Exercise 1: Error Bounds for Interval Arithmetic (4 points)

Let \( f \in \mathbb{C}[x] \) be a polynomial of degree \( d \) with coefficients of absolute value less than \( 2^L \), and let \( \mathbb{F} \) be the set of fixed point numbers with some precision \( \rho > \log d \). Suppose that \( R = [a, b] + i \cdot [c, d] \) is a rectangle of width \( w(R) < \frac{1}{3} \) with vertices in \( \mathbb{F} + i \cdot \mathbb{F} \) and that we compute \( \Box f \) (and \( \tilde{\Box} f \)) using Horner Evaluation and (fixed point) interval arithmetic (with precision \( \rho \)). Then, show that the following holds:

\[
\frac{1}{2} w(\Box f(R)) \leq w(\tilde{\Box} f(R)) < 8 \cdot (d + 1)^2 \cdot 2^L \cdot \text{max}(1, |m_R|)^d \cdot w(R),
\]

where \( m_R \) denotes the center of \( R \).

*Hint: Consider a similar argument as in the proof of Theorem 1.3.3.*

Exercise 2: Fast Division (4 points + 4 bonus points for *)

For a given \( b \in \mathbb{N} \), we recursively define \( x_0 := 2 - \lceil \log b \rceil \) and

\[
x_{i+1} := x_i \cdot (2 - b \cdot x_i) \quad \text{for } i \in \mathbb{N}_{\geq 1}.
\]

Show that, for all \( i \), it holds that:

(a) \( |x_{i+1} - \frac{1}{b}| \leq b \cdot |x_i - \frac{1}{b}|^2 \) and

(b) \( |x_i - \frac{1}{b}| < \frac{1}{b} \cdot 2^{-2^i} \). In particular, it holds that \( |x_i - \frac{1}{b}| < 2^{-L} \) for all \( i \geq \log L \).

(c) * Suppose now that we start with \( y_0 := x_1 = 2^{-\lceil \log b \rceil} \cdot (2 - b \cdot 2^{-\lceil \log b \rceil}) \) and define

\[
y_{i+1} := \text{fl}(y_i \cdot (2 - b \cdot y_i)) \quad \text{for } i \in \mathbb{N}_{\geq 1},
\]

where we consider rounding to the nearest fixed-point number of precision \( \rho_i := 2^{i+1} + 2\lceil \log(b + 1) \rceil \). Then, it holds \( |y_i - \frac{1}{b}| < \frac{1}{b+1} \cdot 2^{-2^i} \) for all \( i \).

*Hint: For (c), use that the error \( 2^{-\rho_i-1} \) that is induced by the rounding in the \((i + 1)\)-st iteration is smaller than \( \frac{2^{-2^i+1}}{(b+1)^2} \). Then, use induction on \( i \) to prove the claim.*
Exercise 3: Computing π (4 points)

For arbitrary $x \in \mathbb{R}$ with $0 \leq x \leq 1$, it holds that

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad (1)$$

Now, for given $L \in \mathbb{N}$, use the above formula and the fact (due to Euler) that

$$\pi = 20 \cdot \arctan(1/7) + 8 \cdot \arctan(3/79)$$

to derive an efficient algorithm (i.e. with a running time polynomial in $L$) for computing a dyadic approximation $\tilde{\pi}$ of $\pi$ to an error less than $2^{-L}$.

*Hint: Estimate the error when considering only the first $k$ summands in (1). Then, proceed with a suitably truncated series.*

Exercise 4: Box Functions for Analytic Functions (4 points)

For any $x \in \mathbb{R}$ with $0 \leq x \leq 1$ and any $k$, there exists a $\xi \in [0, x]$ such that

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{x^{4k}}{(4k)!} \cdot \cos(\xi) \quad \text{(Taylor Series Expansion with Remainder Term)}$$

Use the above formula to derive a box function $\square \cos$ for $\cos$ for intervals $[a, b] \subset [0, 1]$! Can you extend your approach to derive a box function for $\sin x$ and $e^x$. 
