Exercise 1:  Mignotte Polynomials (4 points)

For polynomials $f, g \in \mathbb{C}[x]$ and a disk $\Delta$ in complex space, Rouché’s Theorem states that if

$$|f(z)| > |f(z) - g(z)|$$

for all $z \in \partial \Delta$, with $\partial \Delta$ the boundary of $\Delta$, then $f$ and $g$ have the same number of roots in $\Delta$. Use Rouché’s Theorem to show that, for $n \geq 8$, the so-called Mignotte polynomial

$$f(x) = x^n - (2L \cdot x - 1)^2$$

has two distinct real roots $x_1$ and $x_2$ with $|x_1 - x_2| < 2^{-\frac{nL}{2}} + 1$.

Hint: Use the fact that $g := -(2L \cdot x - 1)^2$ has a root of multiplicity 2 at $m = 2^{-L}$. Then, consider a disc $\Delta$ centered at $m$ and of suitable radius, and compare the values of $|f|$ and $|f - g|$ at the boundary of $\Delta$.

Exercise 2:  Specialization property of resultants (4 points)

(a) Let $\varphi : R \to R'$ be a ring homomorphism. Consider the canonical extension of $\varphi$ to a ring homomorphism between the polynomial rings $R[x]$ and $R'[x]$ given by

$$\bar{\varphi} : R[x] \to R'[x], \quad a_n x^n + \cdots + a_1 x + a_0 \mapsto \varphi(a_n) x^n + \cdots + \varphi(a_1) x + \varphi(a_0).$$

Let $f$ and $g$ be polynomials in $R[x]$. Prove the following specialization theorem for resultants:

If $\bar{\varphi}$ preserves the degrees of $f$ and $g$ (i.e., $\deg \bar{\varphi}(f) = \deg f$ and $\deg \bar{\varphi}(g) = \deg g$), then

$$\text{Res}(\bar{\varphi}(f), \bar{\varphi}(g)) = \varphi(\text{Res}(f, g)).$$

(b) Let $f := y^2 + 2 \cdot x^2 + x \cdot y - 4 \cdot x - 2 \cdot y + 2$ and $g := 3 \cdot x^2 + y^2 - 4 \cdot x$ be two polynomials in $\mathbb{Z}[x]$. Show that $f = g = 0$ has exactly one real solution and determine this solution.

Hint: Consider $f$ and $g$ as polynomials in $R[y]$, with $R = \mathbb{Z}[x]$. Then, use Part (a) with the ring homomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ that maps an $h \in \mathbb{Z}[x]$ to its value $h(x_0)$ at some fixed point $x_0 \in \mathbb{R}$. You should also use the fact that $f(x_0, y)$ and $g(x_0, y)$ have a common (complex) root if and only if their greatest common divisor is non-trivial.

Exercise 3:  Conditions for multiple roots of polynomials (4 points)

- Show that $f = a_2 x^2 + a_1 x + a_0 \in \mathbb{C}[x]$ has a multiple root if and only if $a_1^2 - 4a_0 a_2 = 0$.

- Give a corresponding formula for $f = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \in \mathbb{C}[x]$. 

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Exercise 4: The set of algebraic numbers is a field (6 points + 2 bonus points)

In this exercise, you will show that the set of algebraic numbers
\[
\bar{\mathbb{Q}} := \{ \alpha \in \mathbb{C} : \text{there exists an } f \in \mathbb{Z}[x] \text{ such that } f(\alpha) = 0 \} \subset \mathbb{C}
\]
is a field.

(a) Let \( \alpha, \beta \in \mathbb{C} \) and \( f \) and \( g \) be polynomials in \( \mathbb{Z}[x] \) such that \( f(\alpha) = 0 \) and \( g(\beta) = 0 \). Show how to construct polynomials \( h \in \mathbb{Z}[x] \) that satisfy

- \( h(\alpha + \beta) = 0 \) or \( h(\alpha - \beta) = 0 \), or
- \( h(\alpha \cdot \beta) = 0 \), or
- \( h(1/\alpha) = 0 \), or
- \( h(\sqrt[k]{\alpha}) = 0 \) for some \( k \in \mathbb{N}_{\geq 2} \),

respectively.

\textit{Hint:} Use resultants to show that the coordinates of any solution of a bivariate system \( F(x, y) = G(x, y) = 0 \), with \( F, G \in \mathbb{Z}[x, y] \), is a root of a polynomial with integer coefficients. Then, derive a corresponding bivariate system in \( \alpha \) and \( \gamma \), where \( \gamma = \alpha + \beta, \alpha \cdot \beta, 1/\alpha \), etc.

(b) Determine a polynomial \( f \in \mathbb{Z}[x] \) with \( f(\sqrt{3} - \sqrt[3]{3} + 1) = 0 \).