Exercise 1 (15 points).
Let $V := \mathbb{C}^2$ and $W := \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The group $G := \text{GL}_2$ acts on $V$ via matrix-vector multiplication. Then $G$ also acts on $W$ via

$$g(v_1 \otimes v_2 \otimes v_3) := (gv_1) \otimes (gv_2) \otimes (gv_3).$$

Determine the multiplicities $\text{mult}_\lambda(W)$.

Exercise 2 (15 points).
We have seen in the lecture that for the cyclic group $\mathbb{Z}_3$ there are exactly three isomorphism types of irreducible representations. Let $V = \mathbb{C}^n$. Let $\mathbb{Z}_3$ be generated by $\pi$. The group $\mathbb{Z}_3$ acts on the vector space $V \otimes V \otimes V$ via

$$(\pi)(u \otimes v \otimes w) = (v \otimes w \otimes u)$$

and linear continuation. Determine the three multiplicities of $\mathbb{Z}_3$-irreducibles in $V \otimes V \otimes V$.

Exercise 3 (10 points).
Prove that the border Waring rank of $X_1 X_2$ exceeds 1 by using multiplicities in the coordinate rings of orbit closures as follows. Let $G := \text{GL}_2$. Find a partition $\lambda$ such that

$$\text{mult}_\lambda \mathbb{C}[G(X_1 X_2)] > \text{mult}_\lambda \mathbb{C}[G(X_1^2)].$$