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Computer Algebra

https://www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/winter17/comp-alg/

Assignment sheet 2 due: Monday, November 6

Exercise 1: Error Bounds for Interval Arithmetic (4 points)

Let $f \in \mathbb{C}[x]$ be a polynomial of degree d with coefficients of absolute value less than 2^L, and let $\mathbb{F}$ be the set of fixed point numbers with some precision $\rho > \log d$. Suppose that $R = [a,b] + \mathbf{i} \cdot [c,d]$ is a rectangle of width $w(R) < \frac{1}{d}$ with vertices in $\mathbb{F} + \mathbf{i} \cdot \mathbb{F}$ and that we compute $\square f$ (and $\tilde{\square} f$) using Horner Evaluation and (fixed point) interval arithmetic (with precision ρ). Then, show that the following holds:

$$w(\square f(R)) \leq w(\tilde{\square} f(R)) < 8 \cdot (d+1)^2 \cdot 2^L \cdot \max(1, |m_R|)^d \cdot w(R),$$

where m_R denotes the center of R.

Hint: Consider a similar argument as in the proof of Theorem 1.3.3.

Exercise 2: Fast Division (4 points + 4 bonus points for $\star$)

For a given $b \in \mathbb{N}$, we recursively define $x_0 := 2^{-\lceil \log b \rceil}$ and

$$x_{i+1} := x_i \cdot (2 - b \cdot x_i) \quad \text{for } i \in \mathbb{N}_{\geq 1}.$$

Show that, for all i, it holds that:

(a) $\left|x_{i+1} - \frac{1}{b}\right| \leq b \cdot \left|x_i - \frac{1}{b}\right|^2$ and

(b) $\left|x_i - \frac{1}{b}\right| < \frac{1}{b} \cdot 2^{-2^i}$. In particular, it holds that $\left|x_i - \frac{1}{b}\right| < 2^{-L}$ for all $i \geq \log L$.

(c) $\star$ Suppose now that we start with $y_0 := x_1 = 2^{-\lceil \log b \rceil} \cdot (2 - b \cdot 2^{-\lceil \log b \rceil})$ and define

$$y_{i+1} := \mathrm{fl}(y_i \cdot (2 - b \cdot y_i)) \quad \text{for } i \in \mathbb{N}_{\geq 1},$$

where we consider rounding to the nearest fixed-point number of precision $\rho_i := 2^{i+1} + 2\lceil \log(b+1) \rceil$. Then, it holds $\left|y_i - \frac{1}{b}\right| < \frac{1}{b+1} \cdot 2^{-2^i}$ for all i.

Hint: For (c), use that the error $2^{-\rho_i - 1}$ that is induced by the rounding in the $(i+1)$-st iteration is smaller than $\frac{2^{-2^{i+1}}}{(b+1)^2}$. Then, use induction on i to prove the claim.

Exercise 3: Computing π (4 points)

For arbitrary $x \in \mathbb{R}$ with $0 \leq x \leq 1$, it holds that

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \tag{1}$$

Now, for given $L \in \mathbb{N}$, use the above formula and the fact (due to Euler) that

$$\pi = 20 \cdot \arctan(1/7) + 8 \cdot \arctan(3/79)$$

to derive an efficient algorithm (i.e. with a running time polynomial in L) for computing a dyadic approximation $\tilde{\pi}$ of π to an error less than 2^{-L}.

Hint: Estimate the error when considering only the first k summands in (1). Then, proceed with a suitably truncated series.

Exercise 4: Box Functions for Analytic Functions (4 points)

For any $x \in \mathbb{R}$ with $0 \leq x \leq 1$ and any k, there exists a $\xi \in [0, x]$ such that

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{x^{4k}}{(4k)!} \cdot \cos(\xi) \qquad \text{(Taylor Series Expansion with Remainder Term)}$$

Use the above formula to derive a box function $\square\cos$ for cos for intervals $[a, b] \subset [0, 1]$! Can you extend your approach to derive a box function for $\sin x$ and e^x.