Computing Shortest Non-Trivial Cycles on Orientable Surfaces of Bounded Genus in Almost Linear Time

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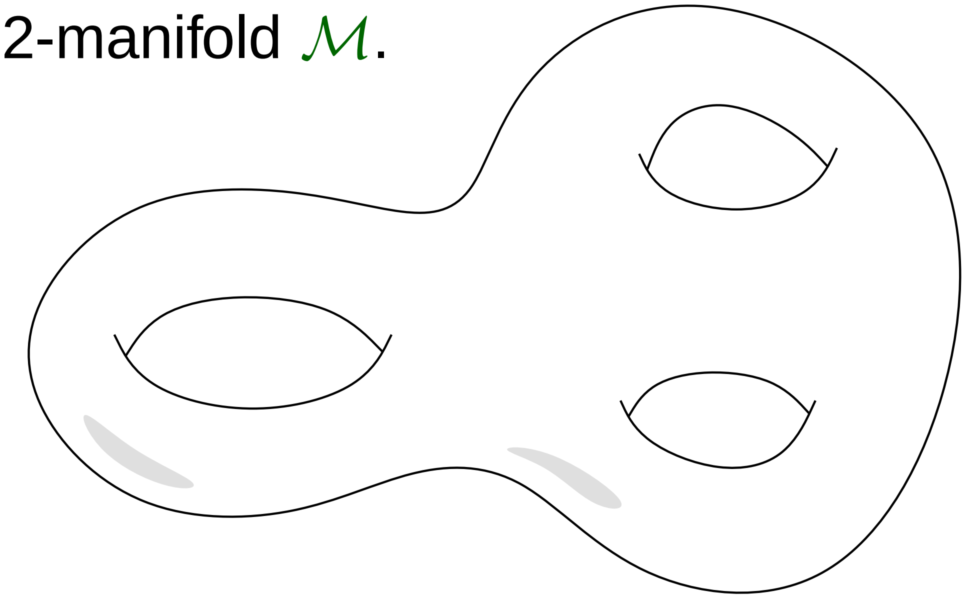
Combinatorial Surfaces

A *surface*: connected, orientable 2-manifold $\mathcal{M}$.



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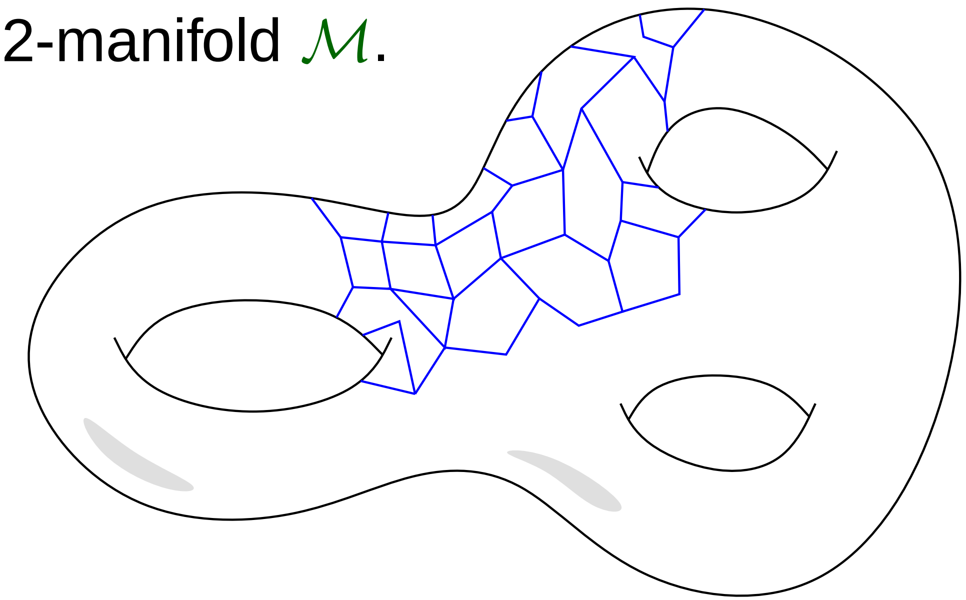

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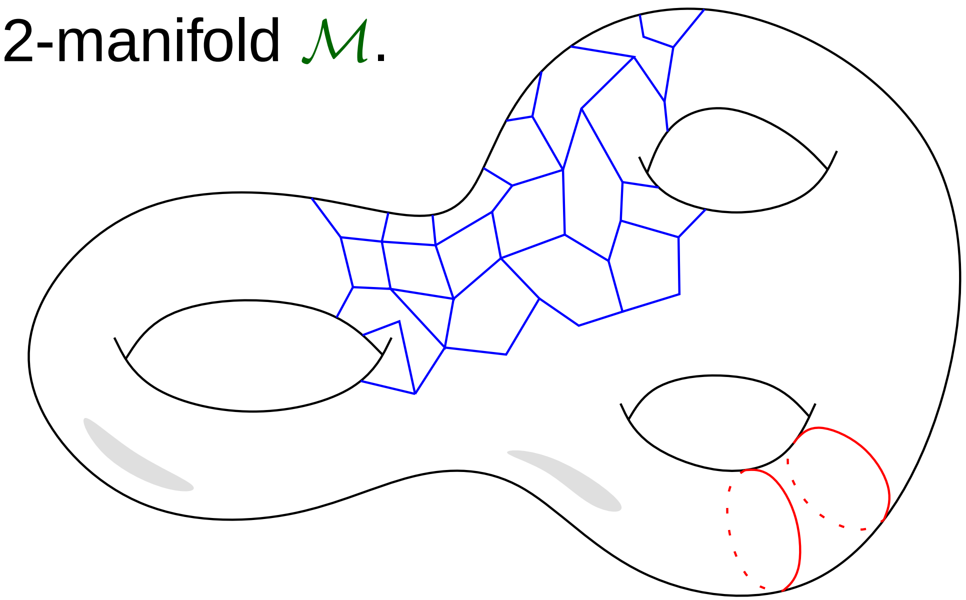

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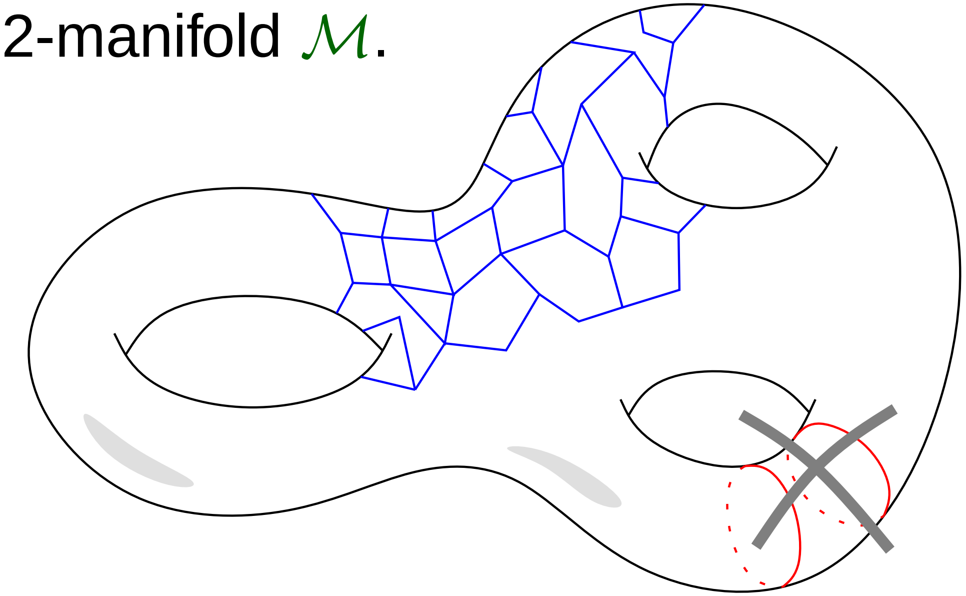

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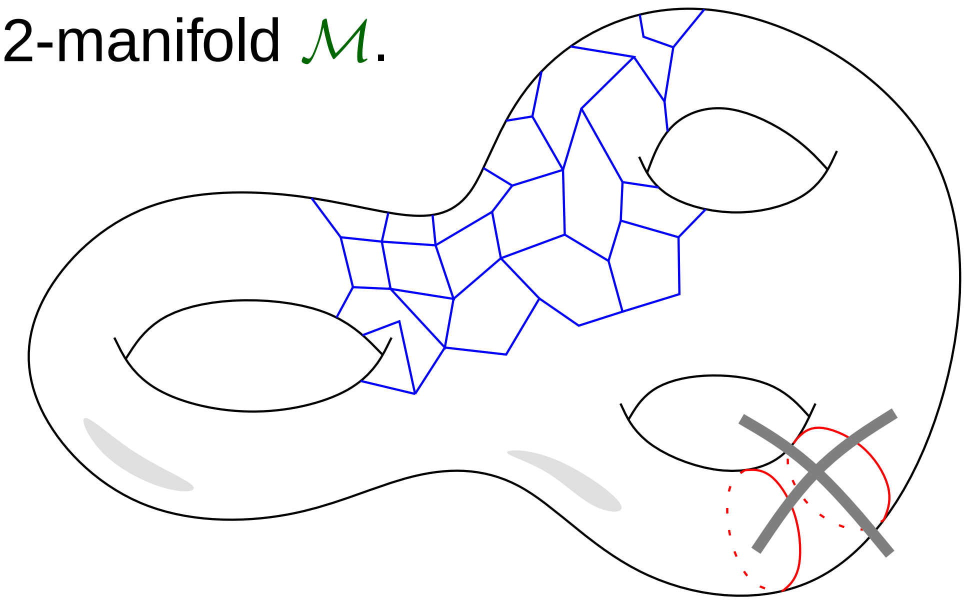

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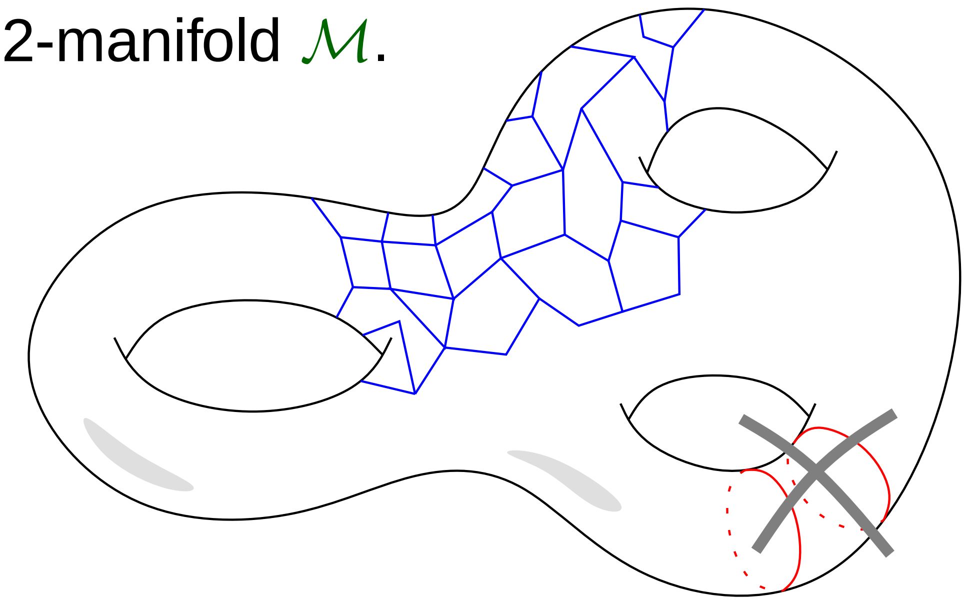

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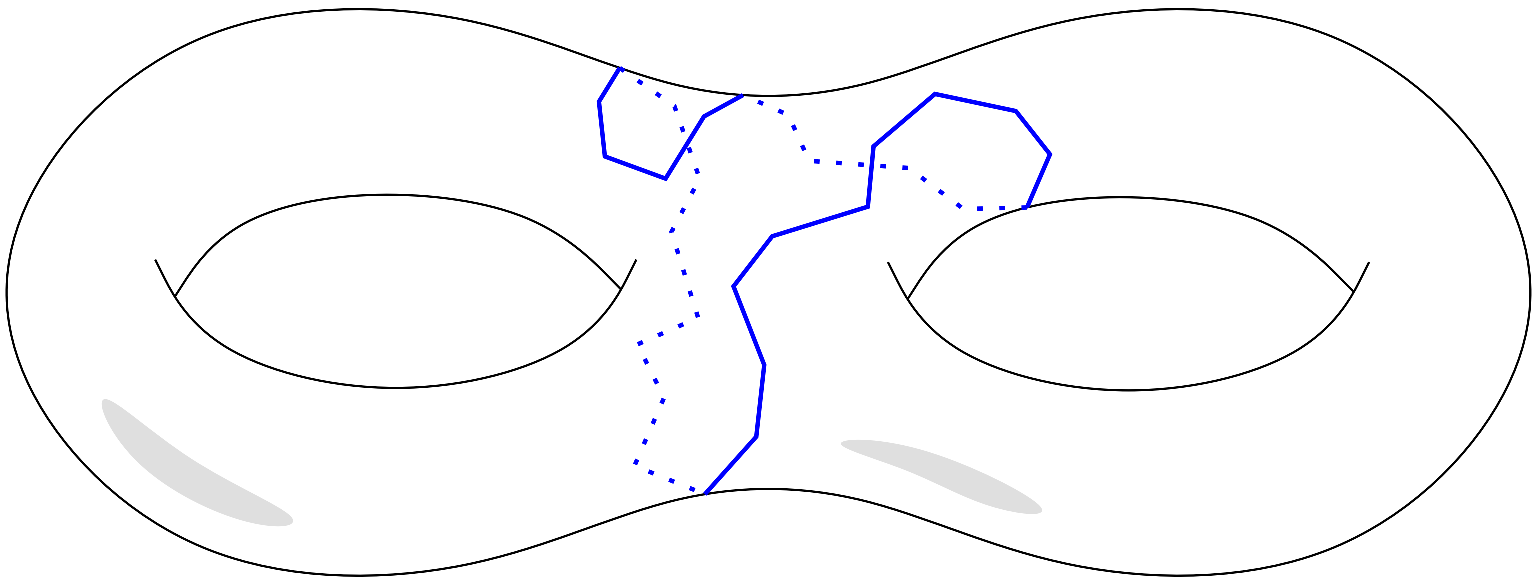
Edges may have weights.

Non-Trivial Cycles

A *cycle* on a surface $\mathcal{M}$ is a closed walk on the defining graph.



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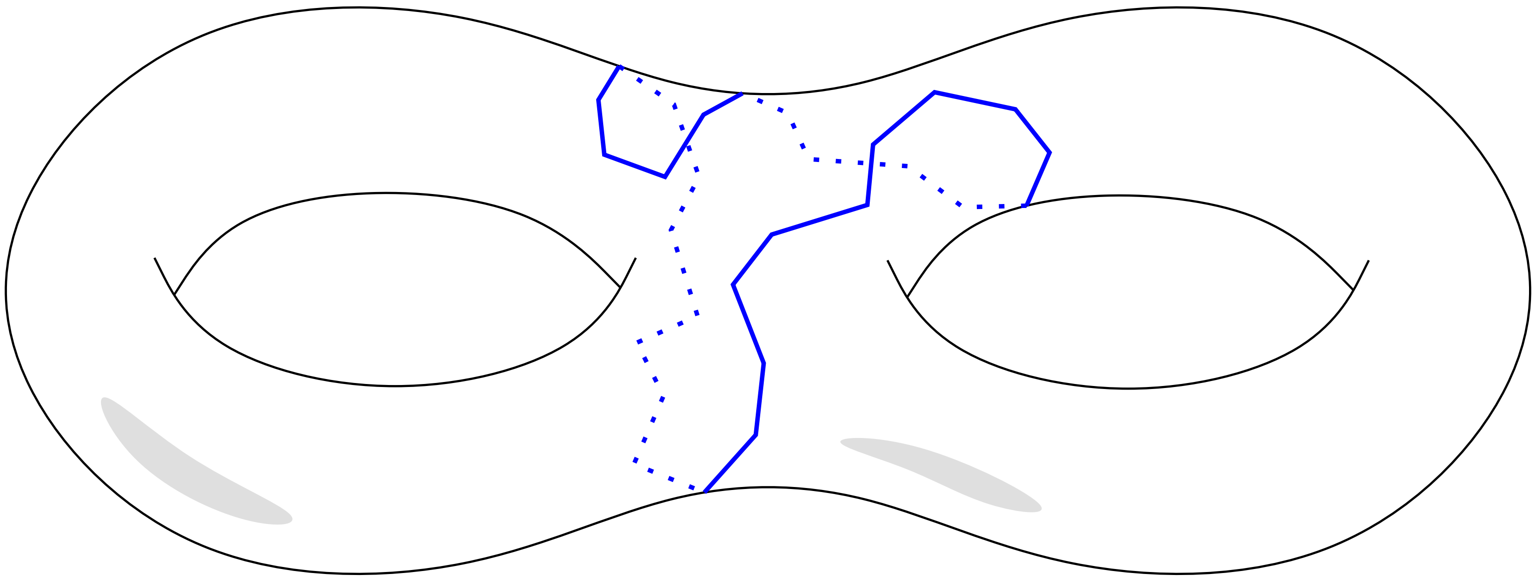

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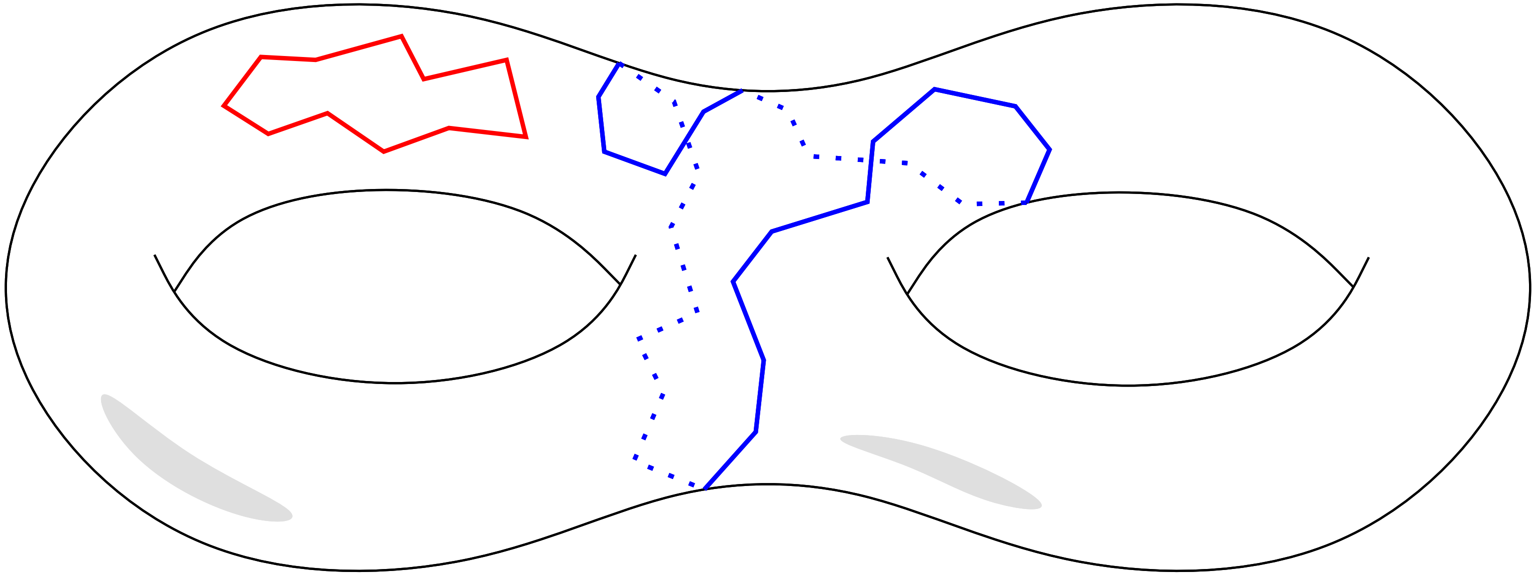

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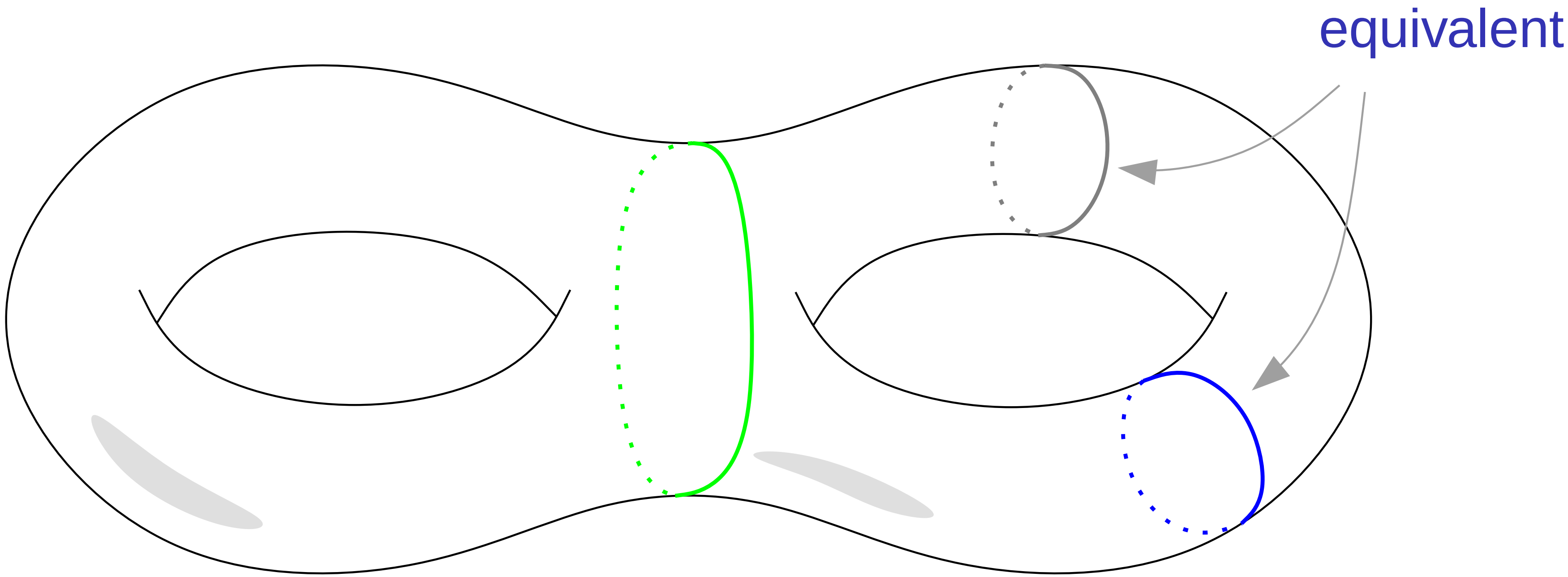

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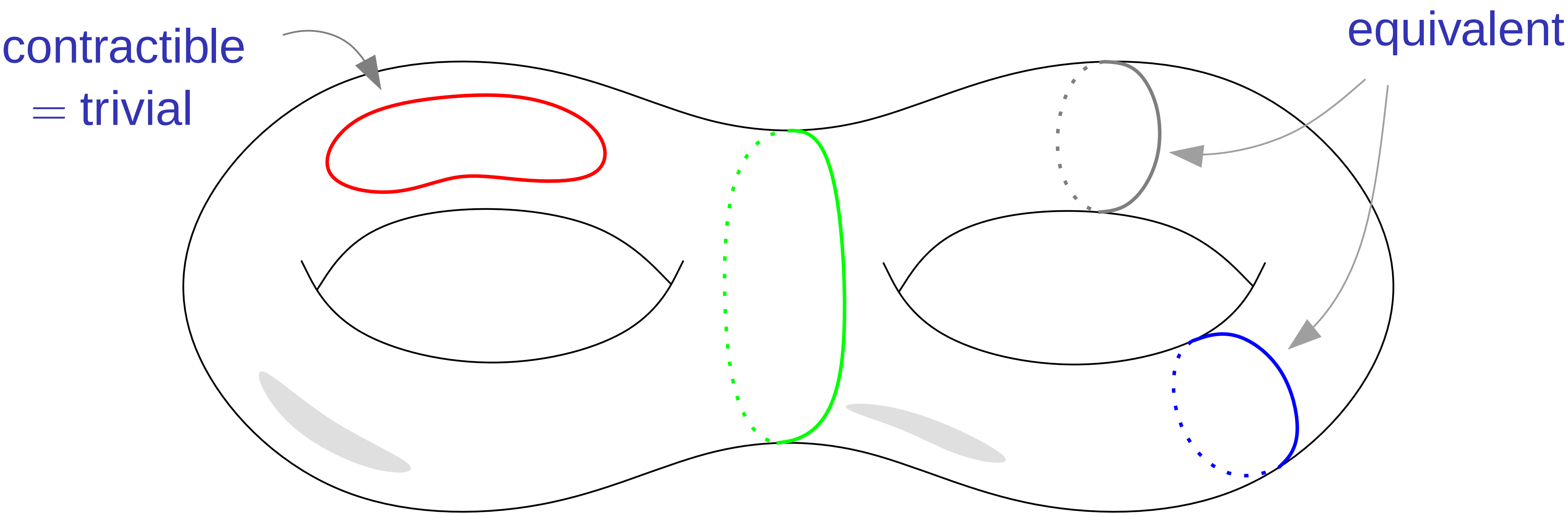

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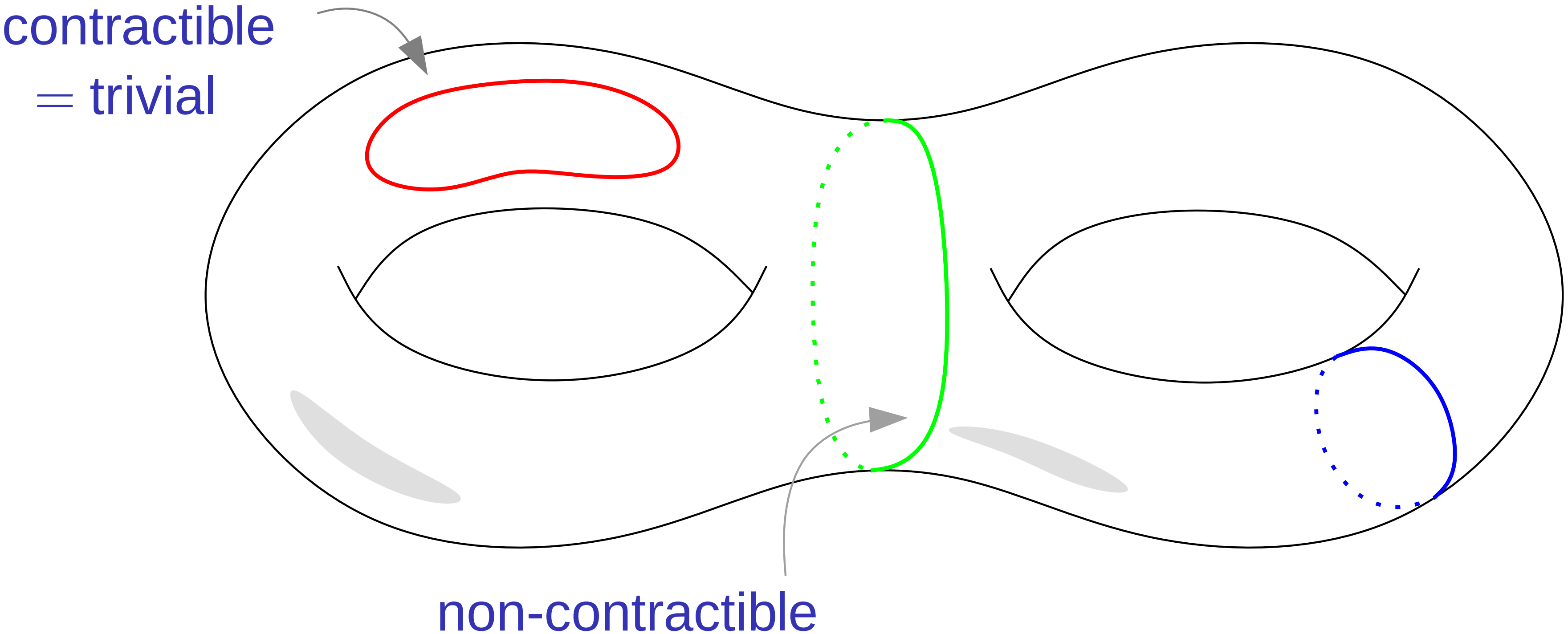

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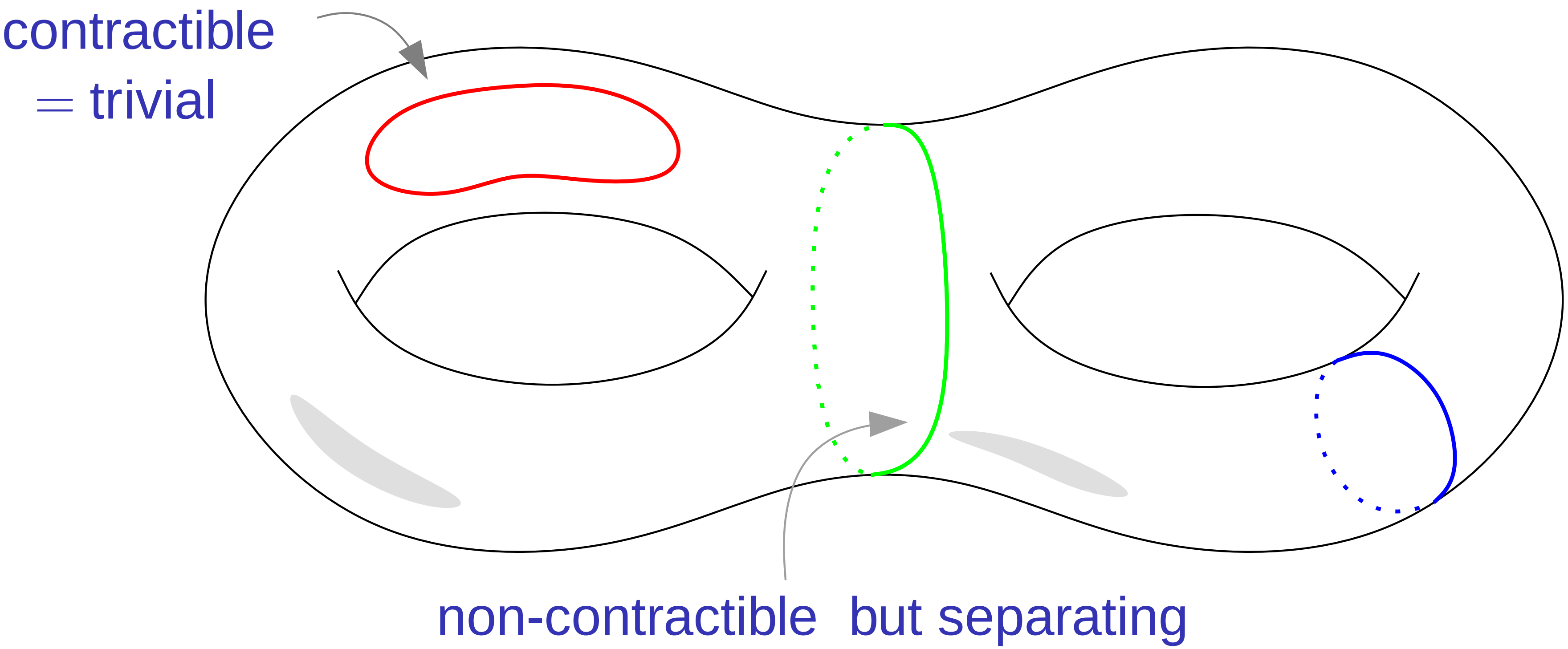

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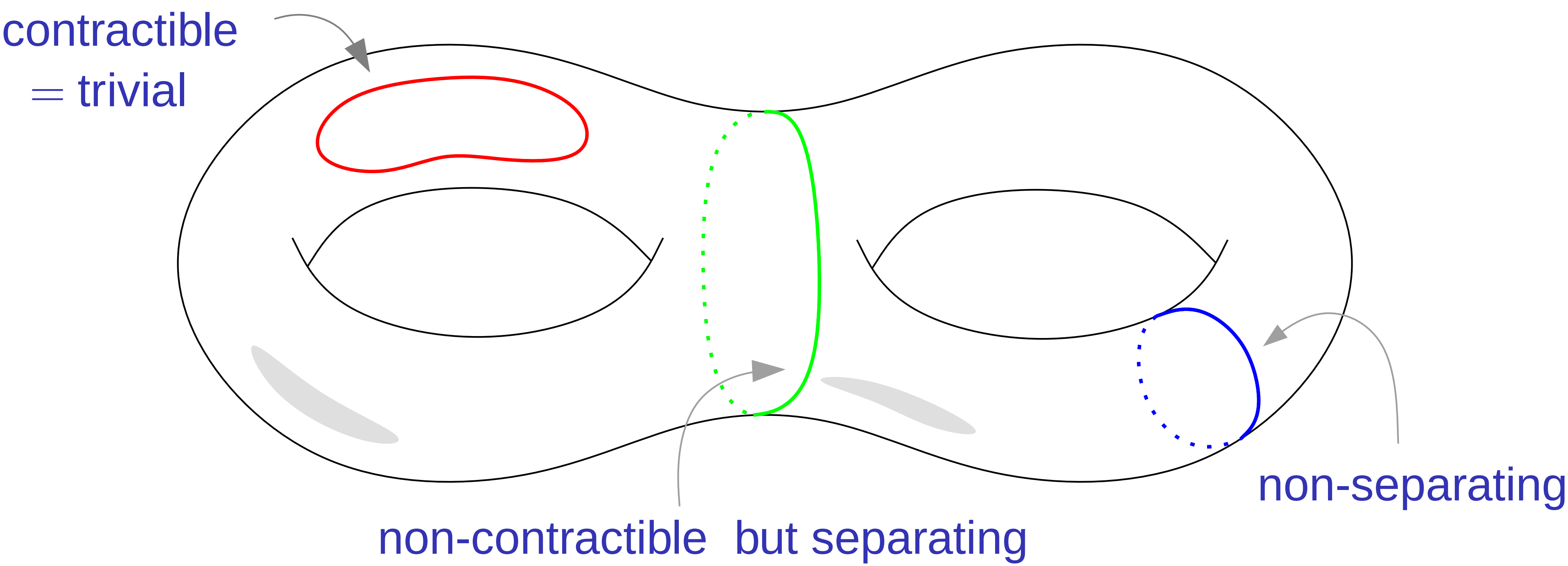

Result

Theorem. **[K, SoCG 2006]**

On orientable surfaces of bounded genus,
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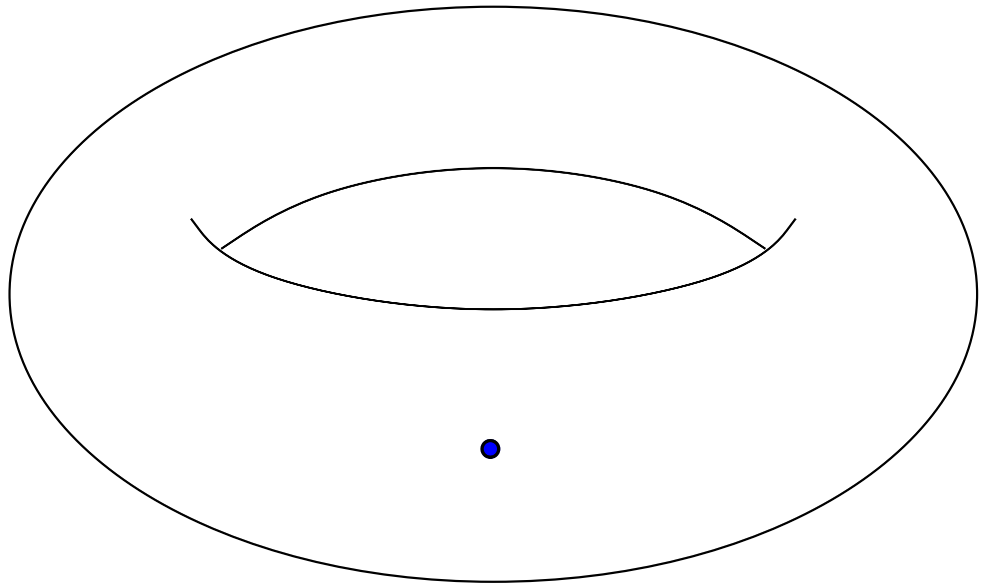
- Computing short non-trivial cycles turned out to be a core problem in computational topology.

Shortest Cycles Through a Basepoint

Erickson & Har-Peled (SoCG 2002): Shortest non-trivial cycle through a given basepoint in $O(n \log n)$ time.



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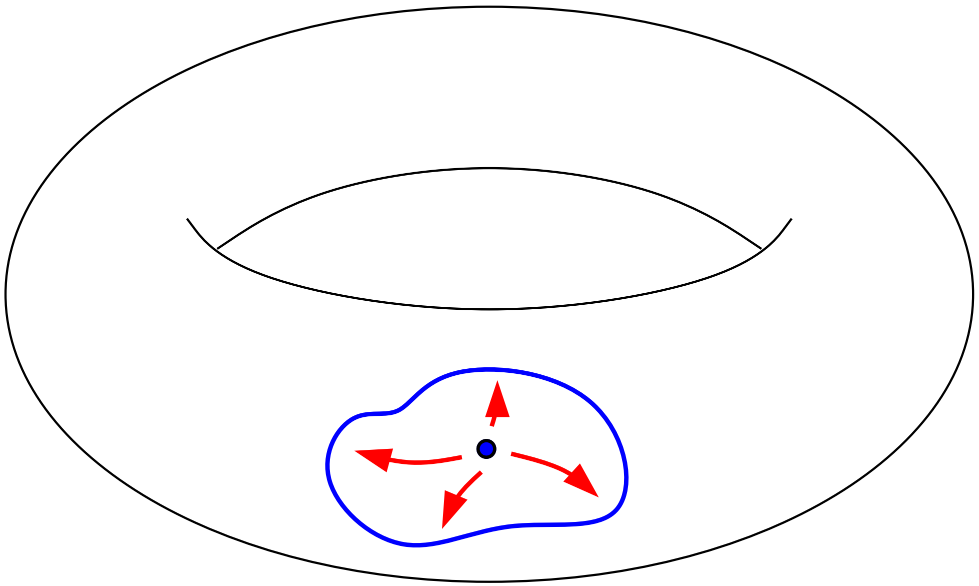

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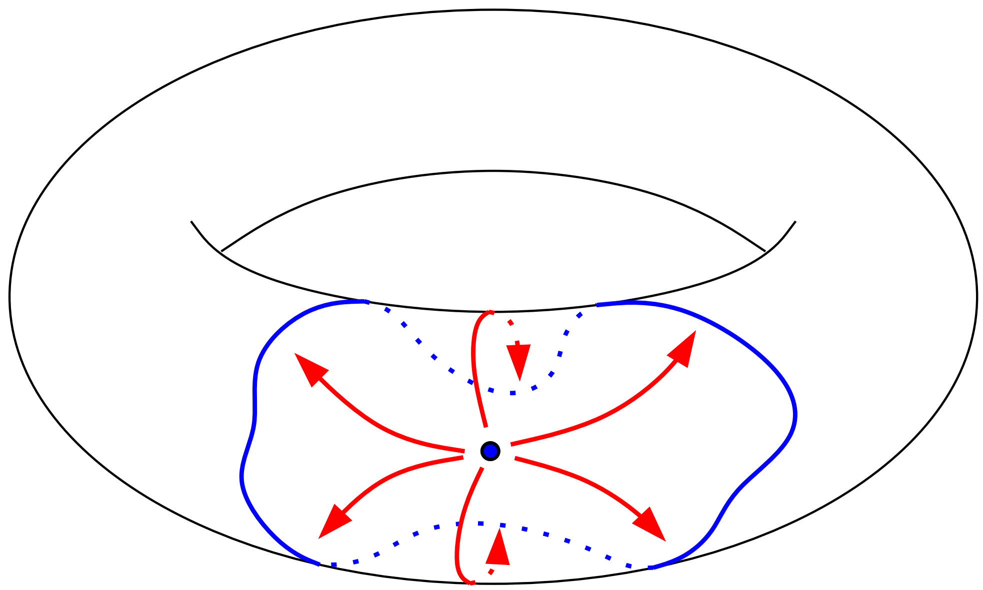

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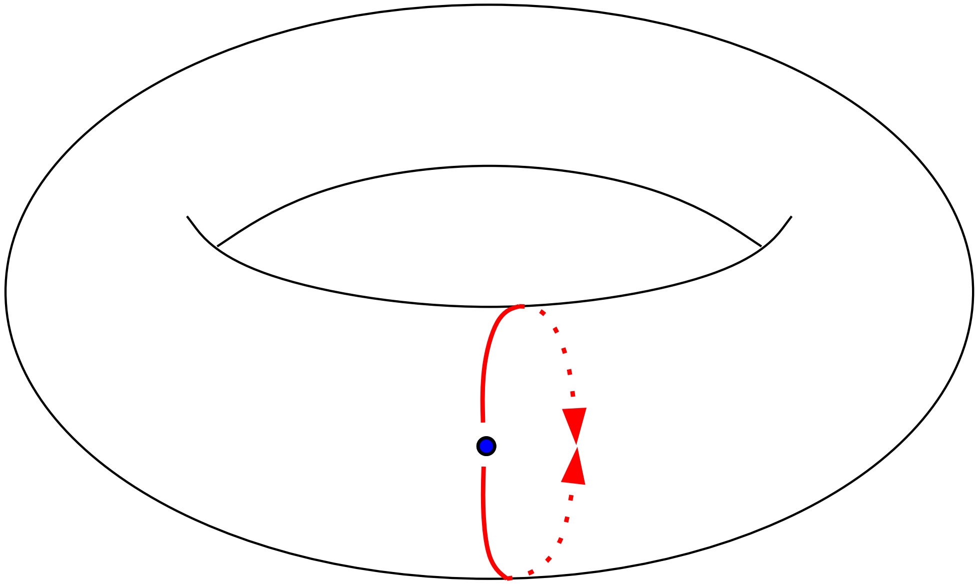

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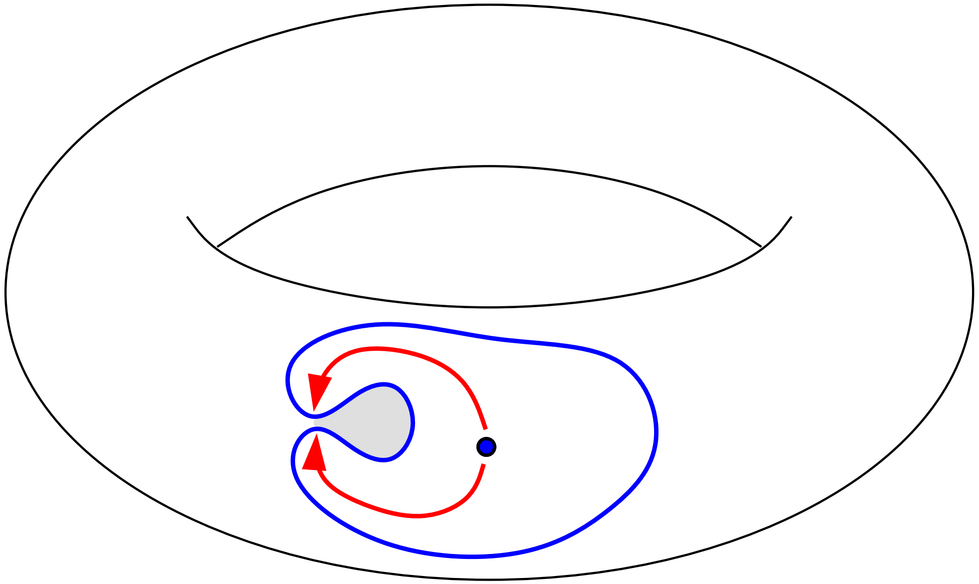

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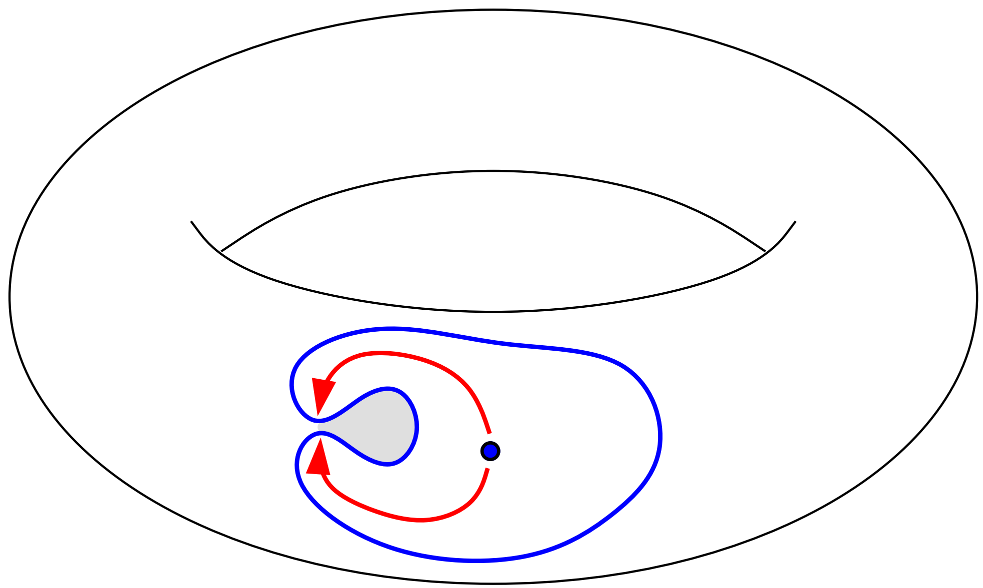

n-fold execution yields globally shortest cycle in $O(n^2 \log n)$ time.

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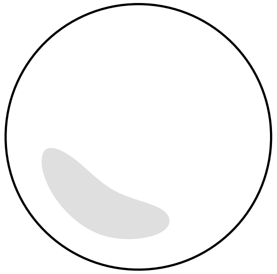

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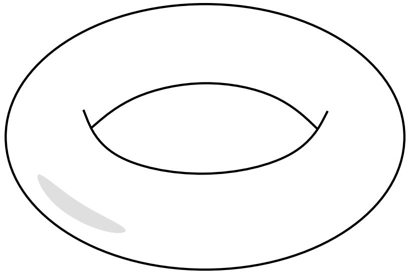

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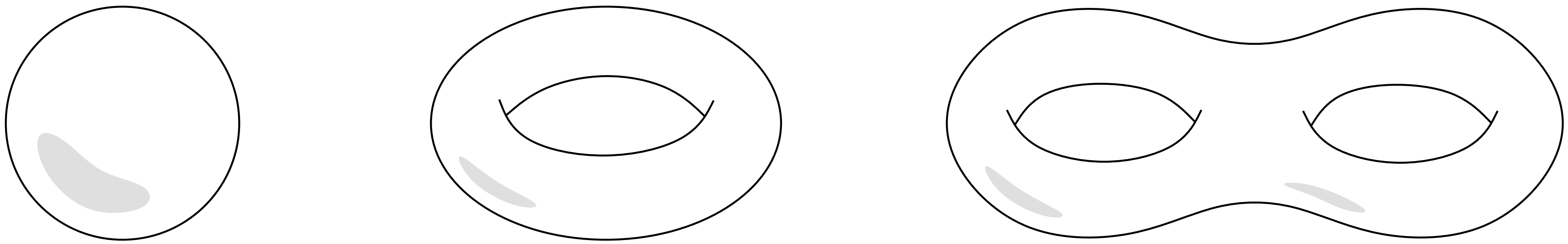

$g = 0$ $g = 1$ $g = 2$

Improvement

Thm. [Cabello & Mohar, ESA 2005] On surfaces of bounded genus, shortest non-contractible cycles can be computed in $O(n^{3/2})$ time. ($O(n^{3/2} \log n)$ for non-separating cycles)

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(*Reminder:* exponential dependence on g!
genus-independent possible in $O(n^2 \log n)$)

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- fix arbitrary basepoint x
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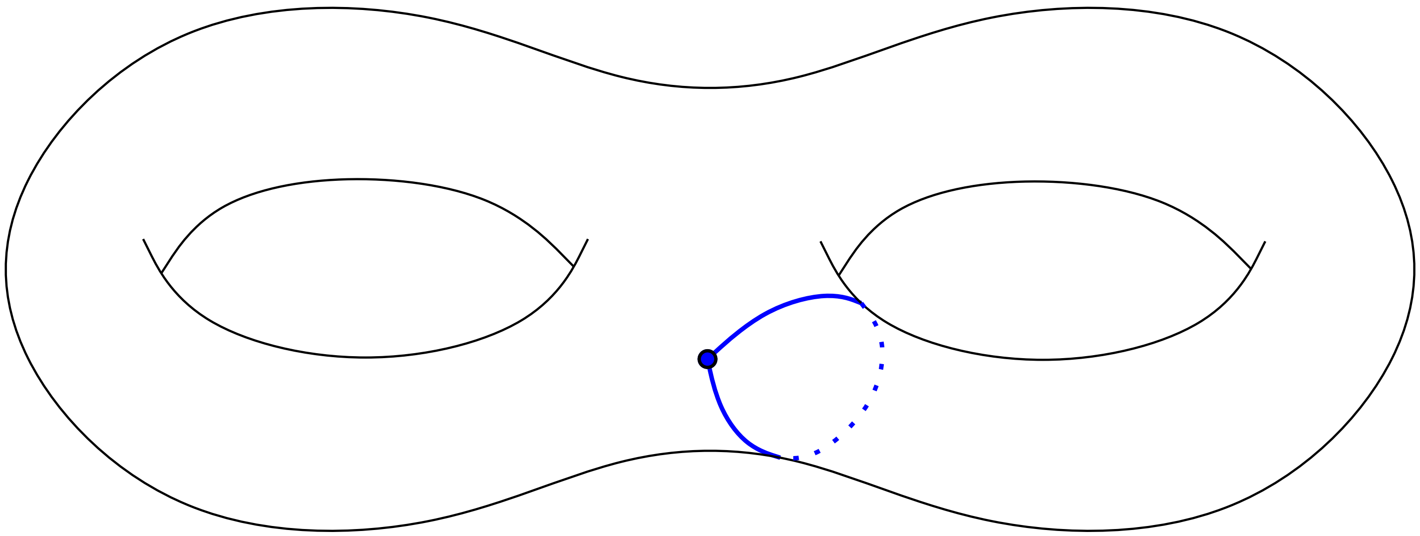

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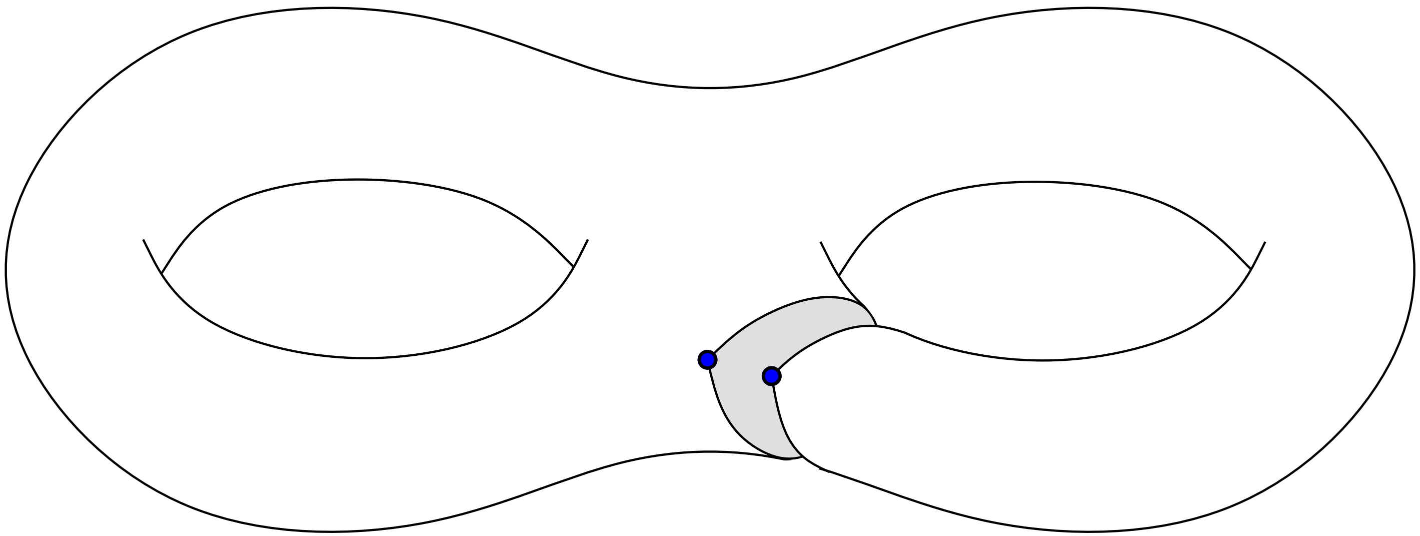

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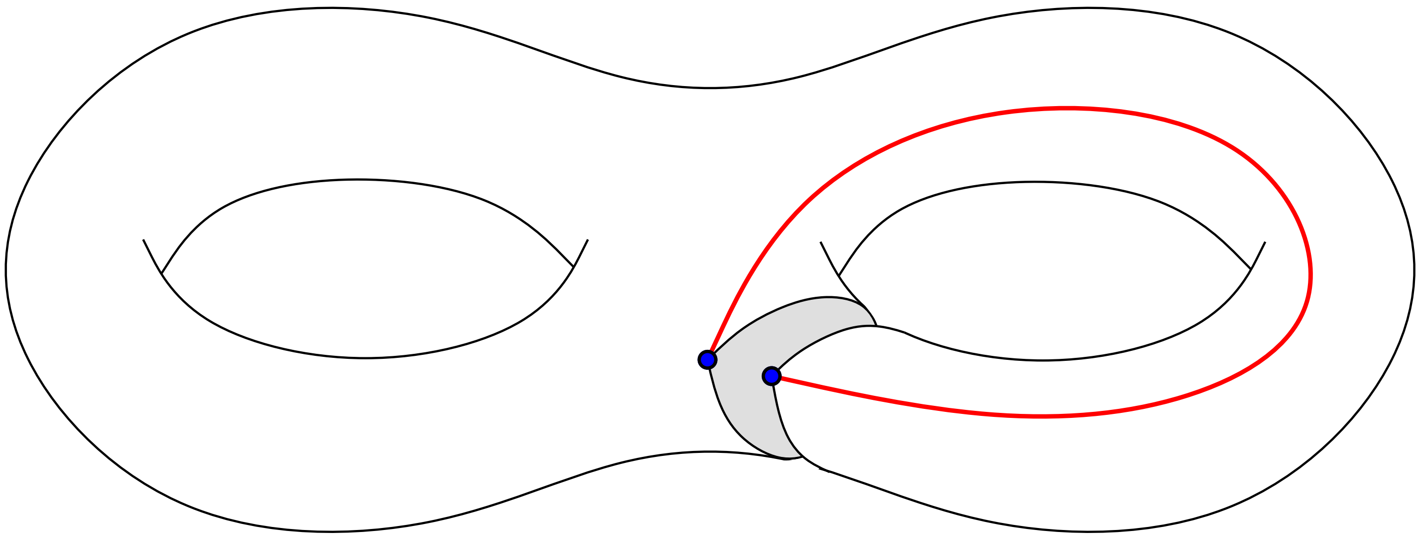

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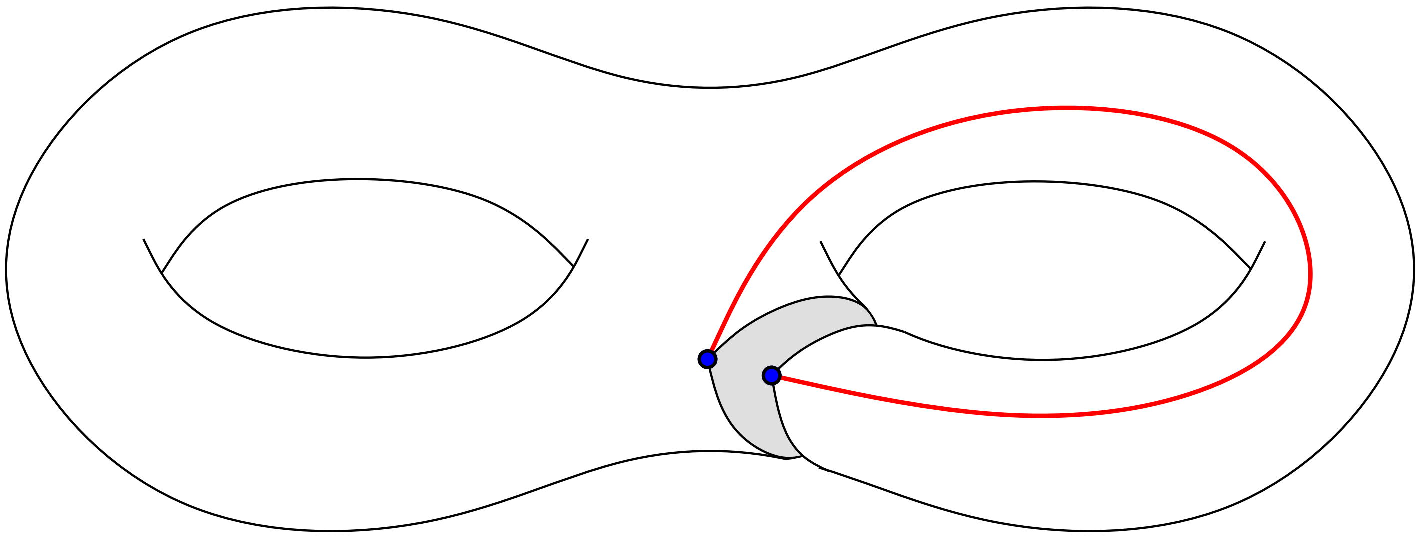

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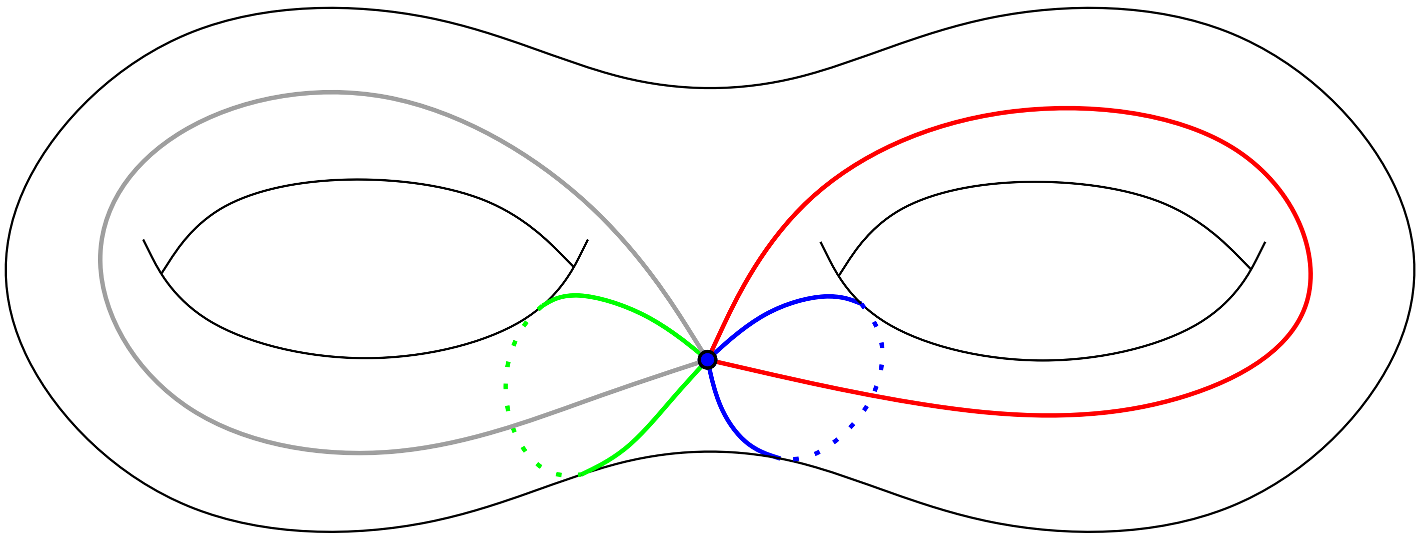
$$\vdots$$

- find shortest non-separating loop ℓ_{2g} in $\mathcal{M}$ ✂ $(\ell_1 \cup \cdots \cup \ell_{2g-1})$ form x to x



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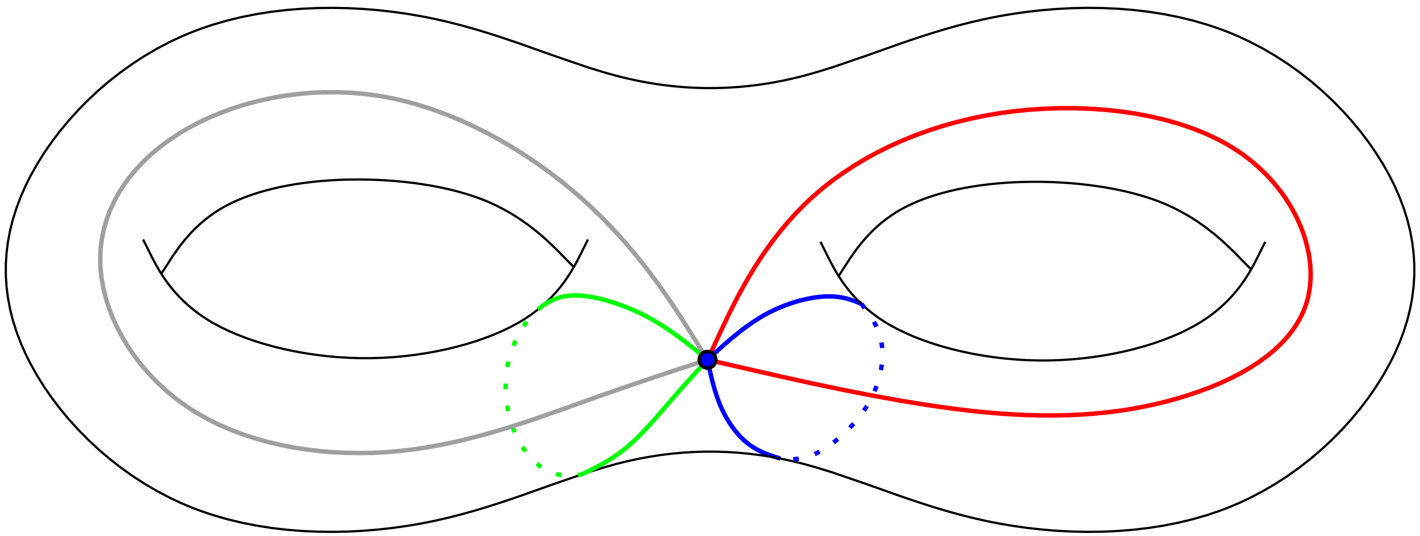

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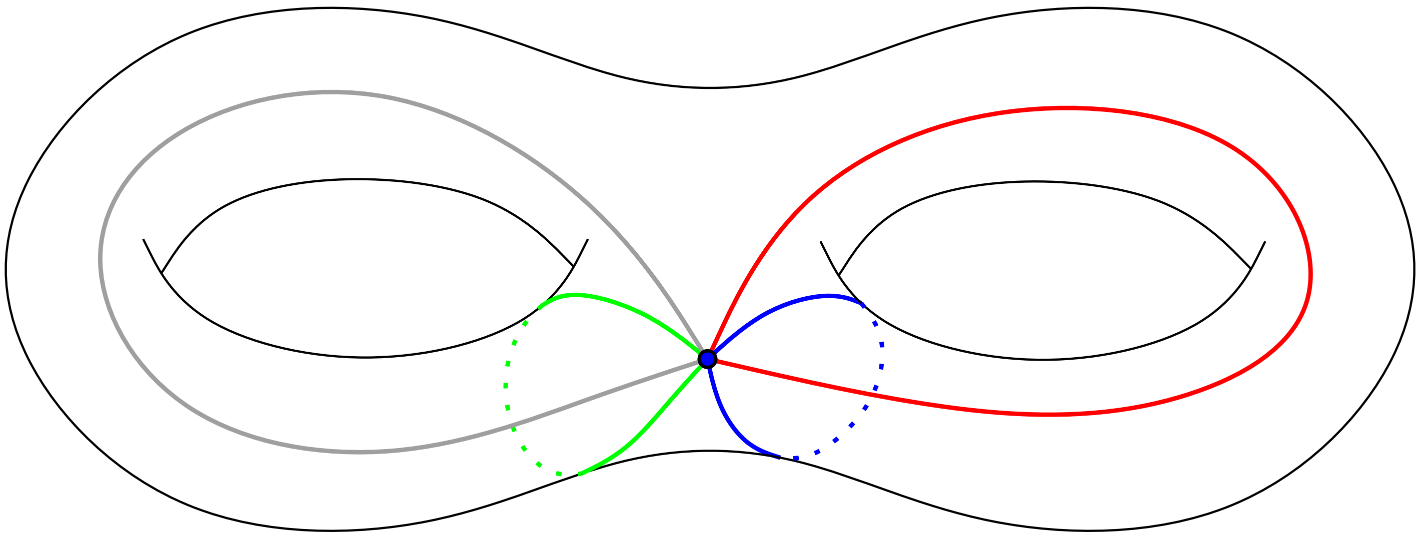

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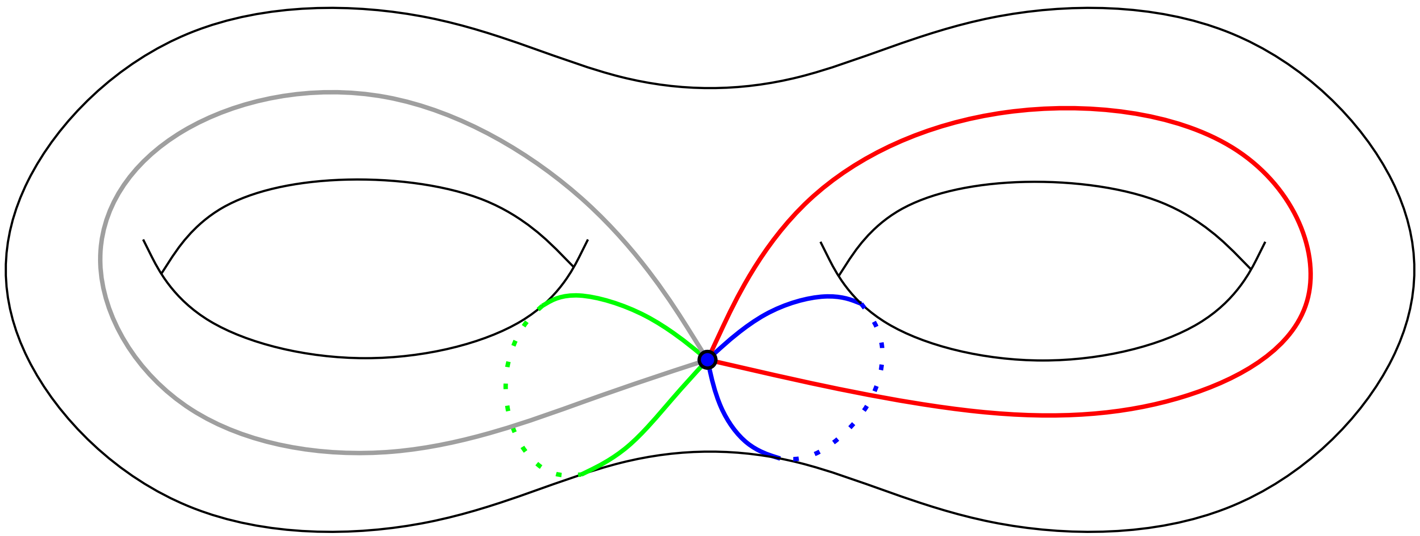
- can be computed in linear time (instead of just $O(gn \log n)$) using *Eppstein's tree-cotree decomposition* [Erickson & Whittlesey, SODA 2005]



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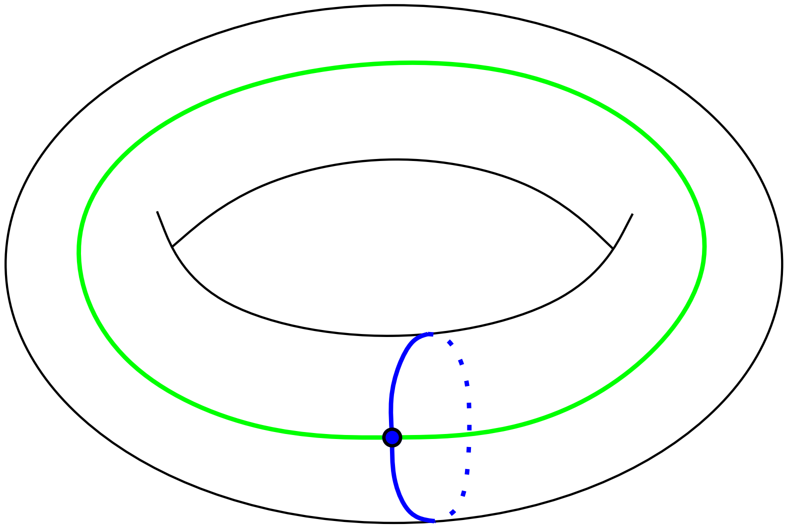

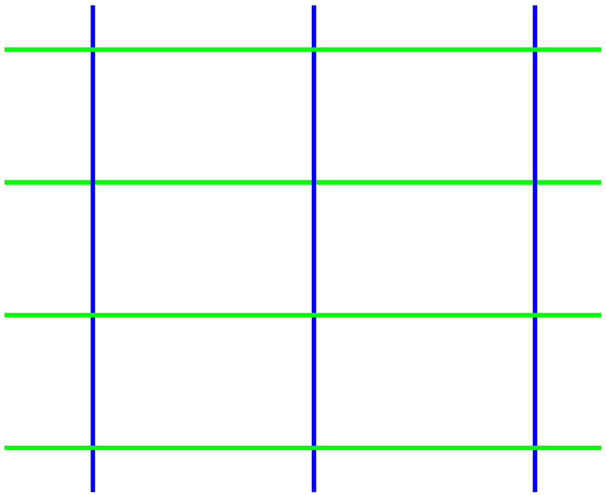

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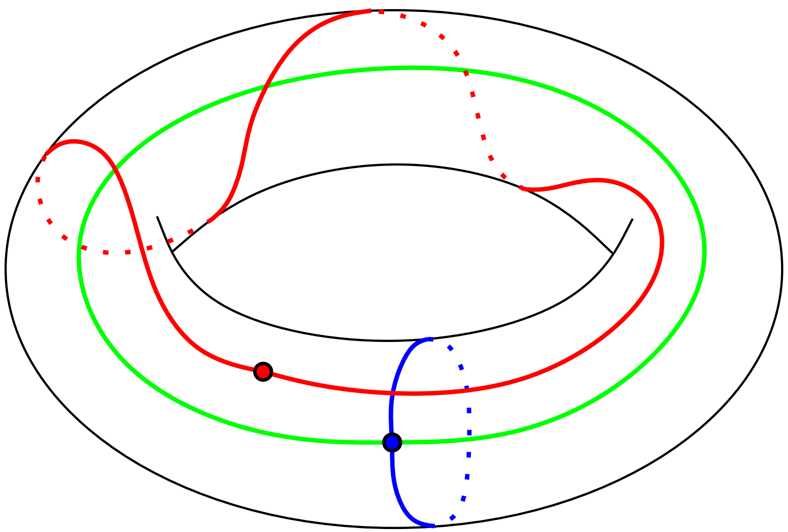

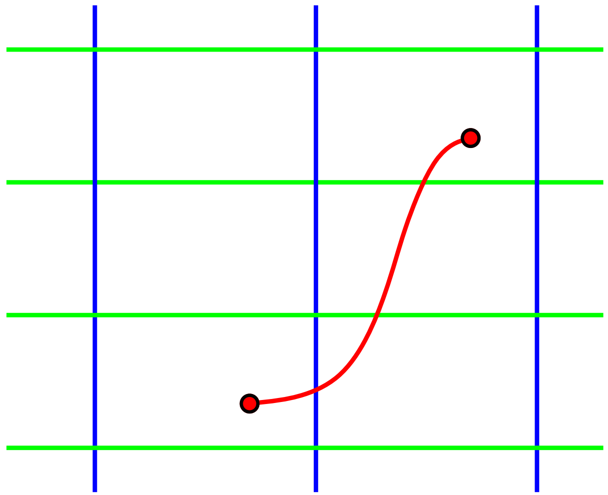

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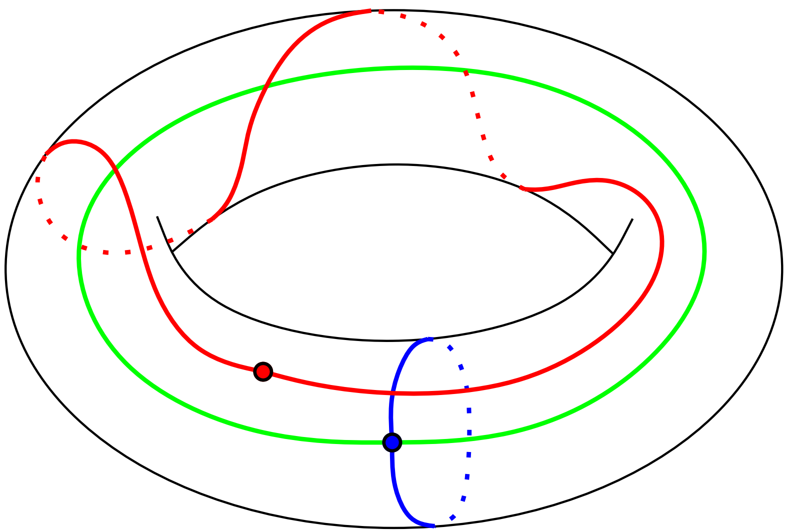
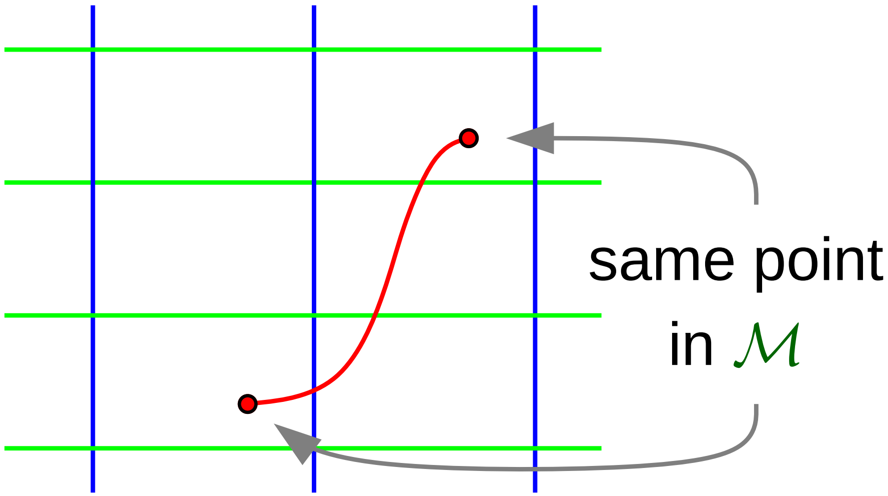
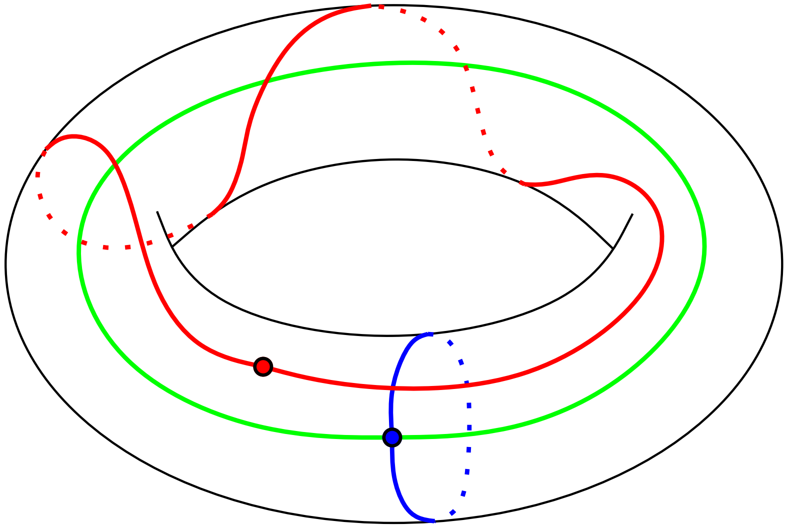


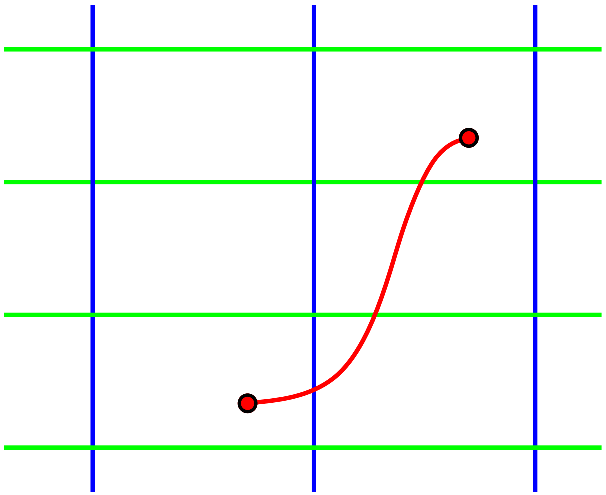


Technical Tools III

Lem. [Cabello & Mohar, ESA 2005] (*Crossing Bound*)
Let $\ell_1, \ldots, \ell_{2g}$ be a minimal system of loops.
Then there exists a shortest non-contractible (non-separating) cycle that crosses each loop ℓ_i at most twice.

Proof via *3-Path Condition* [Thomassen, JCT(A) 1990]





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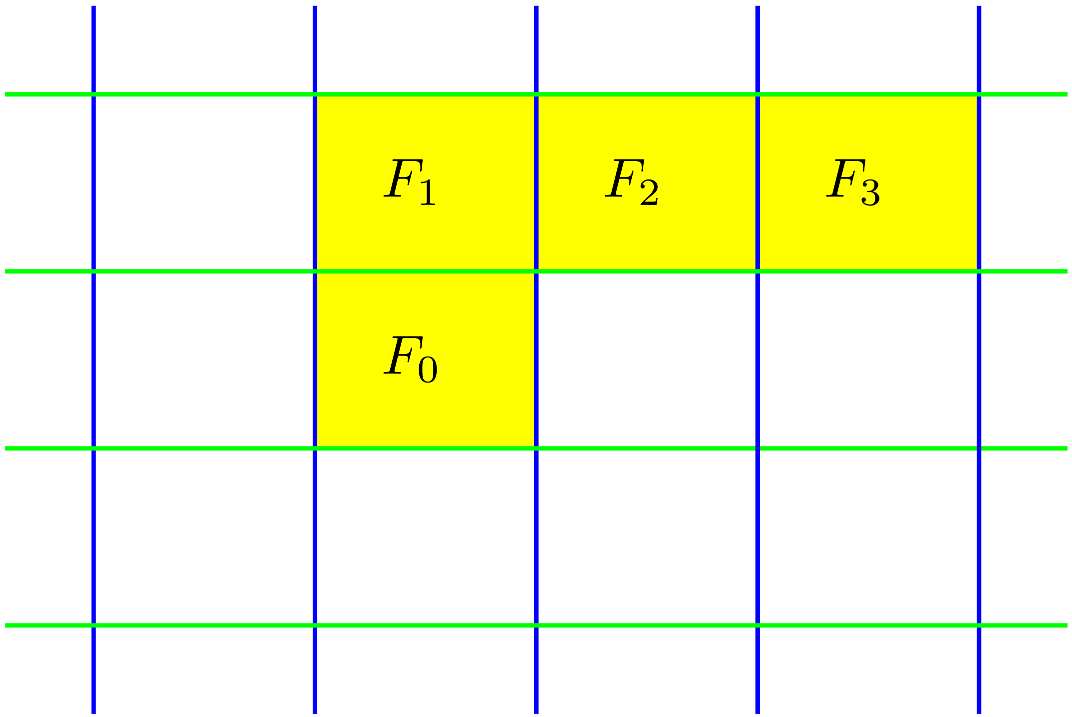

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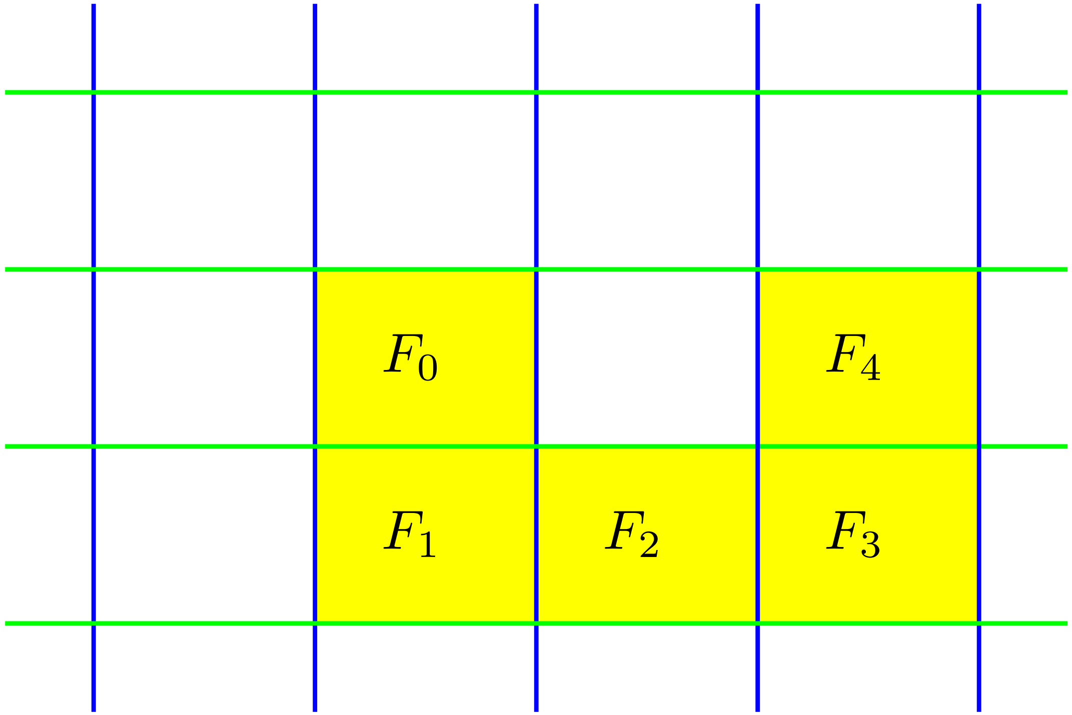

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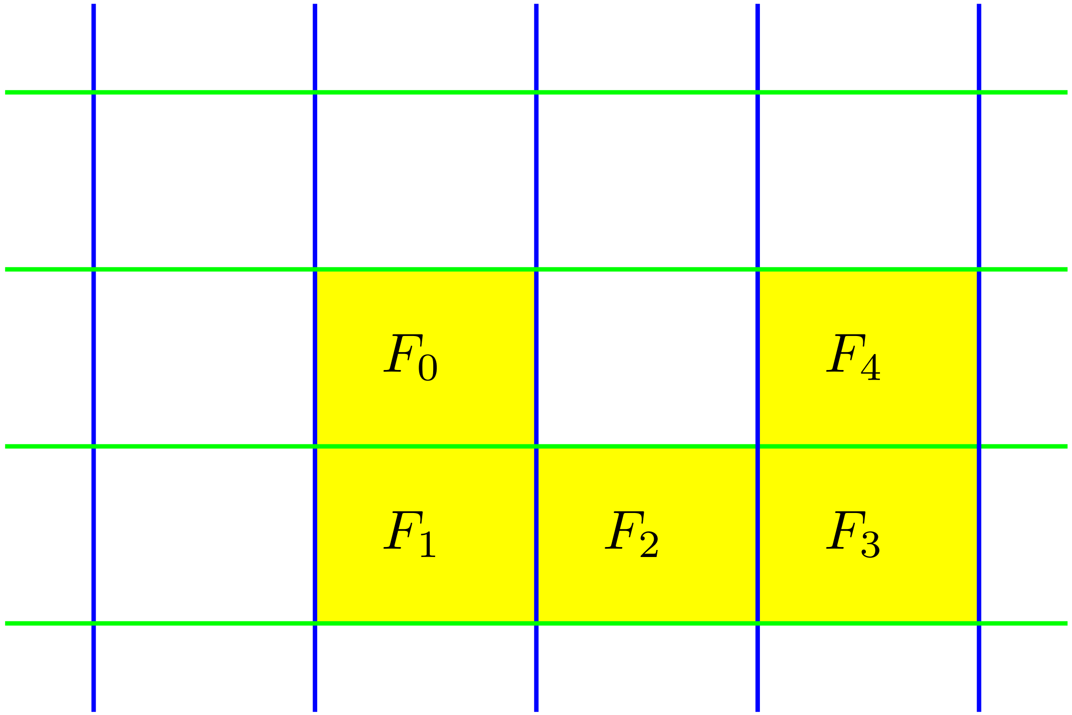
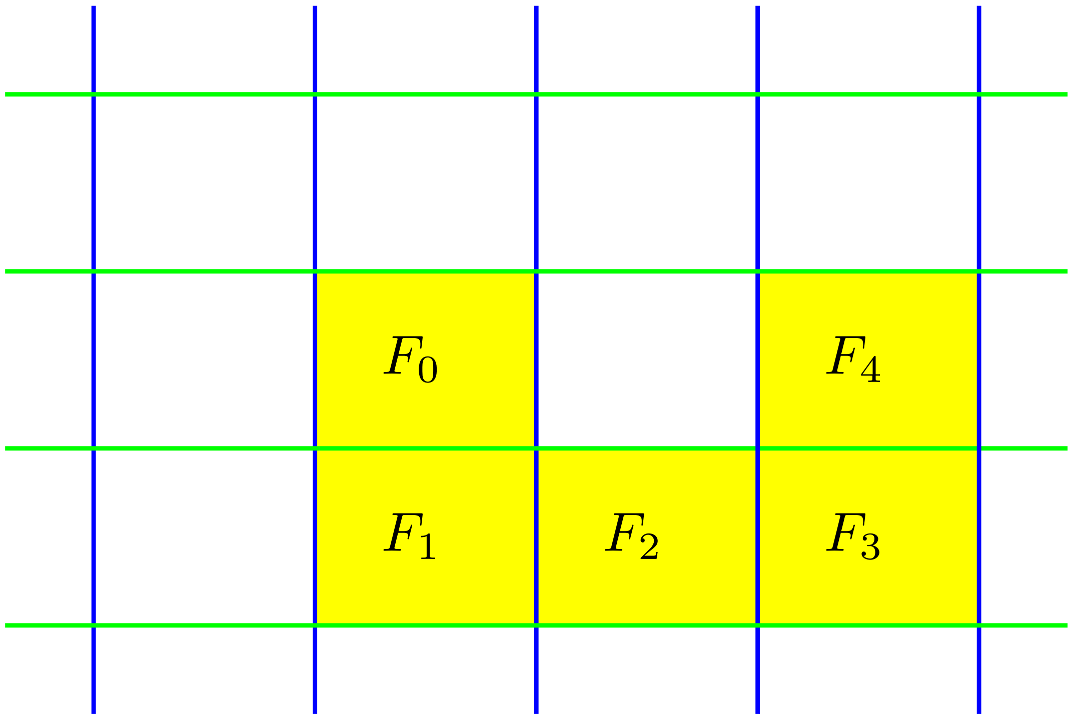
- How to find a shortest cycle in such a meta path?



Meta Paths — *Naive Search*

For each meta path:

- consider n corresponding points pairs u, v



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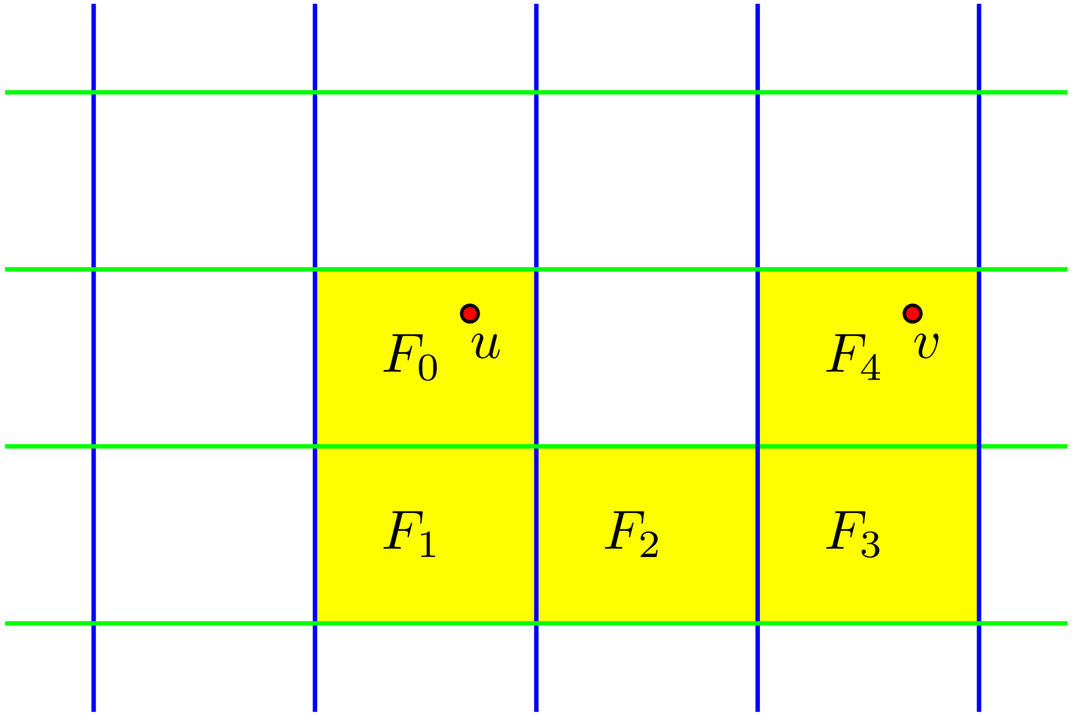

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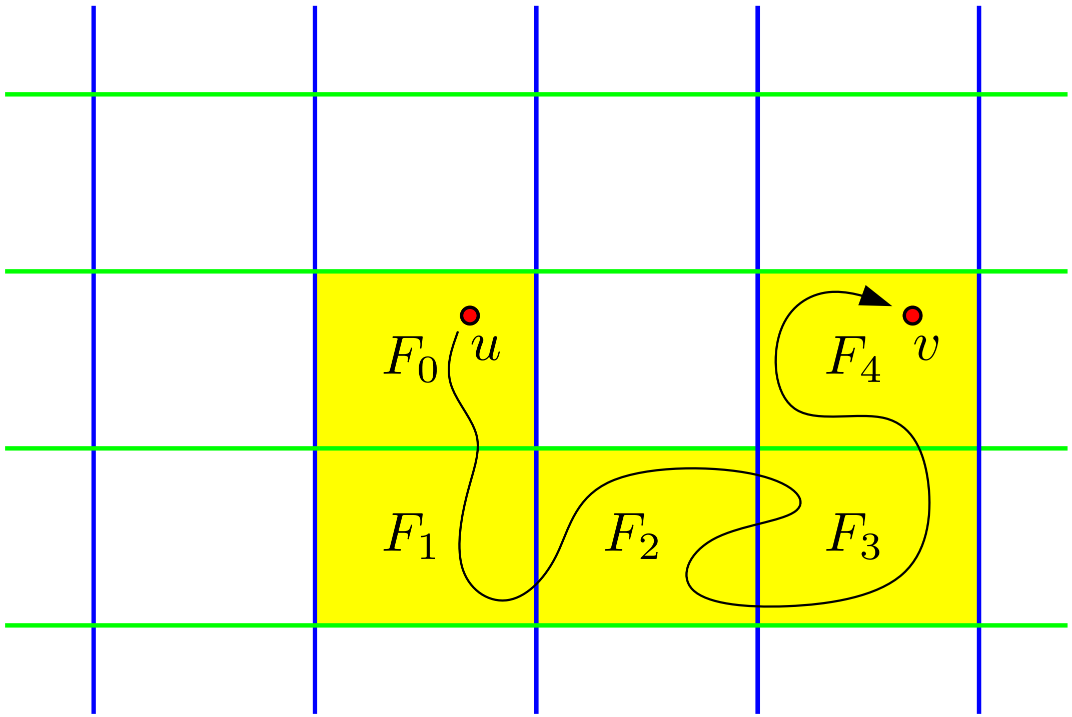

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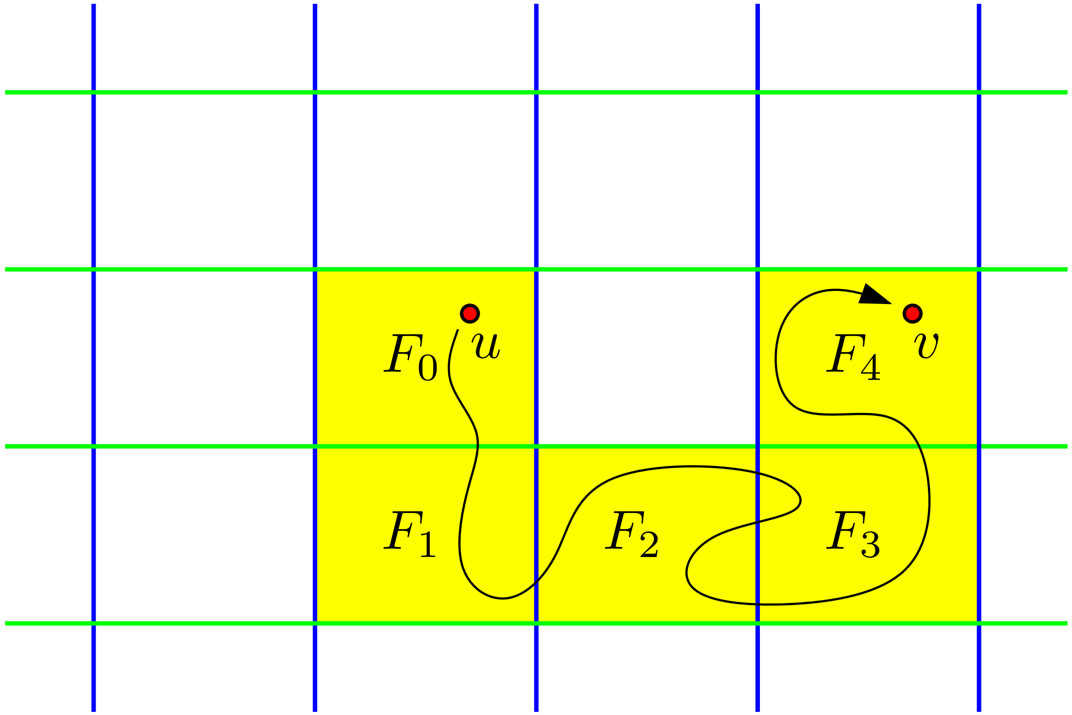
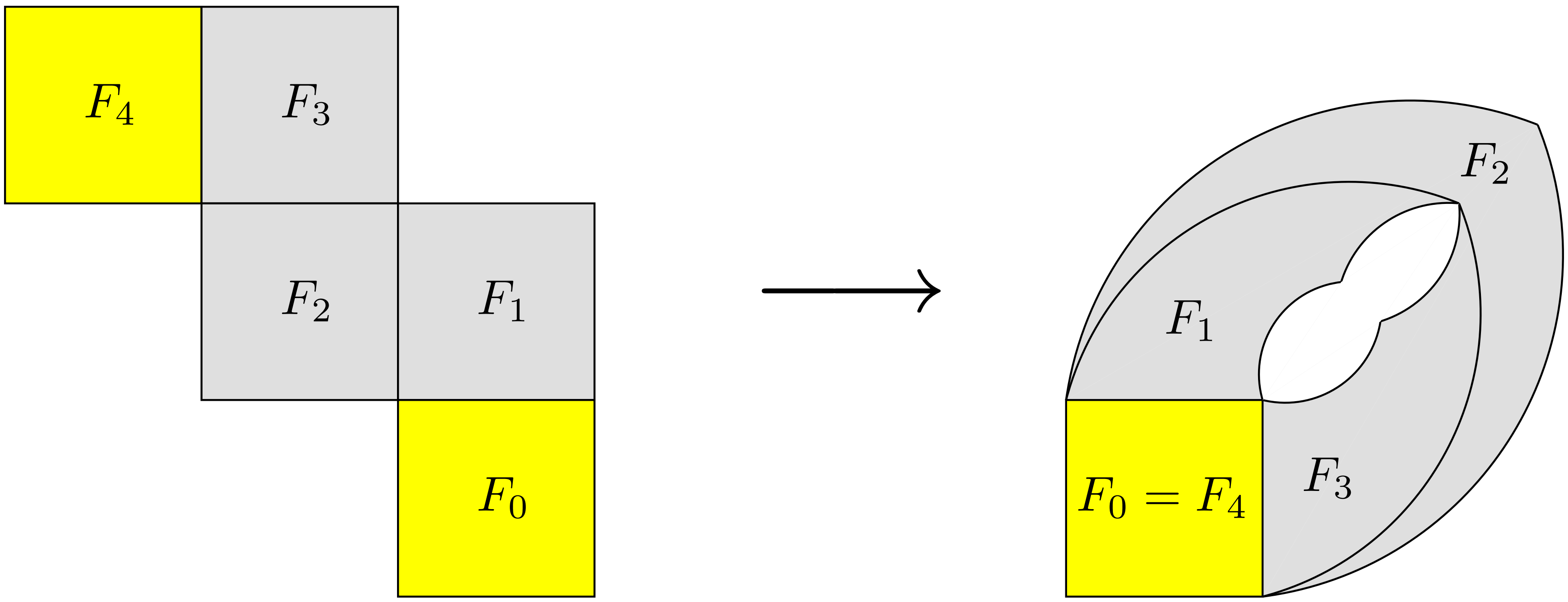
Total (worst-case) running time: $\Omega(n^2)$



Forming Cylinders

Trick: identify the terminal domains of a meta path to form a cylinder



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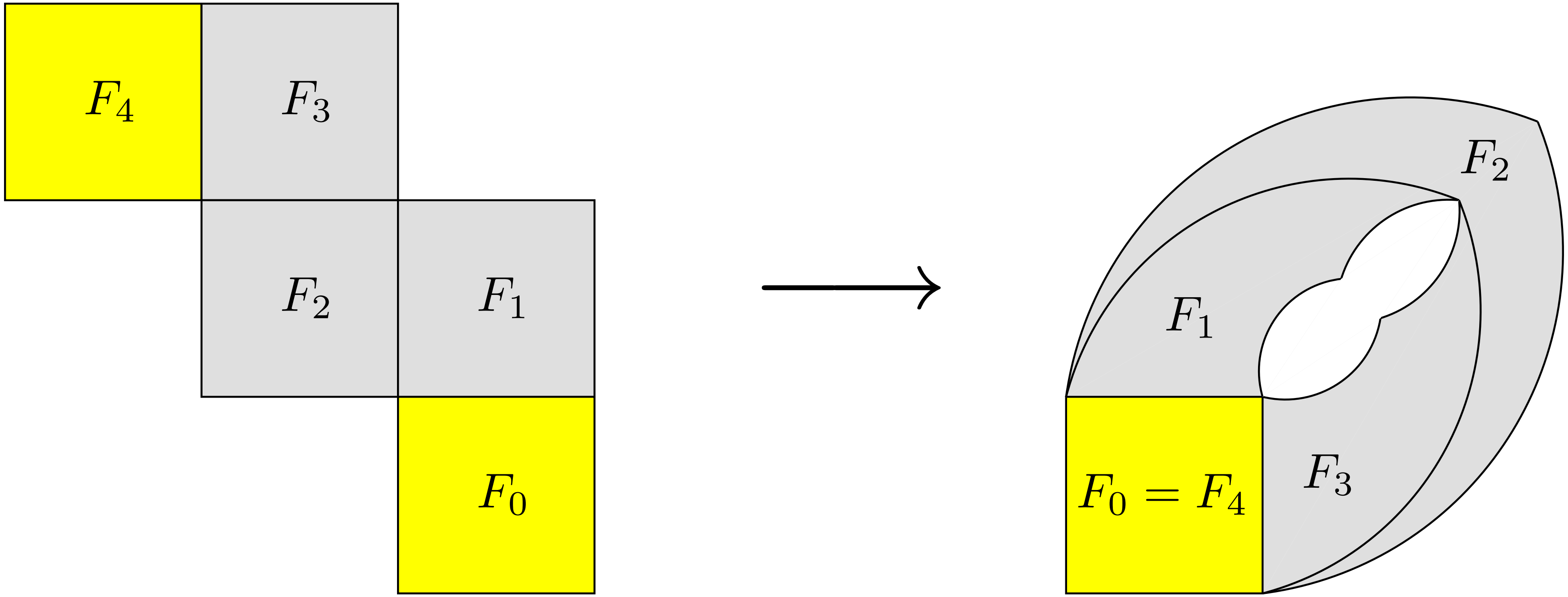
Thm. [Frederickson, 1987] In a planar graph with two vertices s,t, one can find a minimal s-t-cut in $O(n \log n)$ time.



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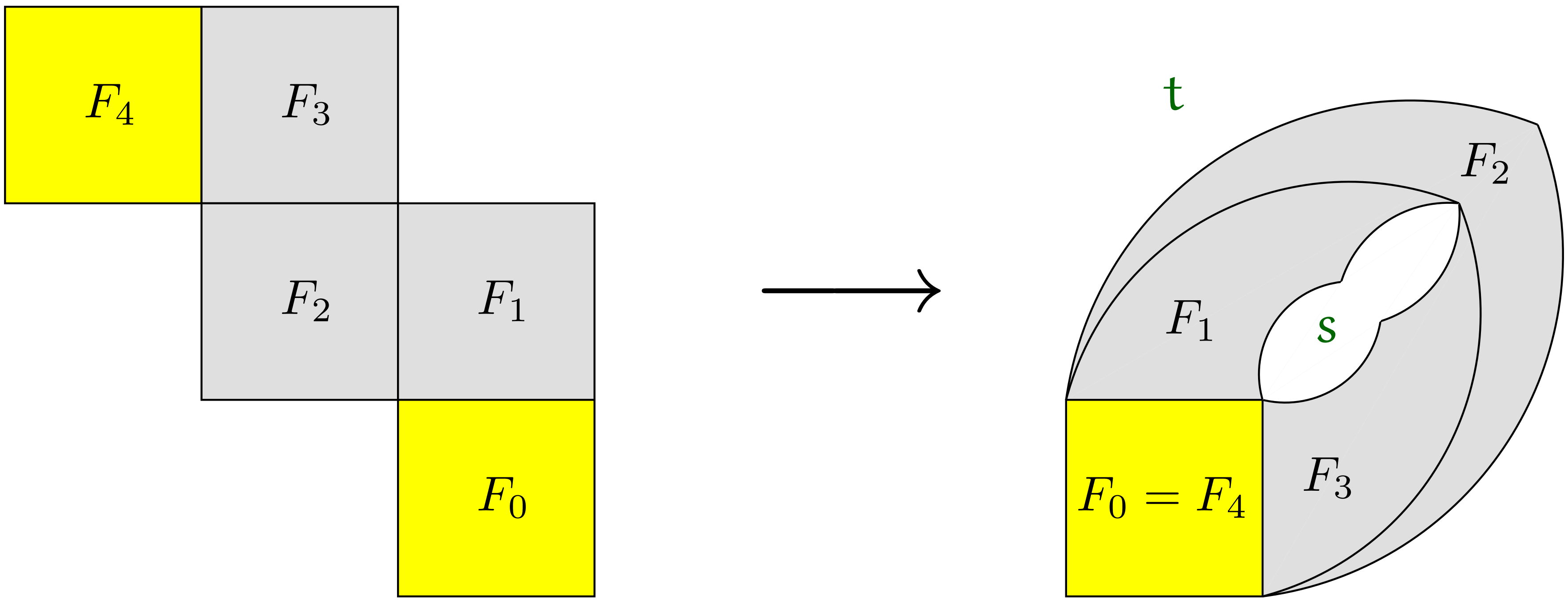

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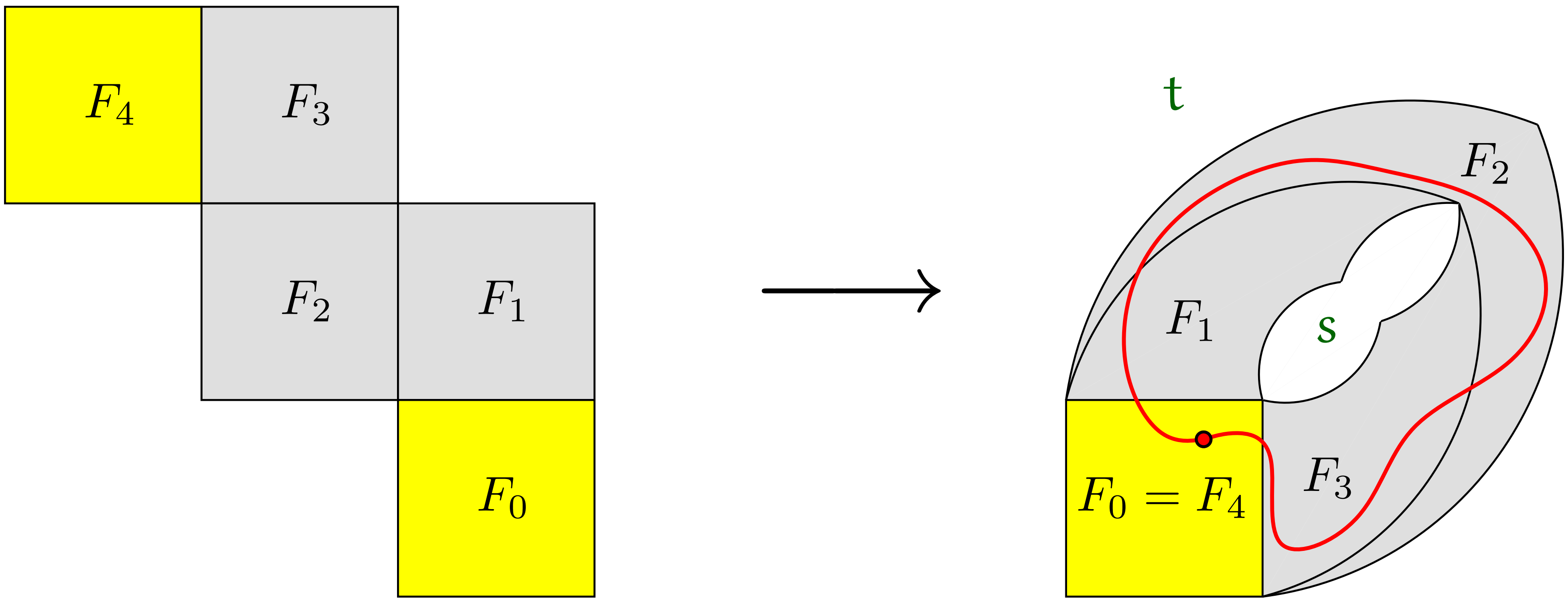

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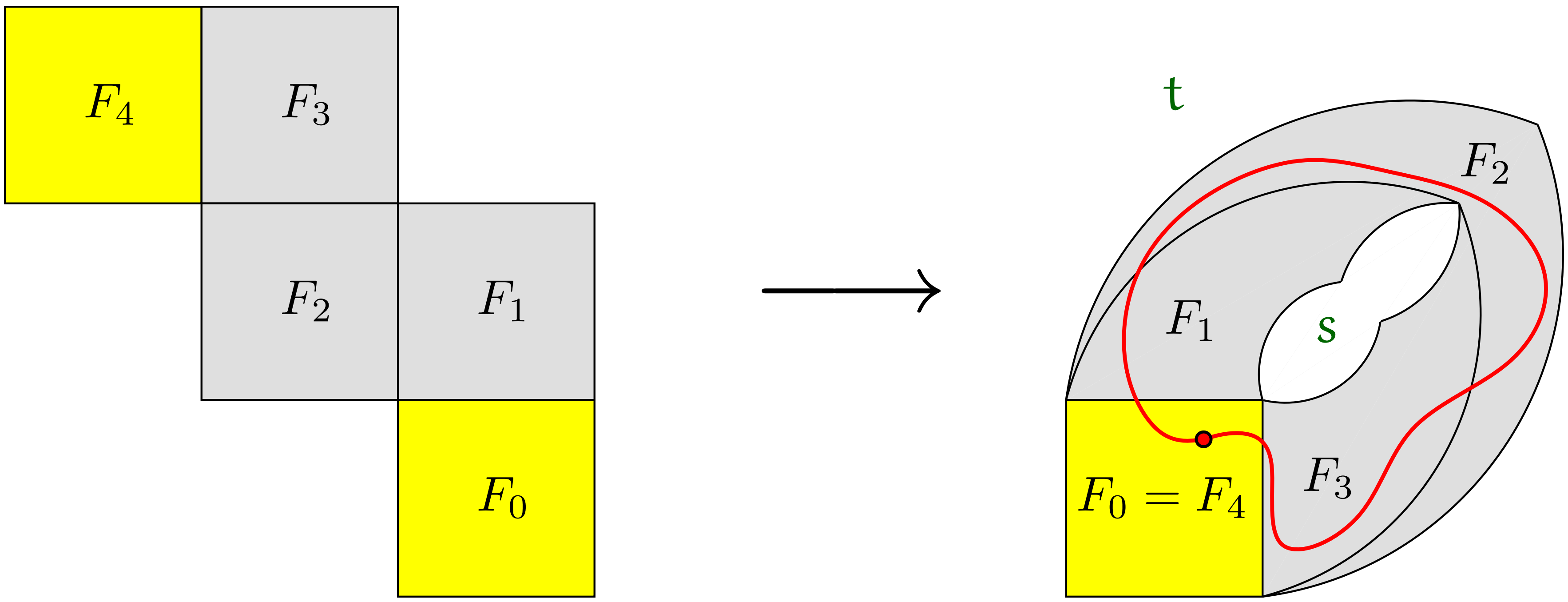
Warning : meta paths must be manyfolds! (can be guaranteed)



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Maybe trade-off possible between n and g?

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Want to cut $\mathcal{M}$ (along several paths) to obtain a (topological) disk.

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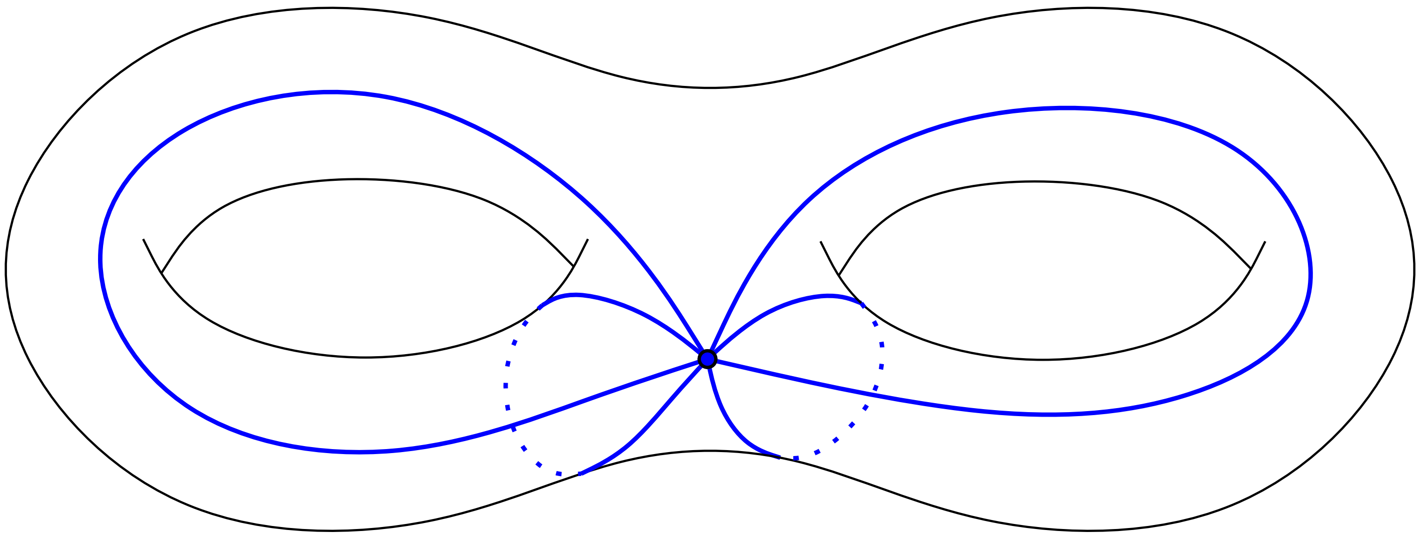

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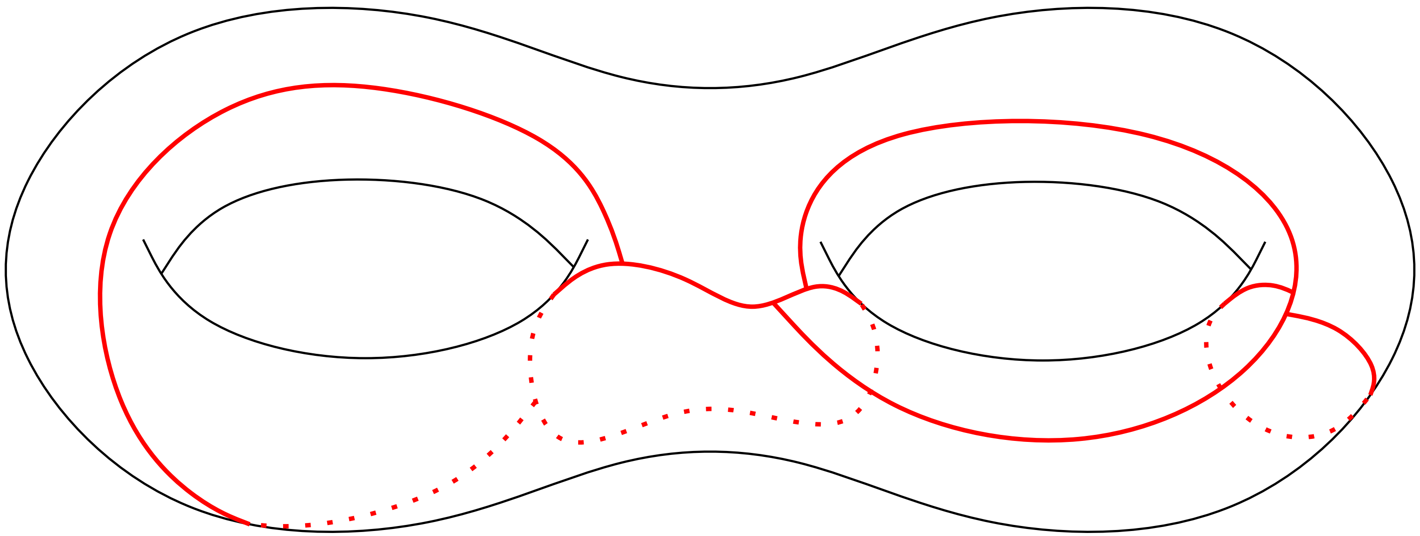

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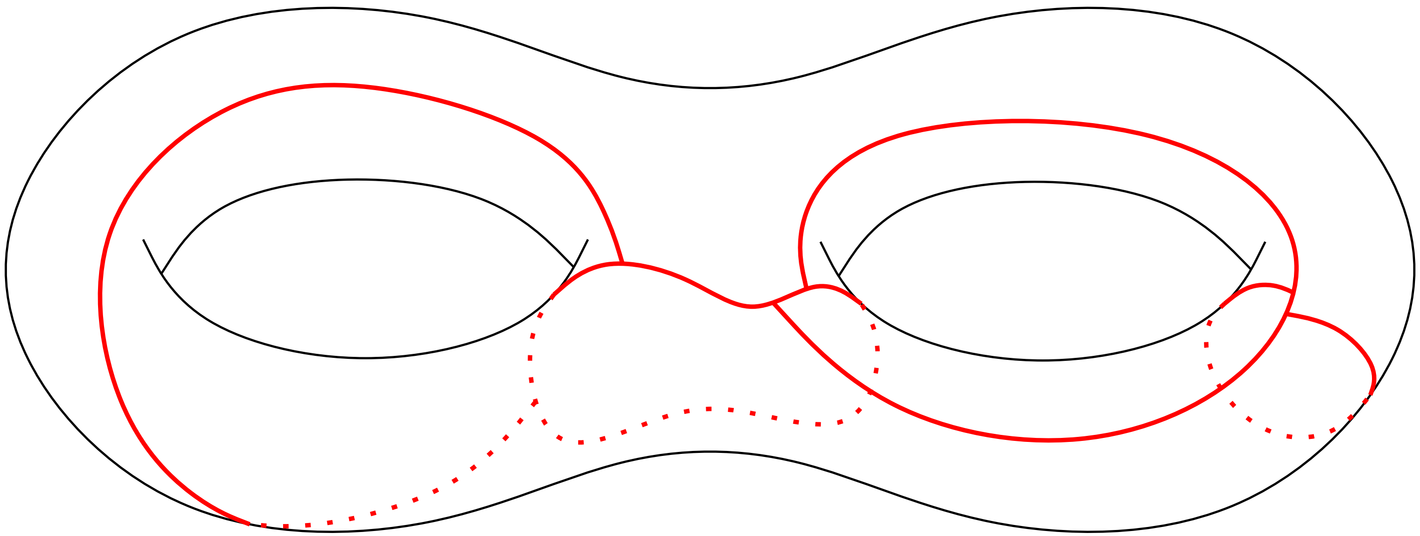

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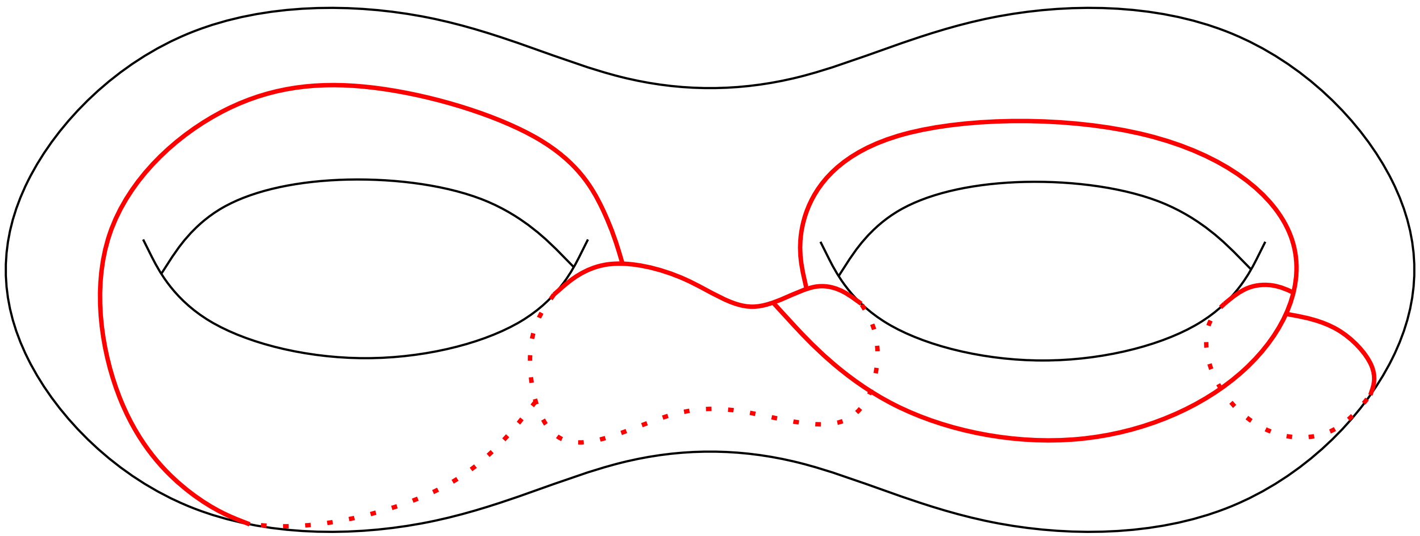

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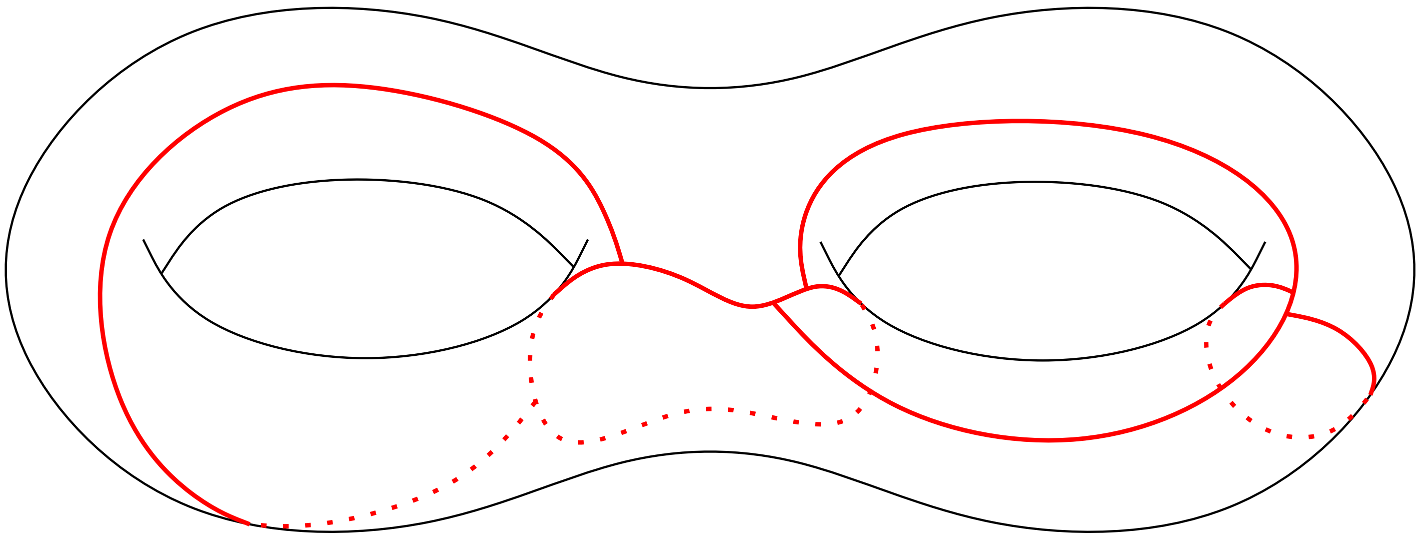

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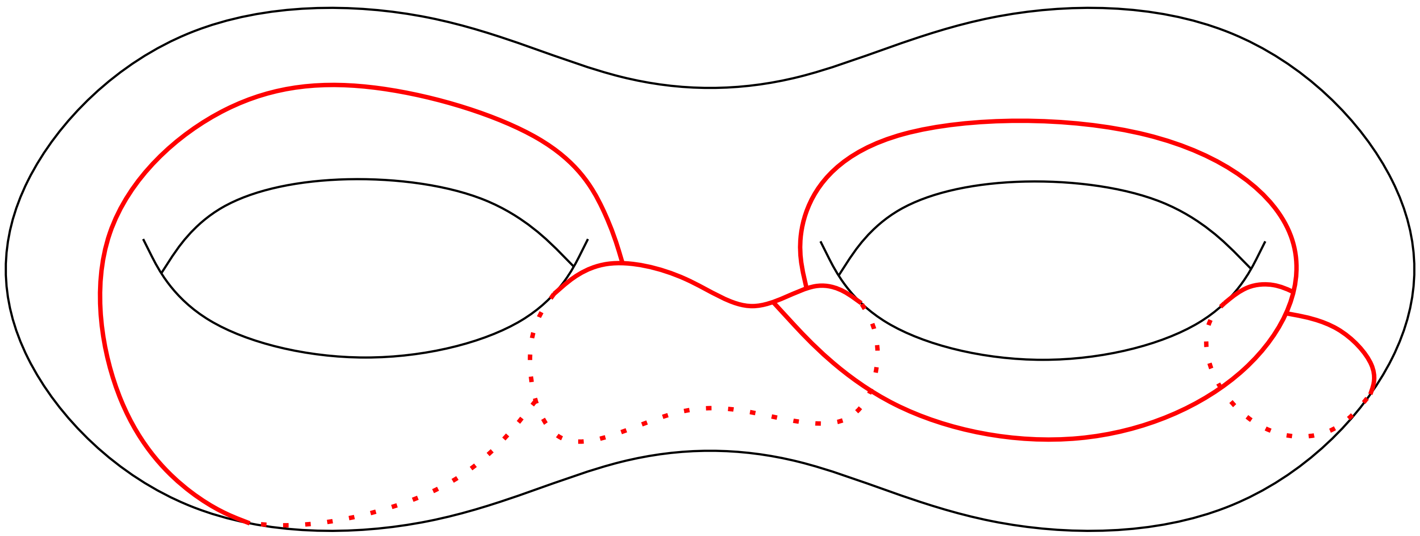
Question: Fixed-parameter tractable? (in g)
Because of similarity to Steiner-tree problem.



Outlook: Cut Graphs

Theorem. [K & Steurer, unpublished / –written]
Approximating the minimum cut graph up to a constant factor is fixed-parameter tractable (in g).