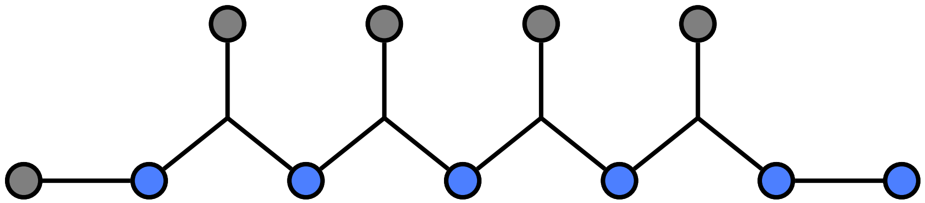

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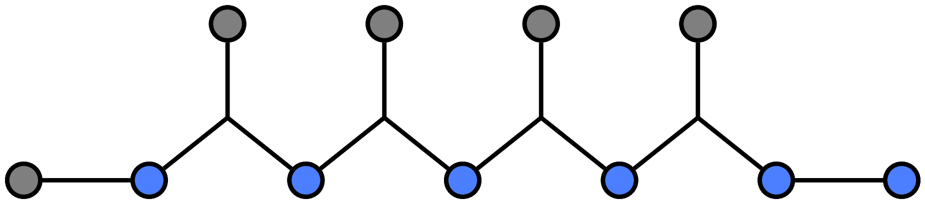

Lemma. Any connected almost-disjoint rank-3 hypergraph with at least two 2-edges is a winner.

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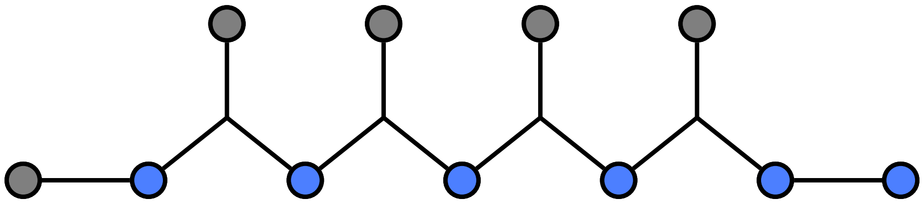

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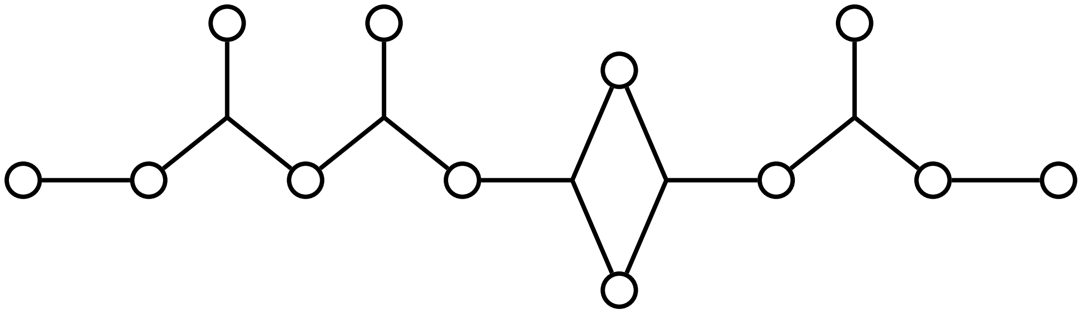

is a loser
(not almost-disjoint)

Decompositions

Lemma. The disjoint union $H = A \dot{\cup} B$ of two hypergraphs is a winner iff one of A and B is a winner.

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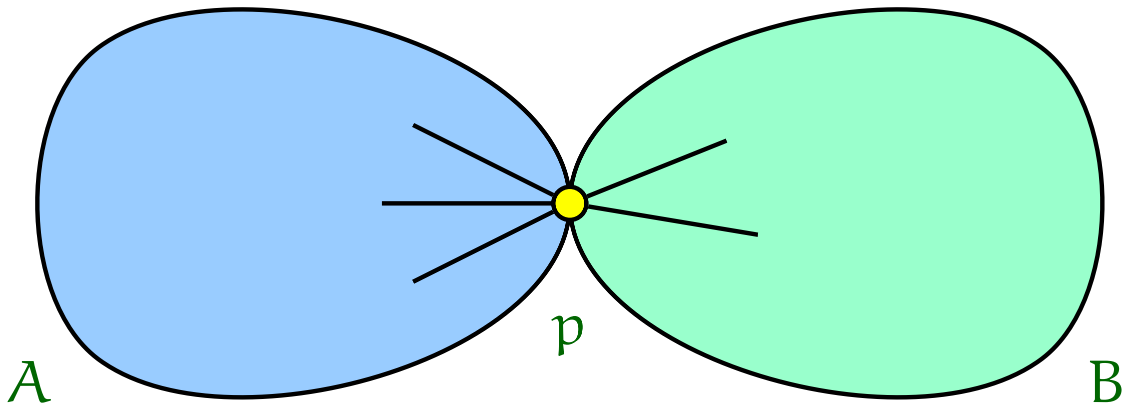
We can extend this result to “almost-disjoint” unions:

Def. A vertex p is an articulation of a hypergraph H if $H = A \cup B$ with $V(A) \cap V(B) = \{p\}$ for non-trivial hypergraphs A and B.



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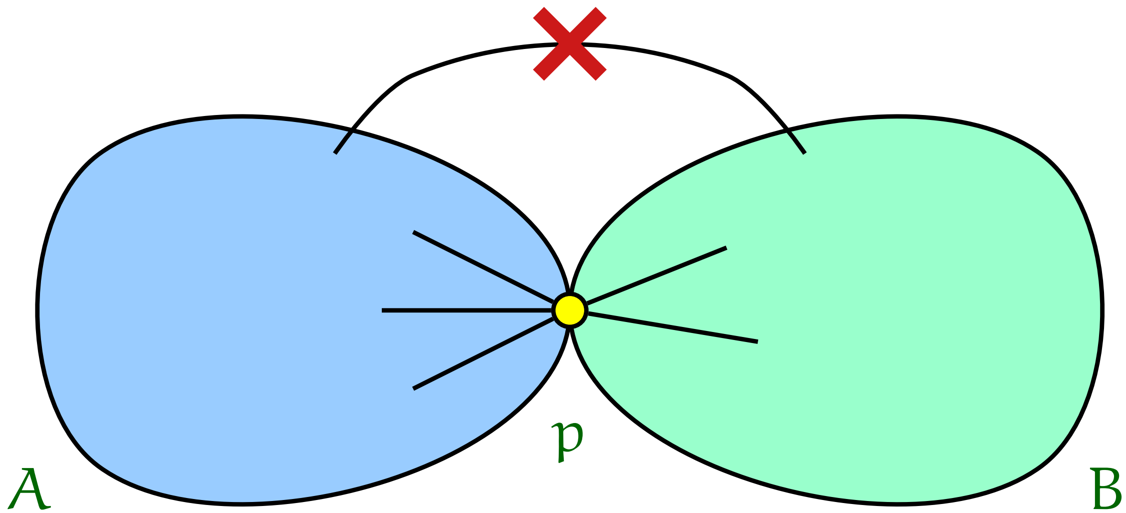

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Articulation Lemma. Let $H = A \cup B$ with $V(A) \cap V(B) = \{p\}$. Then H is a winner iff one of the following holds:

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Corollary. If Maker can win neither on A nor on B alone then playing at the articulation p is definitely an optimal move.

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- extensive case distinctions
  - threats along paths and cycles lead to three essentially different winning blocks for Maker

Towards a General Decomposition Theorem

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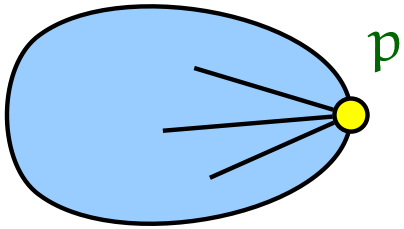

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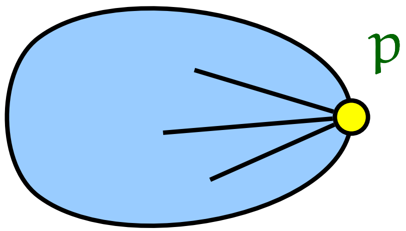
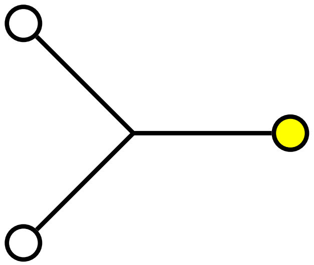
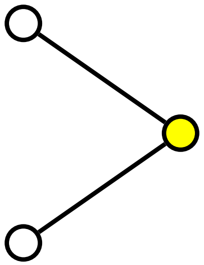
Any  behaves exactly as one of these three:





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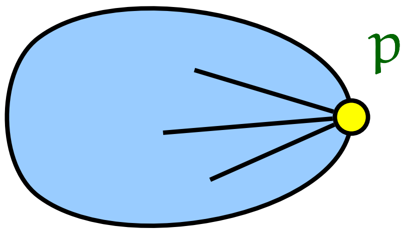
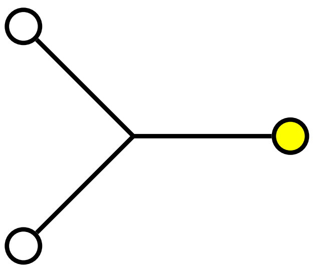
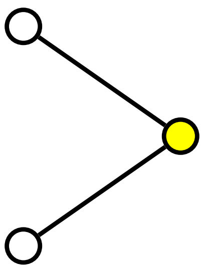
Any  behaves exactly as one of these three:





winner

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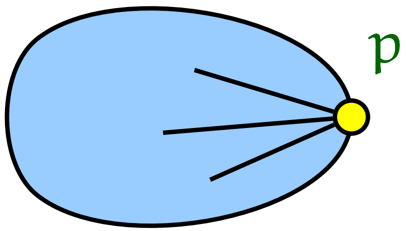
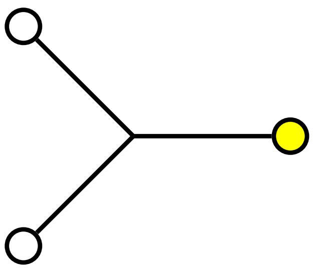
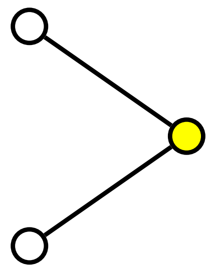
Any  behaves exactly as one of these three:





absolute loser

winner

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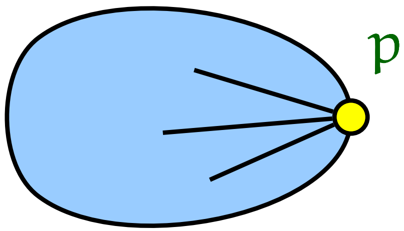
Any 

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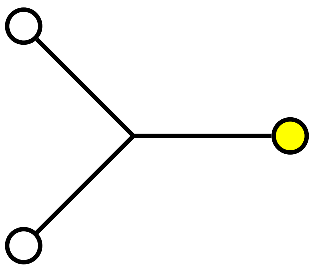
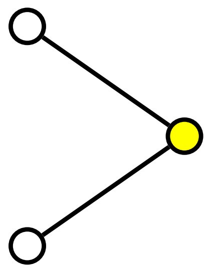


absolute loser

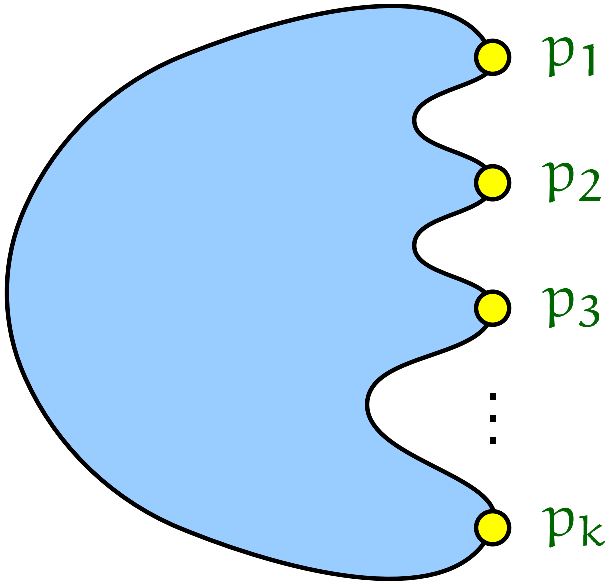
semi-winner



winner

A Poset of “Halfgames”

A k-*pointed hypergraph* contains k marked contact points.



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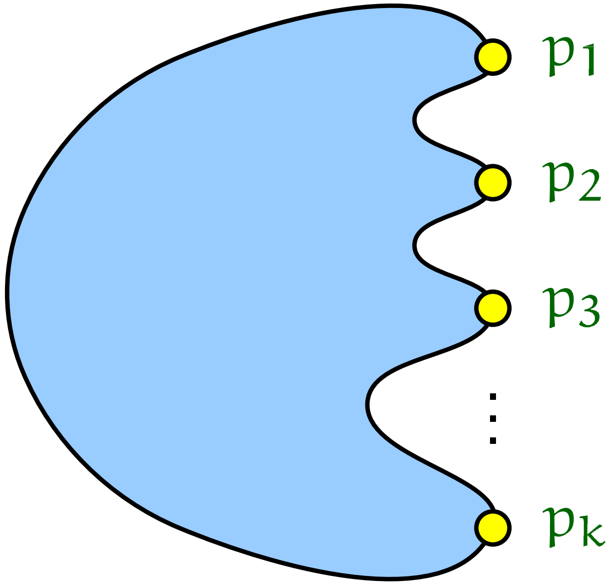
Form the k-*pointed union* $A \sqcup_k X$ of two such hypergraphs by gluing at the points.



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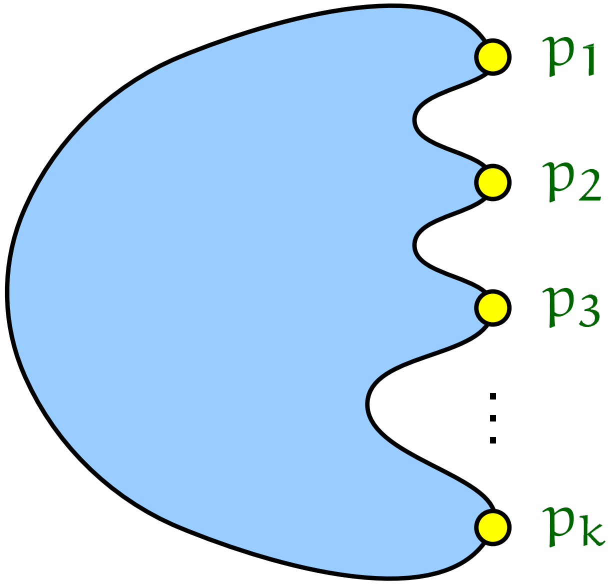

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Let $A \leq B$ for k-ptd h’graphs if for all k-ptd h’graphs X:

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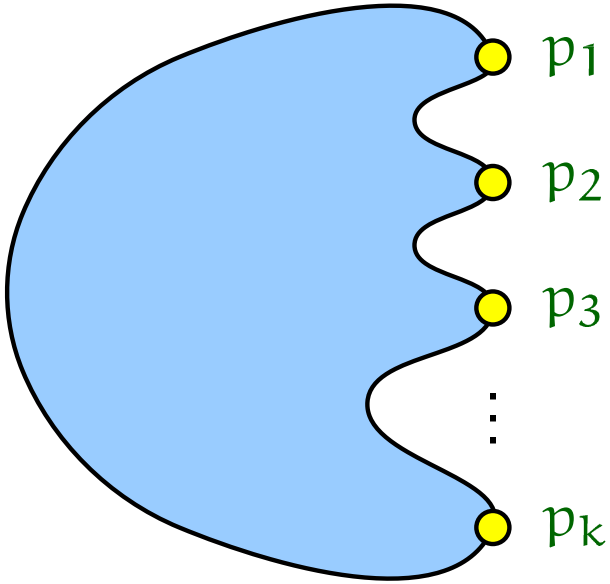

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(after identification of equivalent ptd h'graphs)

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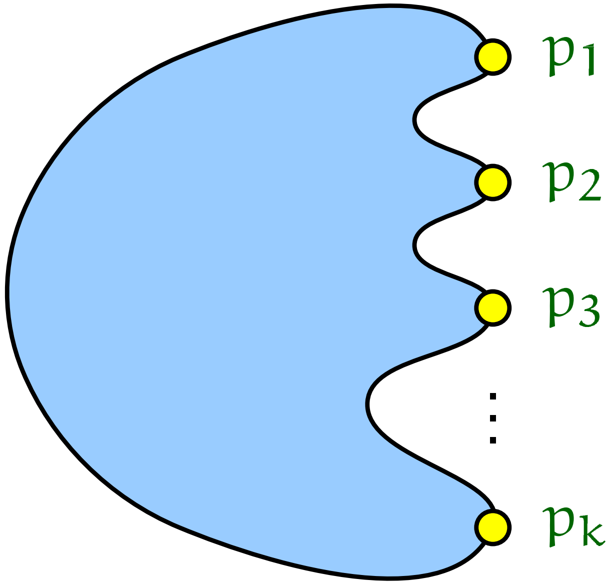

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$\mathcal{H}_1$ is a chain of three elements (Articulation Lemma)

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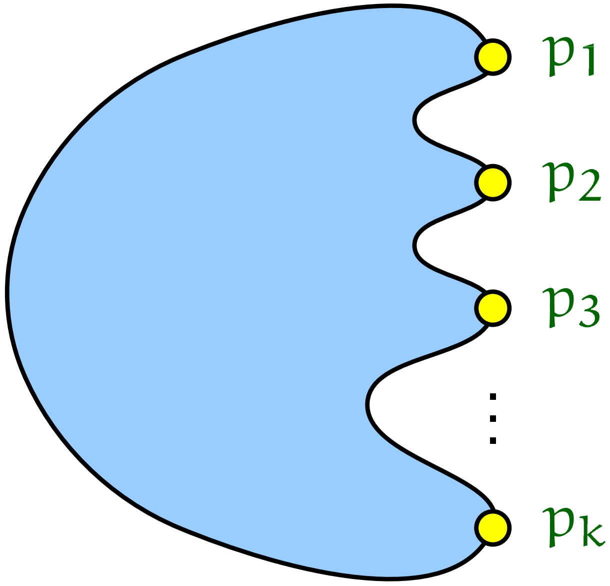

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Conjecture. All $\mathcal{H}_k$ are finite.