Context unification is in PSPACE

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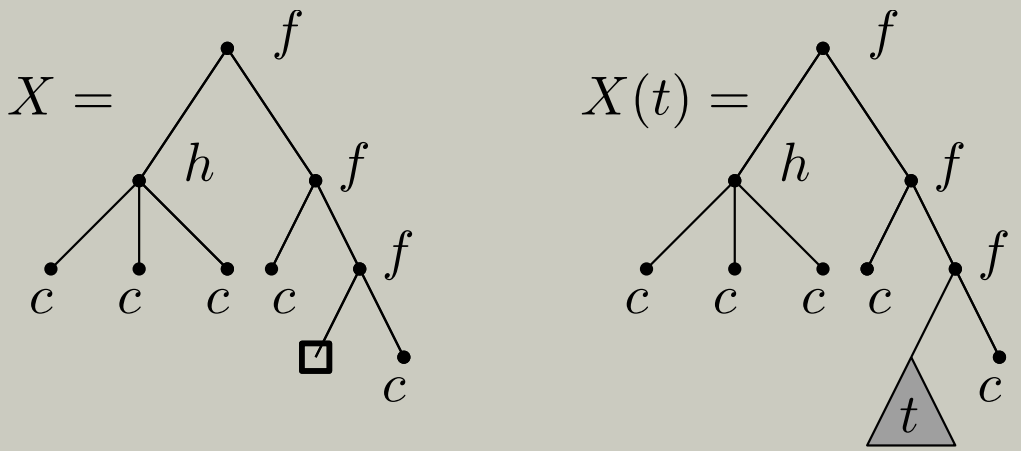
Context unification

Definition (Context unification)

Substitution for X uses argument exactly once.



$$X(f(Y(a), b) = f(Y(b), Y(a))$$

More formally

- signature (fixed arities) $[f, a, c]$
- context variables denoting terms with one 'hole' $[X()]$
- variables denoting closed terms $[x]$
- equations built with them

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- $X(c) = Y(c')$

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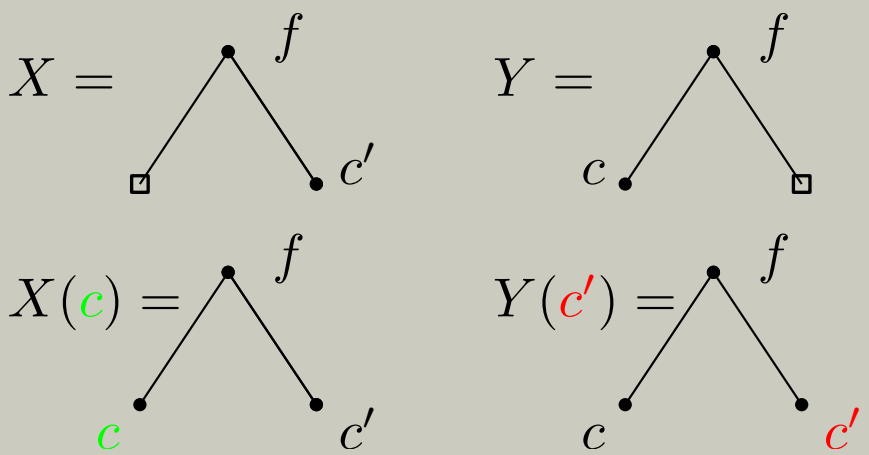

Why

In between important problems:

- word equations , term unification
- second-order unification

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- word equations in PSPACE, term unification in PTIME
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Other connections

- one-step term rewriting
- natural language processing
- linear second-order unification

This talk

Context unification is in PSPACE.
(trivially NP-hard).

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- generalises solution for word equations
- apply compression to both sides
- preserve equality

Leaf compression

- 'Absorb' each leaf by its father (and change the labels).
- Replace $f(c_1, t_2, t_3, \ldots, c_2, \ldots, t_k)$ with $f'(t_2, t_3, \ldots, t_k)$

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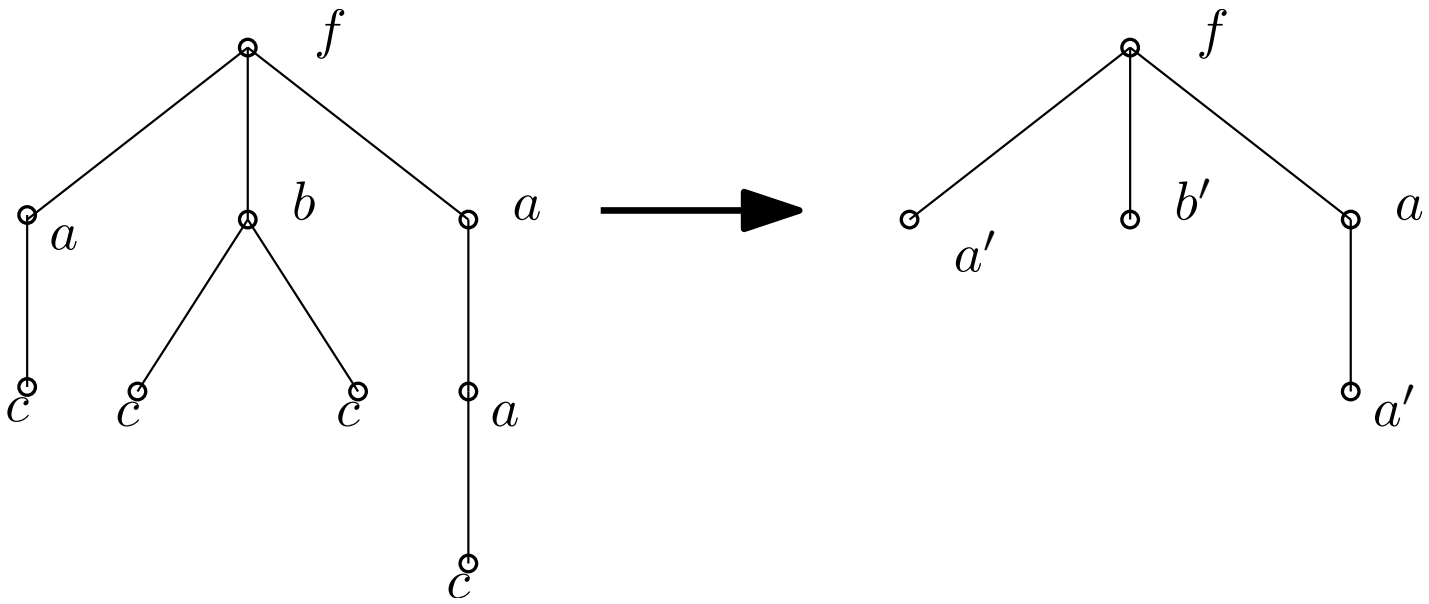

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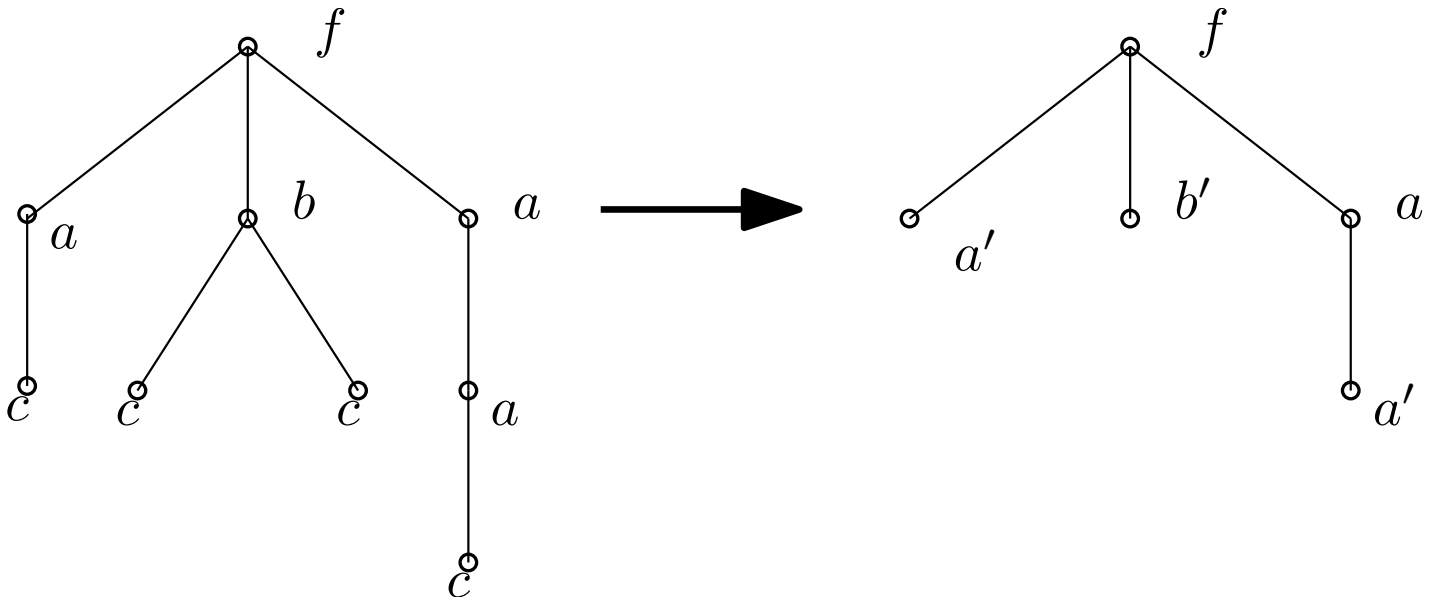

- half of nodes are leaves
- $\mathcal{O}(\log N)$ rounds

Chains compression

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Lemma

1/3 of the nodes are compressed.

Algorithm

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6: **for** $ab \in P$ **do**
7: compress 2-chain ab ▷ 2-chains compression
8: $L_0 \leftarrow$ list of constants, $L_{\geq 1} \leftarrow$ list of other letters in T
9: **for** all $f \in L_{\geq 1}$ and all $c \in L_0$ **do**
10: perform leaf compression (in parallel)

Size analysis

Lemma

In one phase the size drops by a constant factor.

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no chain	we remove all leaves, size halves
single chain	a string, size drops by a constant factor
general	some mix of above

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But how?

Easy case (2-chain compression)

- we replace every explicit ab
- every ab in $S(X)$ ($S(x)$) is replaced implicitly

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Definition (2-chain types)

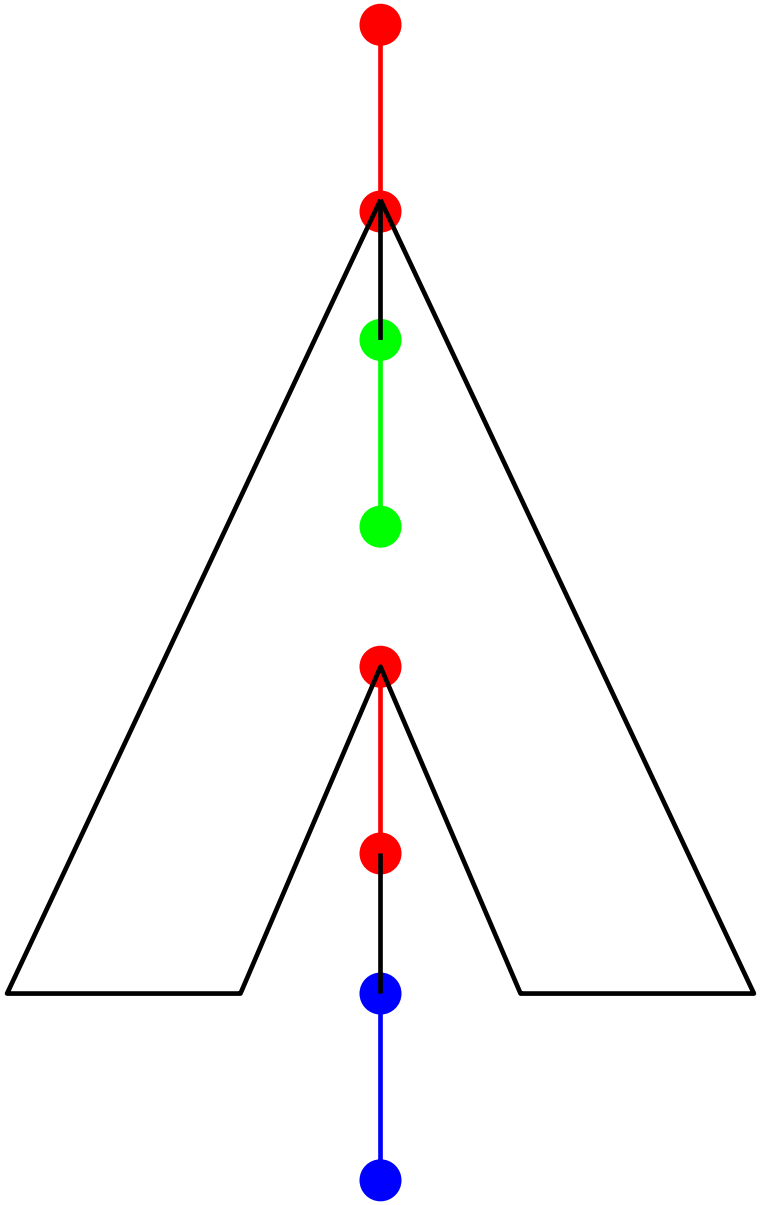
For a solution S the occurrence of ab is

explicit it comes from the equation;

implicit comes solely from $S(X)$ (or $S(x)$);

crossing in other case.

ab is crossing (for S) if it has a crossing occurrence (for S), non-crossing (for S) otherwise.



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Lemma

2ChainNCrComp(a, b) *properly compresses noncrossing 2chains.*

Proof.

Every ab is replaced:

explicit pairs replaced explicitly

implicit pairs replaced implicitly (in the solution)

crossing there are none □

a-chains

Definition (*a*-chain types)

For a solution S the occurrence of a maximal a-chain is

- explicit: it comes from the equation;
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MaxChainNCr(a) *compresses noncrossing a-chains.*

Leaf compression

Definition (father-leaf pair)

For a solution S the occurrence of *father-leaf pair* (f, c) is

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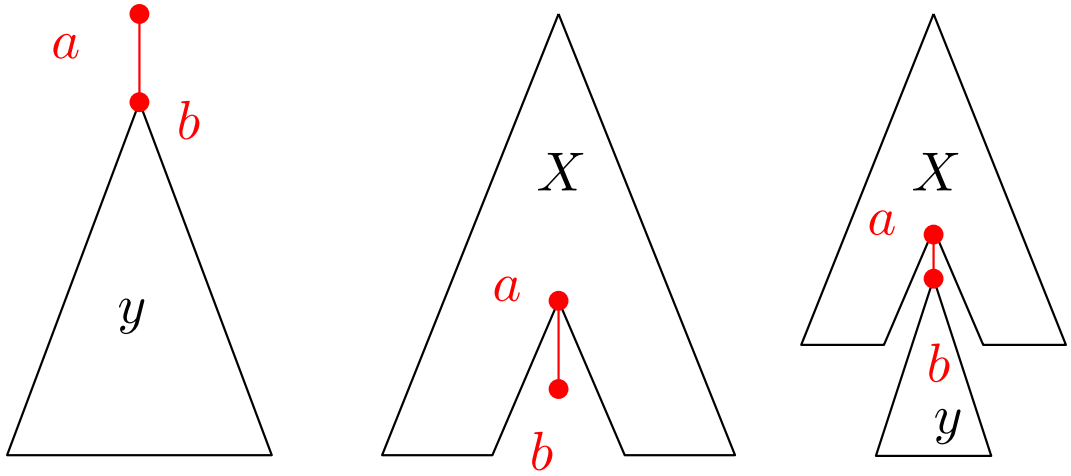
Lemma

LeafNCr *performs leaf compression when there is no crossing father-leaf pair* (f, c).

Uncrossing



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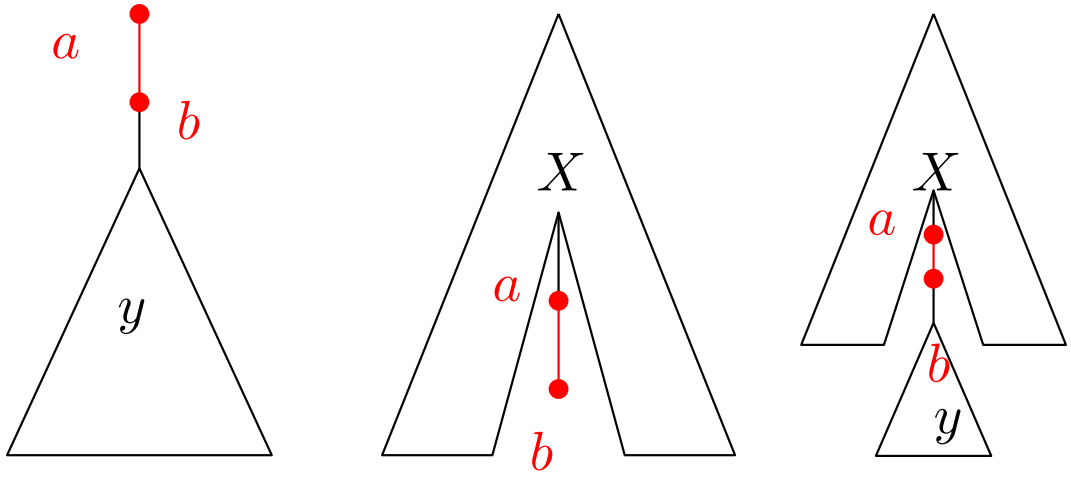

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  implicitly change $S(Y) = bt, S(X) = t'a$ to $S(Y) = t$, $S(X) = t'$

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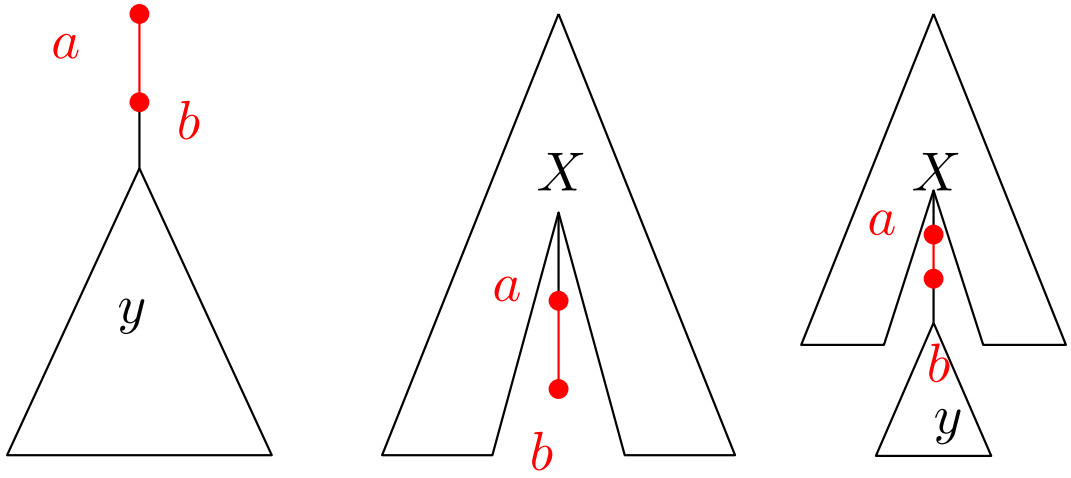

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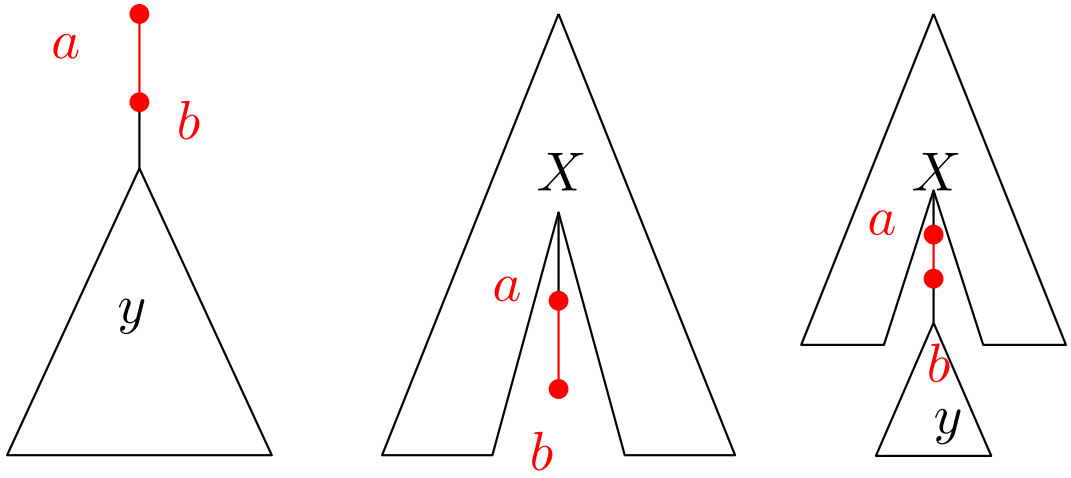

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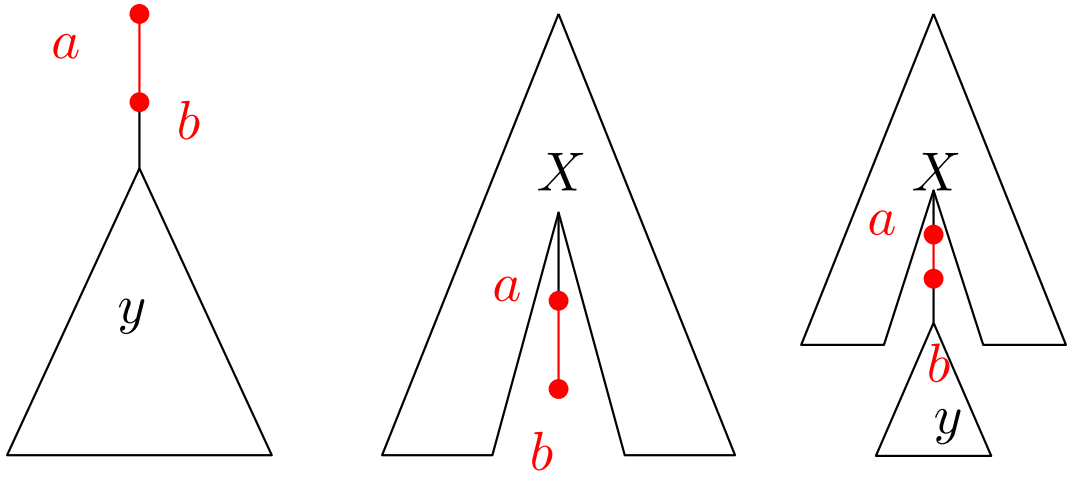

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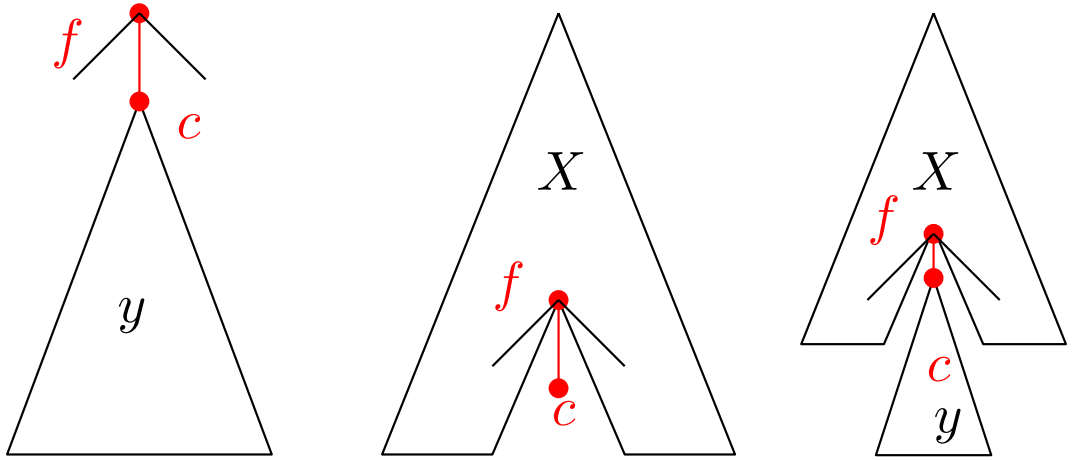
Compress it!

Crossing *a*-chains?

a is a crossing chain $\Leftrightarrow$ *aa* is a crossing 2-chain

- Crossing *a*-block: similar to crossing *ab* (equiv. to crossing *aa*).
- pop whole *a*-‘prefix’ and *a*-‘suffix’

Uncrossing father-leaf pair



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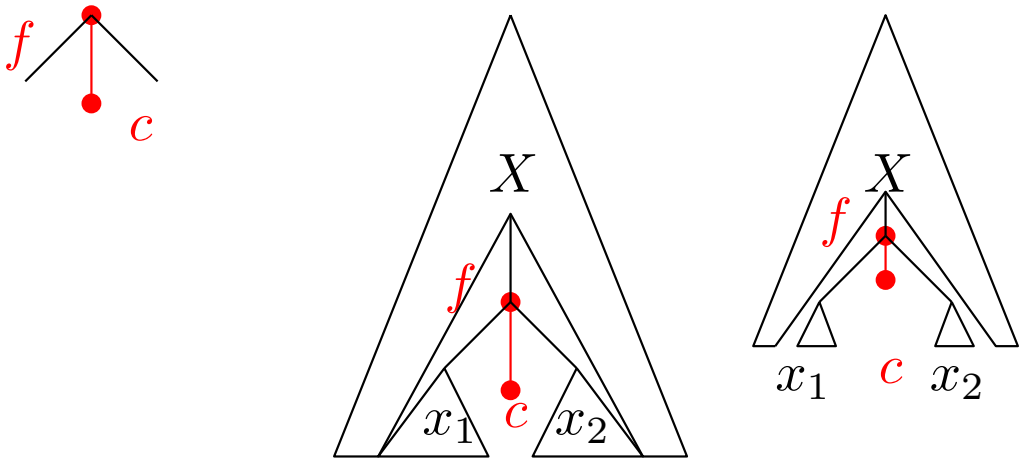

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- replace X with $X(f(x_1, \ldots x_{i-1}, \cdot, x_{i+1}, \ldots, x_\ell))$

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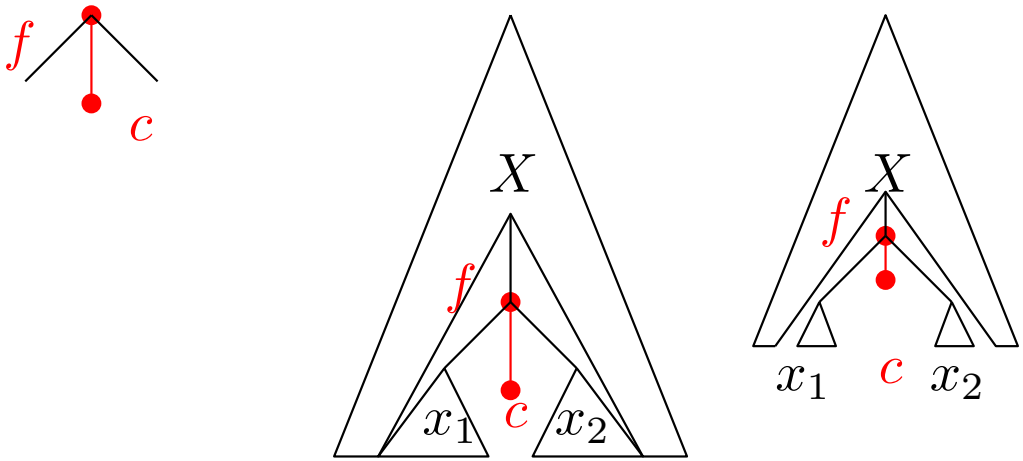

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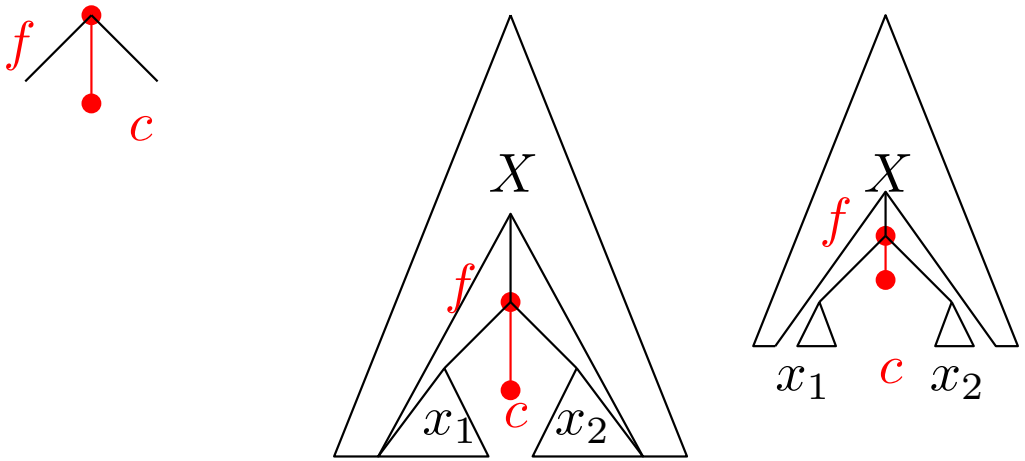

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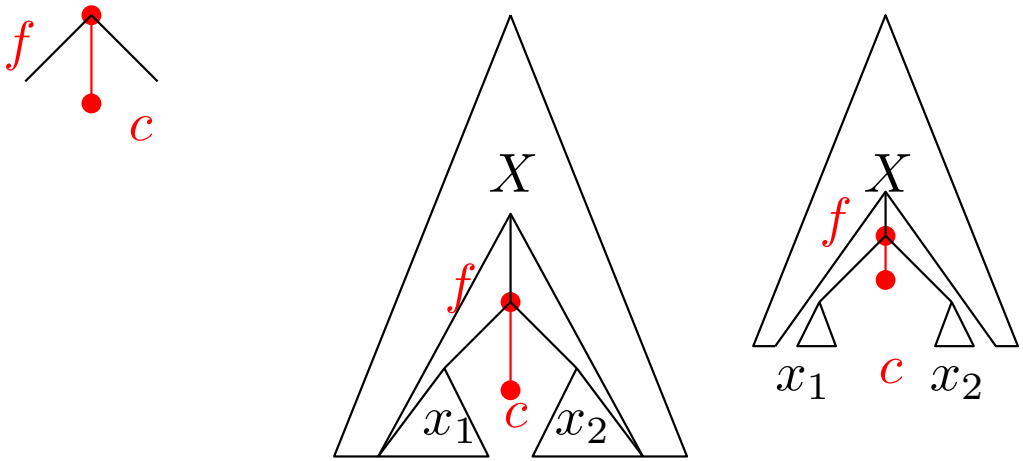

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- there are no crossing (f, c) father-leaf pairs
- compress them

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12:    L_0 ← list of constants, L_≥1 ← list of other letters in T
13:    for noncrossing f ∈ L_≥1, c_1, ..., c_k ∈ L_0 do
14:        leaf compressions for f, c_1, ..., c_k
15:    for remaining f ∈ L_≥1 and c_1, ..., c_k ∈ L_0 do
16:        uncross and leaf compress f, c_1, ..., c_k
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Controlling the new variables

Lemma

There are at most

- *n context variables*
- *kn variables*

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Proof.

- we do not introduce new context variables
- we can associate each ‘new’ variable with context variable at most $(k - 1)$ are associated with one context variable □

Equation size

- there are kn variables and n context variables
- each uncrossing: k letters per variable (so $\mathcal{O}(k^2 n)$)
- there are $\mathcal{O}(kn)$ uncrossings
- $\mathcal{O}(k^3 n^2)$ new letters

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