Least and greatest solutions of equations over sets of integers

Artur Jeż Alexander Okhotin

Wrocław, Poland Turku, Finland

23 August 2010 A. D.

Resolved systems of language equations

$$\begin{cases} X_1 = \varphi_1(X_1, \ldots, X_n) \\ \vdots \\ X_n = \varphi_n(X_1, \ldots, X_n) \end{cases}$$

- X_i: subset of Ω^*.
- φ_i: variables, constants, operations on languages.

Resolved systems of language equations

$$\begin{cases} X_1 &= \varphi_1(X_1, \dots, X_n) \\ &\vdots \\ X_n &= \varphi_n(X_1, \dots, X_n) \end{cases}$$

- X_i: subset of Ω^*.
- φ_i: variables, constants, operations on languages.

- studied by Ginsburg and Rice ($\cup$, $\cdot$), semantics of CFG
- extended by Okhotin to ($\cap$, $\cup$ and $\cdot$), defines conjunctive grammars

Resolved systems of language equations

$$\begin{cases} X_1 = \varphi_1(X_1, \dots, X_n) \\ \vdots \\ X_n = \varphi_n(X_1, \dots, X_n) \end{cases}$$

- X_i: subset of Ω^*.
- φ_i: variables, constants, operations on languages.

- studied by Ginsburg and Rice ($\cup$, $\cdot$), semantics of CFG
- extended by Okhotin to ($\cap$, $\cup$ and $\cdot$), defines conjunctive grammars

- interested in $(S_1, \dots, S_n)$ which are
  - least: $S_i \subseteq S'_i$ for every other solution $(S'_1, \dots, S'_n)$
  - greatest: $S_i \supseteq S'_i$ for every other solution $(S'_1, \dots, S'_n)$
- guaranteed to exist (Tarski's fixpoint theorem).

Example

Example

$$X = XX \cup \{a\}X\{b\} \cup \{\epsilon\}$$

Example

Example

$$X = XX \cup \{a\}X\{b\} \cup \{\epsilon\}$$

Least solution: the Dyck language.
Greatest solution: $\{a, b\}^*$.

Language equations—results

- Language equations over Ω, with $|\Omega| \geqslant 2$.

Language equations—results

- Language equations over Ω, with $|\Omega| \geqslant 2$.

Theorem (Okhotin, ICALP 2003)

$L \subseteq \Omega^*$ is given by unique (least, greatest) solution of a system with $\{\cup, \cap, \sim, \cdot\}$ and equations of the form $\varphi(X_1, \ldots, X_n) = \psi(X_1, \ldots, X_n)$

if and only if

L is recursive (r.e., co-r.e.)

Language equations—results

- Language equations over Ω, with $|\Omega| \geqslant 2$.

Theorem (Okhotin, ICALP 2003)

$L \subseteq \Omega^*$ is given by unique (least, greatest) solution of a system with $\{\cup, \cap, \sim, \cdot\}$ and equations of the form $\varphi(X_1, \ldots, X_n) = \psi(X_1, \ldots, X_n)$

if and only if

L is recursive (r.e., co-r.e.)

Theorem (Kunc, STACS 2005)

There exists a finite L such that the greatest solution of

$$LX = XL$$

for $X \subseteq \{a, b\}^*$ is co-r.e.-hard.

Simple case and equations over sets of numbers

- simple case: $\Omega = \{a\}$.

Simple case and equations over sets of numbers

- simple case: $\Omega = \{a\}$.
- $\{\cup, \cdot\}$: regular

Simple case and equations over sets of numbers

- simple case: $\Omega = \{a\}$.
- $\{\cup, \cdot\}$: regular
- $\{\cdot, {}^c\}$: non-regular [Leiss 1994]

Simple case and equations over sets of numbers

- simple case: $\Omega = \{a\}$.
- $\{\cup, \cdot\}$: regular
- $\{\cdot, {}^c\}$: non-regular [Leiss 1994]
- $\{\cup, \cap, \cdot\}$: ?

Simple case and equations over sets of numbers

- simple case: $\Omega = \{a\}$.
- $\{\cup, \cdot\}$: regular
- $\{\cdot, {}^c\}$: non-regular [Leiss 1994]
- $\{\cup, \cap, \cdot\}$: ?

- only length matters: $a^n \longleftrightarrow$ number n

Simple case and equations over sets of numbers

- simple case: $\mathbb{N}$
- $\{\cup, \cdot\}$: periodic
- $\{\cdot, {}^c\}$: non-periodic [Leiss 1994]
- $\{\cup, \cap, \cdot\}$: ?

$$\begin{cases} X_1 = \varphi_1(X_1, \dots, X_n) \\ \vdots \\ X_n = \varphi_n(X_1, \dots, X_n) \end{cases}$$

- X_i: subset of $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.
- φ_i: variables, singleton constants, operations on sets

$$X + Y = \{x + y \mid x \in X, y \in Y\}$$

Using positional notation

- Numbers in base-k notation: strings over $\Omega_k = \{0, 1, \ldots, k-1\}$.

Using positional notation

- Numbers in base-k notation: strings over $\Omega_k = \{0, 1, \dots, k-1\}$.
- $(a_\ell \dots a_0)_k$: number denoted by $a_\ell \dots a_0$ in base-k notation

Using positional notation

- Numbers in base-k notation: strings over $\Omega_k = \{0, 1, \ldots, k-1\}$.
- $(a_\ell \ldots a_0)_k$: number denoted by $a_\ell \ldots a_0$ in base-k notation
- Set of numbers $\leftrightarrow$ formal language over Ω_k

Using positional notation

- Numbers in base-k notation: strings over $\Omega_k = \{0, 1, \ldots, k-1\}$.
- $(a_\ell \ldots a_0)_k$: number denoted by $a_\ell \ldots a_0$ in base-k notation
- Set of numbers $\leftrightarrow$ formal language over Ω_k

Example (Jeż, DLT 2007)

$$X_1 = (X_2 + X_2 \cap X_1 + X_3) \cup \{1\}$$
$$X_2 = (X_{12} + X_2 \cap X_1 + X_1) \cup \{2\}$$
$$X_3 = (X_{12} + X_{12} \cap X_1 + X_2) \cup \{3\}$$
$$X_{12} = X_3 + X_3 \cap X_1 + X_2$$

Using positional notation

- Numbers in base-k notation: strings over $\Omega_k = \{0, 1, \ldots, k-1\}$.
- $(a_\ell \ldots a_0)_k$: number denoted by $a_\ell \ldots a_0$ in base-k notation
- Set of numbers $\leftrightarrow$ formal language over Ω_k

Example (Jeż, DLT 2007)

$$X_1 = (X_2 + X_2 \cap X_1 + X_3) \cup \{1\}$$
$$X_2 = (X_{12} + X_2 \cap X_1 + X_1) \cup \{2\}$$
$$X_3 = (X_{12} + X_{12} \cap X_1 + X_2) \cup \{3\}$$
$$X_{12} = X_3 + X_3 \cap X_1 + X_2$$

$$((10^*)_4, (20^*)_4, (30^*)_4, (120^*)_4)$$

Using positional notation

- Numbers in base-k notation: strings over $\Omega_k = \{0, 1, \ldots, k-1\}$.
- $(a_\ell \ldots a_0)_k$: number denoted by $a_\ell \ldots a_0$ in base-k notation
- Set of numbers $\leftrightarrow$ formal language over Ω_k

Example (Jeż, DLT 2007)

$$\begin{aligned} X_1 &= (X_2 + X_2 \cap X_1 + X_3) \cup \{1\} \\ X_2 &= (X_{12} + X_2 \cap X_1 + X_1) \cup \{2\} \\ X_3 &= (X_{12} + X_{12} \cap X_1 + X_2) \cup \{3\} \\ X_{12} &= X_3 + X_3 \cap X_1 + X_2 \end{aligned}$$

$$((10^*)_4, (20^*)_4, (30^*)_4, (120^*)_4)$$

- $X_2 + X_2 = (20^*)_4 + (20^*)_4 =$

Using positional notation

- Numbers in base-k notation: strings over $\Omega_k = \{0, 1, \dots, k-1\}$.
- $(a_\ell \dots a_0)_k$: number denoted by $a_\ell \dots a_0$ in base-k notation
- Set of numbers $\leftrightarrow$ formal language over Ω_k

Example (Jeż, DLT 2007)

$$X_1 = (X_2 + X_2 \cap X_1 + X_3) \cup \{1\}$$
$$X_2 = (X_{12} + X_2 \cap X_1 + X_1) \cup \{2\}$$
$$X_3 = (X_{12} + X_{12} \cap X_1 + X_2) \cup \{3\}$$
$$X_{12} = X_3 + X_3 \cap X_1 + X_2$$

$$((10^*)_4, (20^*)_4, (30^*)_4, (120^*)_4)$$

- $X_2 + X_2 = (20^*)_4 + (20^*)_4 = (10^+)_4 \cup$

Using positional notation

- Numbers in base-k notation: strings over $\Omega_k = \{0, 1, \ldots, k-1\}$.
- $(a_\ell \ldots a_0)_k$: number denoted by $a_\ell \ldots a_0$ in base-k notation
- Set of numbers $\leftrightarrow$ formal language over Ω_k

Example (Jeż, DLT 2007)

$$X_1 = (X_2 + X_2 \cap X_1 + X_3) \cup \{1\}$$
$$X_2 = (X_{12} + X_2 \cap X_1 + X_1) \cup \{2\}$$
$$X_3 = (X_{12} + X_{12} \cap X_1 + X_2) \cup \{3\}$$
$$X_{12} = X_3 + X_3 \cap X_1 + X_2$$

$$((10^*)_4, (20^*)_4, (30^*)_4, (120^*)_4)$$

- $X_2 + X_2 = (20^*)_4 + (20^*)_4 = (10^+)_4 \cup (20^*20^*)_4$

Using positional notation

- Numbers in base-k notation: strings over $\Omega_k = \{0, 1, \dots, k-1\}$.
- $(a_\ell \dots a_0)_k$: number denoted by $a_\ell \dots a_0$ in base-k notation
- Set of numbers $\leftrightarrow$ formal language over Ω_k

Example (Jeż, DLT 2007)

$$X_1 = (X_2 + X_2 \cap X_1 + X_3) \cup \{1\}$$
$$X_2 = (X_{12} + X_2 \cap X_1 + X_1) \cup \{2\}$$
$$X_3 = (X_{12} + X_{12} \cap X_1 + X_2) \cup \{3\}$$
$$X_{12} = X_3 + X_3 \cap X_1 + X_2$$

$$((10^*)_4, (20^*)_4, (30^*)_4, (120^*)_4)$$

- $X_2 + X_2 = (20^*)_4 + (20^*)_4 = (10^+)_4 \cup (20^*20^*)_4$
- $X_1 + X_3 = (10^*)_4 + (30^*)_4 =$

Using positional notation

- Numbers in base-k notation: strings over $\Omega_k = \{0, 1, \dots, k-1\}$.
- $(a_\ell \dots a_0)_k$: number denoted by $a_\ell \dots a_0$ in base-k notation
- Set of numbers $\leftrightarrow$ formal language over Ω_k

Example (Jeż, DLT 2007)

$$X_1 = (X_2 + X_2 \cap X_1 + X_3) \cup \{1\}$$
$$X_2 = (X_{12} + X_2 \cap X_1 + X_1) \cup \{2\}$$
$$X_3 = (X_{12} + X_{12} \cap X_1 + X_2) \cup \{3\}$$
$$X_{12} = X_3 + X_3 \cap X_1 + X_2$$

$$((10^*)_4, (20^*)_4, (30^*)_4, (120^*)_4)$$

- $X_2 + X_2 = (20^*)_4 + (20^*)_4 = (10^+)_4 \cup (20^*20^*)_4$
- $X_1 + X_3 = (10^*)_4 + (30^*)_4 = (10^+)_4 \cup$

Using positional notation

- Numbers in base-k notation: strings over $\Omega_k = \{0, 1, \ldots, k-1\}$.
- $(a_\ell \ldots a_0)_k$: number denoted by $a_\ell \ldots a_0$ in base-k notation
- Set of numbers $\leftrightarrow$ formal language over Ω_k

Example (Jeż, DLT 2007)

$$X_1 = (X_2 + X_2 \cap X_1 + X_3) \cup \{1\}$$
$$X_2 = (X_{12} + X_2 \cap X_1 + X_1) \cup \{2\}$$
$$X_3 = (X_{12} + X_{12} \cap X_1 + X_2) \cup \{3\}$$
$$X_{12} = X_3 + X_3 \cap X_1 + X_2$$

$$((10^*)_4, (20^*)_4, (30^*)_4, (120^*)_4)$$

- $X_2 + X_2 = (20^*)_4 + (20^*)_4 = (10^+)_4 \cup (20^*20^*)_4$
- $X_1 + X_3 = (10^*)_4 + (30^*)_4 = (10^+)_4 \cup (10^*30^*)_4 \cup (30^*10^*)_4$

Using positional notation

- Numbers in base-k notation: strings over $\Omega_k = \{0, 1, \ldots, k-1\}$.
- $(a_\ell \ldots a_0)_k$: number denoted by $a_\ell \ldots a_0$ in base-k notation
- Set of numbers $\leftrightarrow$ formal language over Ω_k

Example (Jeż, DLT 2007)

$$X_1 = (X_2 + X_2 \cap X_1 + X_3) \cup \{1\}$$
$$X_2 = (X_{12} + X_2 \cap X_1 + X_1) \cup \{2\}$$
$$X_3 = (X_{12} + X_{12} \cap X_1 + X_2) \cup \{3\}$$
$$X_{12} = X_3 + X_3 \cap X_1 + X_2$$

$$((10^*)_4, (20^*)_4, (30^*)_4, (120^*)_4)$$

- $X_2 + X_2 = (20^*)_4 + (20^*)_4 = (10^+)_4 \cup (20^*20^*)_4$
- $X_1 + X_3 = (10^*)_4 + (30^*)_4 = (10^+)_4 \cup (10^*30^*)_4 \cup (30^*10^*)_4$
- $(X_2 + X_2) \cap (X_1 + X_3) = (10^+)_4$

Equations over sets of natural numbers—results

Theorem (Jeż, Okhotin, CSR 2007)

Let M be a one-way real-time cellular automaton. Then $(L(M))_k$ is the unique solution of a resolved system of equations over $\mathbb{N}$ with $\{\cup, \cap, +\}$:

$$\begin{cases} X_1 &= \varphi_1(X_1, \ldots, X_n) \\ &\vdots \\ X_n &= \varphi_n(X_1, \ldots, X_n) \end{cases}$$

Equations over sets of natural numbers—results

Theorem (Jeż, Okhotin, CSR 2007)

Let M be a one-way real-time cellular automaton. Then $(L(M))_k$ is the unique solution of a resolved system of equations over $\mathbb{N}$ with $\{\cup, \cap, +\}$:

$$\begin{cases} X_1 = \varphi_1(X_1, \ldots, X_n) \\ \vdots \\ X_n = \varphi_n(X_1, \ldots, X_n) \end{cases}$$

Theorem (Jeż, Okhotin, ICALP 2008)

$S \subseteq \mathbb{N}$ is given by unique (least, greatest) solution of a system with $\{\cup, +\}$ and equations of the form

$$\varphi(X_1, \ldots, X_n) = \psi(X_1, \ldots, X_n)$$

if and only if S is recursive (r.e., co-r.e.).

Obvious upper bound for greatest solutions

Obvious upper bound for greatest solutions

Set equation translates into formulas:

$$X_i = X_j + X_k \iff (\forall n)\big[n \in X_i \leftrightarrow (\exists n', n'')n = n' + n'' \wedge n' \in X_j \wedge n'' \in X_k\big]$$

system is turned into arithmetical formula $Eq(X_1, \dots, X_n)$

Obvious upper bound for greatest solutions

Set equation translates into formulas:

$$X_i = X_j + X_k \iff (\forall n)\big[n \in X_i \leftrightarrow (\exists n', n'')n = n' + n'' \wedge n' \in X_j \wedge n'' \in X_k\big]$$

system is turned into arithmetical formula $Eq(X_1, \ldots, X_n)$

Solution $(S_1, \ldots, S_n)$:

- greatest: $\varphi(x) = (\exists X_1) \ldots (\exists X_n) Eq(X_1, \ldots, X_n) \wedge x \in X_1 \quad (\Sigma^1_1)$
- least: $\varphi'(x) = (\forall X_1) \ldots (\forall X_n) Eq(X_1, \ldots, X_n) \rightarrow x \in X_1 \quad (\Pi^1_1)$
- unique: $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$

Obvious upper bound for greatest solutions

Set equation translates into formulas:

$$X_i = X_j + X_k \iff (\forall n)\big[n \in X_i \leftrightarrow (\exists n', n'')n = n' + n'' \land n' \in X_j \land n'' \in X_k\big]$$

system is turned into arithmetical formula $Eq(X_1, \dots, X_n)$

Solution $(S_1, \dots, S_n)$:

- greatest: $\varphi(x) = (\exists X_1) \dots (\exists X_n) Eq(X_1, \dots, X_n) \land x \in X_1 \quad (\Sigma^1_1)$
- least: $\varphi'(x) = (\forall X_1) \dots (\forall X_n) Eq(X_1, \dots, X_n) \to x \in X_1 \quad (\Pi^1_1)$
- unique: $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$

Theorem (Jeż, Okhotin, STACS 2010)

$S \subseteq \mathbb{Z}$ is given by unique solution of a system with $\{\cup, +\}$
if and only if
S is a Δ^1_1-set.

New results

$$\begin{cases} X_1 = \varphi_1(X_1, \ldots, X_n) \\ \vdots \\ X_n = \varphi_n(X_1, \ldots, X_n) \end{cases} \quad (*)$$

Theorem

$S \subseteq \mathbb{Z}$ is given by the greatest solution of a system (*) with $\{\cup, \cap, +\}$
if and only if
S is a Σ^1_1-set.

Theorem

$S \subseteq \mathbb{Z}$ is given by the least solution of a system (*) with $\{\cup, \cap, +\}$
if and only if
S is a r.e.-set.

Least solutions are r.e. sets

Definition (fixpoint iteration)

- $S^0 = (\emptyset, \dots, \emptyset)$
- $S^{n+1} = \varphi(S^n)$
- $S^\omega = \bigcup_{i \geqslant 0} S^i$

Least solutions are r.e. sets

Definition (fixpoint iteration)

- $S^0 = (\emptyset, \dots, \emptyset)$
- $S^{n+1} = \varphi(S^n)$
- $S^\omega = \bigcup_{i \geqslant 0} S^i$

Definition ($\cup$-continuous operations)

An operation φ is $\cup$-continuous,
if for every ascending sequence of sets

$$\bigcup_{i \geqslant 0} \varphi(A_n) = \varphi\Big(\bigcup_{i \geqslant 0} A_n\Big)$$

Theorem (folklore)

For $\cup$-continuous and monotone φ, the S^ω is the least fixpoint of φ.

Least solutions are r.e. sets

Definition (fixpoint iteration)

- $S^0 = (\emptyset, \dots, \emptyset)$
- $S^{n+1} = \varphi(S^n)$
- $S^\omega = \bigcup_{i \geqslant 0} S^i$

Definition ($\cup$-continuous operations)

An operation φ is $\cup$-continuous,
if for every ascending sequence of sets

$$\bigcup_{i \geqslant 0} \varphi(A_n) = \varphi\Big(\bigcup_{i \geqslant 0} A_n\Big)$$

Theorem (folklore)

For $\cup$-continuous and monotone φ, the S^ω is the least fixpoint of φ.

Least solutions are r.e. sets

Definition (fixpoint iteration)

- $S^0 = (\emptyset, \dots, \emptyset)$
- $S^{n+1} = \varphi(S^n)$
- $S^\omega = \bigcup_{i \geqslant 0} S^i$

Definition ($\cup$-continuous operations)

An operation φ is $\cup$-continuous,
if for every ascending sequence of sets

$$\bigcup_{i \geqslant 0} \varphi(A_n) = \varphi\Big(\bigcup_{i \geqslant 0} A_n\Big)$$

Theorem (folklore)

For $\cup$-continuous and monotone φ, the S^ω is the least fixpoint of φ.

- $\cup$, $\cap$, $+$ for sets of integers are $\cup$-continuous.

Algorithm for membership in the least solution

Construct S_i for consecutive i and check the membership for them.

Constructing r.e. sets

- Turing machine (Turing, 1936)

Constructing r.e. sets

- Turing machine (Turing, 1936)
- VALC(T): intersection of two LinCFLs (Hartmanis, 1967; Baker, Book, 1978)

$$\mathrm{VALC}(T) = \{C_T(w)1w \mid w \in L(T)\}$$
$$C_T(w) \in \{3,6\}^*$$

Constructing r.e. sets

- Turing machine (Turing, 1936)
- VALC(T): intersection of two LinCFLs (Hartmanis, 1967; Baker, Book, 1978)

$$\mathrm{VALC}(T) = \{C_T(w)1w \mid w \in L(T)\}$$
$$C_T(w) \in \{3,6\}^*$$

- LinCFLs $\to$ resolved equations over sets of numbers (Jeż, Okhotin, CSR 2007)

Constructing r.e. sets

- Turing machine (Turing, 1936)
- VALC(T): intersection of two LinCFLs (Hartmanis, 1967; Baker, Book, 1978)

$$\mathrm{VALC}(T) = \{C_T(w)1w \mid w \in L(T)\}$$
$$C_T(w) \in \{3,6\}^*$$

- LinCFLs $\to$ resolved equations over sets of numbers (Jeż, Okhotin, CSR 2007)
- deleting $C_T(w)$ to obtain $\{(1w)_7 \mid w \in L(T)\}$:

$$E(S) = \{(1w)_7 \mid \exists x \in \{3,6\}^* : (x1w)_7 \in S\}$$

(Jeż, Okhotin, STACS 2010)

Constructing r.e. sets

- Turing machine (Turing, 1936)
- VALC(T): intersection of two LinCFLs (Hartmanis, 1967; Baker, Book, 1978)

$$\mathrm{VALC}(T) = \{C_T(w)1w \mid w \in L(T)\}$$
$$C_T(w) \in \{3,6\}^*$$

- LinCFLs $\to$ resolved equations over sets of numbers (Jeż, Okhotin, CSR 2007)
- deleting $C_T(w)$ to obtain $\{(1w)_7 \mid w \in L(T)\}$:

$$E(S) = \{(1w)_7 \mid \exists x \in \{3,6\}^* : (x1w)_7 \in S\}$$

(Jeż, Okhotin, STACS 2010)

- deleting leading 1

Existential quantifier

Theorem (Jeż, Okhotin, STACS 2010)

For $S \subseteq (\{3,6\}^+ 1 \Omega_7^*)_7$

$$\left[S - (\{3,6\}^+ 0^*)_7\right] \cap (1\Omega_7^*)_7 = \{(1y)_7 \mid \exists x \in \{3,6\}^+ (x1y)_7 \in S\}$$

Existential quantifier

Theorem (Jeż, Okhotin, STACS 2010)

For $S \subseteq (\{3,6\}^+ 1 \Omega_7^*)_7$

$$\left[S - (\{3,6\}^+ 0^*)_7\right] \cap (1\Omega_7^*)_7 = \{(1y)_7 \mid \exists x \in \{3,6\}^+ (x1y)_7 \in S\}$$

- Goal:

$$\begin{array}{rccccccccc} & x_1 & x_2 & \dots & x_\ell & 1 & w_1 & w_2 & \dots & w_{\ell'} \\ - & x_1 & x_2 & \dots & x_\ell & 0 & 0 & 0 & \dots & 0 \\ \hline & & & & & 1 & w_1 & w_2 & \dots & w_{\ell'} \end{array}$$

Existential quantifier

Theorem (Jeż, Okhotin, STACS 2010)

For $S \subseteq (\{3,6\}^+ 1 \Omega_7^*)_7$

$$\left[S - (\{3,6\}^+ 0^*)_7\right] \cap (1\Omega_7^*)_7 = \{(1y)_7 \mid \exists x \in \{3,6\}^+ (x1y)_7 \in S\}$$

- Goal:

$$\begin{array}{rcccccccccc} & x_1 & x_2 & \dots & x_\ell & 1 & w_1 & w_2 & \dots & w_{\ell'} \\ - & x_1 & x_2 & \dots & x_\ell & 0 & 0 & 0 & \dots & 0 \\ \hline & & & & & 1 & w_1 & w_2 & \dots & w_{\ell'} \end{array}$$

- Reality: $\big((x1y)_7 - (x'0\overbrace{0\dots0}^{?})_7\big) \cap (1\Omega_7^*)_7$

Existential quantifier

Theorem (Jeż, Okhotin, STACS 2010)

For $S \subseteq (\{3,6\}^+ 1\Omega_7^*)_7$

$$\left[S - (\{3,6\}^+ 0^*)_7\right] \cap (1\Omega_7^*)_7 = \{(1y)_7 \mid \exists x \in \{3,6\}^+ (x1y)_7 \in S\}$$

- Goal:

$$\begin{array}{ccccccccc} & x_1 & x_2 & \dots & x_\ell & 1 & w_1 & w_2 & \dots & w_{\ell'} \\ - & x_1 & x_2 & \dots & x_\ell & 0 & 0 & 0 & \dots & 0 \\ \hline & & & & & 1 & w_1 & w_2 & \dots & w_{\ell'} \end{array}$$

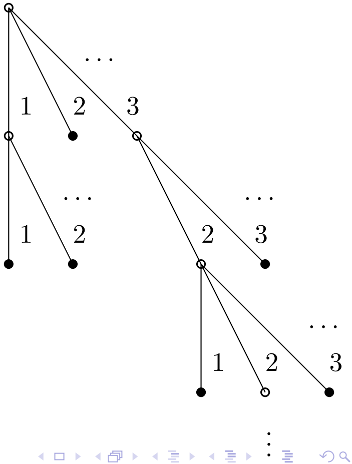
- Reality: $\big((x1y)_7 - (x'0\overbrace{0\dots0}^{?})_7\big) \cap (1\Omega_7^*)_7$
- example of a bad case

$$\begin{array}{cccccccccc} & x_1 & x_2 & \dots & 6 & x_\ell & 1 & w_1 & w_2 & \dots & w_{\ell'} \\ - & x_1 & x_2 & \dots & 3 & x_\ell & 0 & 0 & 0 & \dots & 0 \\ \hline & & & & 3 & 0 & 1 & w_1 & w_2 & \dots & w_{\ell'} \end{array}$$

Σ_1^1-hard problem for trees

Basic Σ_1^1-hard problem

Does a tree of a countable degree (given by a TM recognizing its prefixes) has an infinite path?

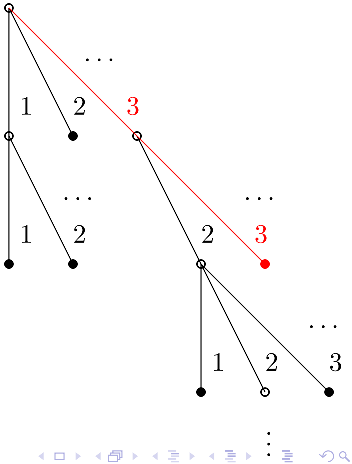


Σ_1^1-hard problem for trees

Basic Σ_1^1-hard problem

Does a tree of a countable degree (given by a TM recognizing its prefixes) has an infinite path?

- sequences of numbers

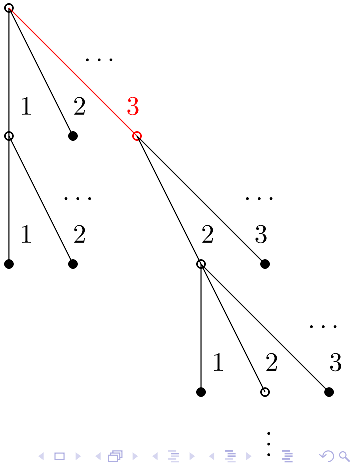


Σ_1^1-hard problem for trees

Basic Σ_1^1-hard problem

Does a tree of a countable degree (given by a TM recognizing its prefixes) has an infinite path?

- sequences of numbers
- closed under taking prefixes

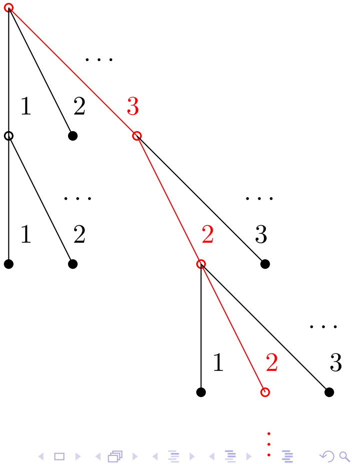


Σ_1^1-hard problem for trees

Basic Σ_1^1-hard problem

Does a tree of a countable degree (given by a TM recognizing its prefixes) has an infinite path?

- sequences of numbers
- closed under taking prefixes
- infinite path: infinite sequence,

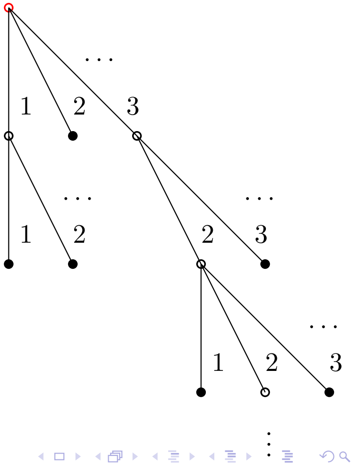


Σ_1^1-hard problem for trees

Basic Σ_1^1-hard problem

Does a tree of a countable degree (given by a TM recognizing its prefixes) has an infinite path?

- sequences of numbers
- closed under taking prefixes
- infinite path: infinite sequence, s.t. all finite prefixes are in the tree

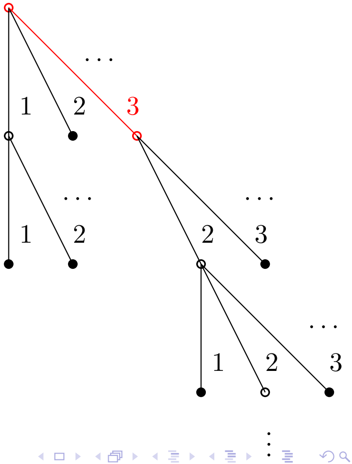


Σ^1_1-hard problem for trees

Basic Σ^1_1-hard problem

Does a tree of a countable degree (given by a TM recognizing its prefixes) has an infinite path?

- sequences of numbers
- closed under taking prefixes
- infinite path: infinite sequence, s.t. all finite prefixes are in the tree

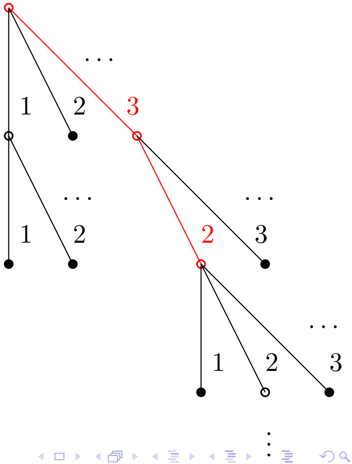


Σ^1_1-hard problem for trees

Basic Σ^1_1-hard problem

Does a tree of a countable degree (given by a TM recognizing its prefixes) has an infinite path?

- sequences of numbers
- closed under taking prefixes
- infinite path: infinite sequence, s.t. all finite prefixes are in the tree

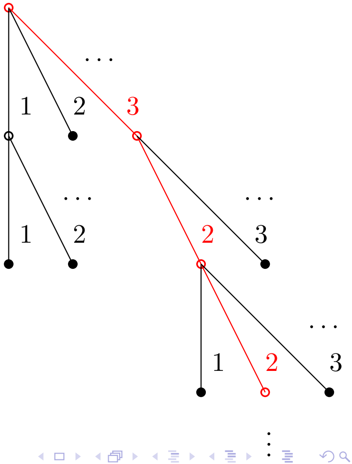


Σ_1^1-hard problem for trees

Basic Σ_1^1-hard problem

Does a tree of a countable degree (given by a TM recognizing its prefixes) has an infinite path?

- sequences of numbers
- closed under taking prefixes
- infinite path: infinite sequence, s.t. all finite prefixes are in the tree

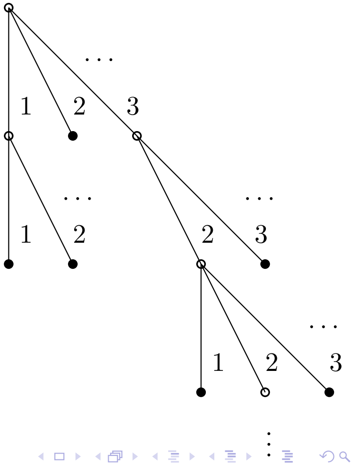


Σ_1^1-hard problem for trees

Basic Σ_1^1-hard problem

Does a tree of a countable degree (given by a TM recognizing its prefixes) has an infinite path?

- sequences of numbers
- closed under taking prefixes
- infinite path: infinite sequence, s.t. all finite prefixes are in the tree



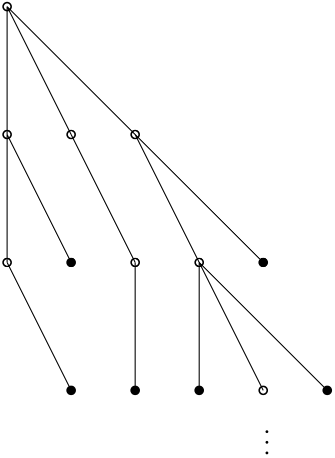
Encoding a tree as a set of integers

Sequence $(n_1, \ldots, n_k)$ becomes $(1x_k1x_{k-1}1\ldots1x_11)_7$, where $x_i \in \{3,6\}^+$ represents n_i in binary.

Basic idea

Idea

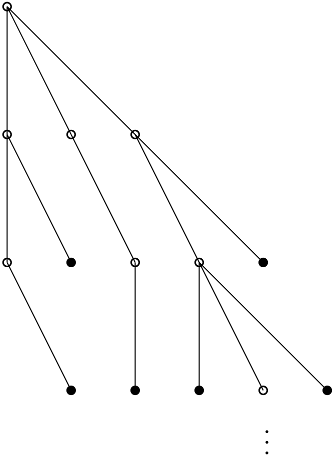
- operator which
  - preserves infinite paths
  - modifies finite paths



Basic idea

Idea

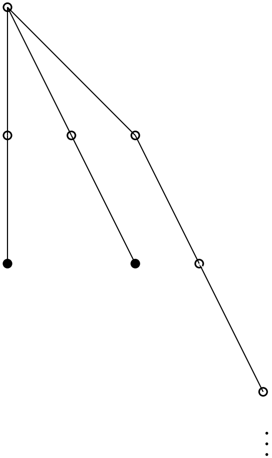
- operator which
  - preserves infinite paths
  - modifies finite paths
- cut leaves



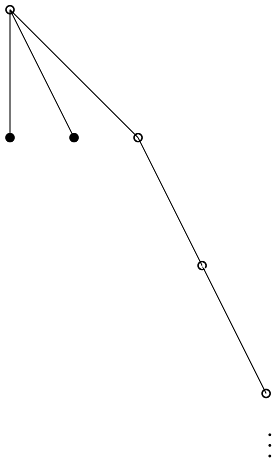
Basic idea

Idea

- operator which
  - preserves infinite paths
  - modifies finite paths
- cut leaves



Basic idea



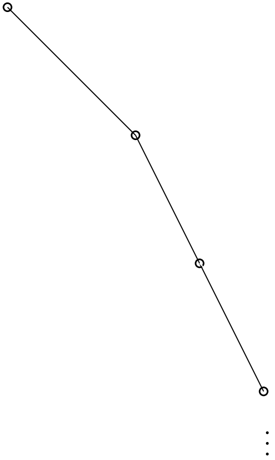
Idea

- operator which
  - preserves infinite paths
  - modifies finite paths
- cut leaves

Basic idea

Idea

- operator which
  - preserves infinite paths
  - modifies finite paths
- cut leaves
- each finite path disappears
- infinite paths survive



Formalisation

How to cut leaves?

$\varphi(S) = \{(1w)_7 \mid \exists x \in \{3, 6\}^* \ (1x1w)_7 \in S\}$
The same expression as applied previously to VALC.

Formalisation

How to cut leaves?

$\varphi(S) = \{(1w)_7 \mid \exists x \in \{3,6\}^* \ (1x1w)_7 \in S\}$
The same expression as applied previously to VALC.

$$X = \mathit{Tree} \cap \varphi(X)$$

By the Tarski's Fixpoint Theorem the greatest solution exists

Formalisation

How to cut leaves?

$\varphi(S) = \{(1w)_7 \mid \exists x \in \{3,6\}^* \ (1x1w)_7 \in S\}$
The same expression as applied previously to VALC.

$$X = \mathit{Tree} \cap \varphi(X)$$

By the Tarski's Fixpoint Theorem the greatest solution exists

- there is a constructive proof

- $T^0 = (\mathbb{Z}, \dots, \mathbb{Z})$
- $T^\alpha = \varphi(T^{\alpha-1})$, α: succesor ordinal
- $T^\alpha = \bigsqcap_{\gamma<\alpha} T^\gamma$, α: limit ordinal

Formalisation

How to cut leaves?

$\varphi(S) = \{(1w)_7 \mid \exists x \in \{3,6\}^* \, (1x1w)_7 \in S\}$
The same expression as applied previously to VALC.

$$X = \mathit{Tree} \cap \varphi(X)$$

By the Tarski's Fixpoint Theorem the greatest solution exists

- there is a constructive proof
- allows induction

- $T^0 = (\mathbb{Z}, \dots, \mathbb{Z})$
- $T^\alpha = \varphi(T^{\alpha-1})$, α: succesor ordinal
- $T^\alpha = \bigsqcap_{\gamma<\alpha} T^\gamma$, α: limit ordinal

Formalisation

How to cut leaves?

$\varphi(S) = \{(1w)_7 \mid \exists x \in \{3,6\}^* \; (1x1w)_7 \in S\}$
The same expression as applied previously to VALC.

$$X = \mathit{Tree} \cap \varphi(X)$$

By the Tarski's Fixpoint Theorem the greatest solution exists

- there is a constructive proof
- allows induction
- generalised height

- $T^0 = (\mathbb{Z}, \dots, \mathbb{Z})$
- $T^\alpha = \varphi(T^{\alpha-1})$, α: succesor ordinal
- $T^\alpha = \bigsqcap_{\gamma<\alpha} T^\gamma$, α: limit ordinal

Formalisation

How to cut leaves?

$\varphi(S) = \{(1w)_7 \mid \exists x \in \{3,6\}^* \ (1x1w)_7 \in S\}$
The same expression as applied previously to VALC.

$$X = \mathit{Tree} \cap \varphi(X)$$

By the Tarski's Fixpoint Theorem the greatest solution exists

- there is a constructive proof
- allows induction
- generalised height

- $T^0 = (\mathbb{Z}, \dots, \mathbb{Z})$
- $T^\alpha = \varphi(T^{\alpha-1})$, α: succesor ordinal
- $T^\alpha = \prod_{\gamma<\alpha} T^\gamma$, α: limit ordinal

Correctness of the construction

The greatest solution is non-empty iff the tree has an infinite path.

Can be improved to represent all Σ^1_1-sets.