Exercises for Introduction to geometric complexity theory


Exercise sheet 1 Solutions  Due: Wednesday, May 03, 2017

Total points : 40

Exercise 1 (10 points) Use the discriminant polynomial to show that the Waring rank of $X^2 + XY + Y^2$ is at least 2.

Solution : The discriminant of the given polynomial is $-3 \neq 0$ and thus its Waring rank cannot be 1.

Exercise 2 (10 points) We have seen polynomials whose Waring rank exceeds their border Waring rank. In contrast to this observation prove that the set of Waring rank 1 polynomials is $C$-closed.

Solution : Let $W_1 \overset{\text{def}}{=} \{ g \in \mathbb{C}[x_1, x_2, \ldots, x_n]_d \mid \text{Waring rank of } g \text{ is } d \}$. We need to show that $W_1 = W_1$. Let $h \in W_1$ be some element. We need to show that $h \in W_1$. Thus there exists a sequence $\{h_i\}$ with $h_i \in W_1$ and $\lim_{i \to \infty} h_i = h$. We also have $h_i = g_i^d = (\alpha_{i1}x_1 + \alpha_{i2}x_2 + \ldots + \alpha_{in}x_n)^d$. If the $\lim_{i \to \infty} g_i = g$ exists then it is clear that $h = g^d$ because the powering by $d$ is a continuous function. But $\lim_{i \to \infty} g_i = g$ may not exist. For an example where this limit does not exist, consider $h_j = (\omega^j(x_1 + x_2 + \ldots + x_n))^d$, here $\omega = e^{\frac{2\pi i}{d}}$ is a $d$th primitive root of unity. In this case $\lim_{i \to \infty} h_i = (x_1 + x_2 + \ldots + x_n)^d$ exists but $\omega^j(x_1 + x_2 + \ldots + x_n) = g_j$ obviously does not converge. To solve this exercise we need to prove the following lemma.

Lemma 1. If $\{\alpha_i^d\}$ is a sequence of complex numbers such that $\lim_{i \to \infty} \alpha_i^d = \beta$. Then there exists an infinite sub-sequence $\{\gamma_i^d\}$ of $\{\alpha_i^d\}$ such that $\lim_{i \to \infty} \gamma_i = \gamma$ exists and $\gamma^d = \beta$.

Proof. Let $\beta_1, \beta_2, \ldots, \beta_d$ be the roots of the polynomial equation $f(x) = x^d - \beta = 0$. Define the sequence $\{\eta_i\}$ by $\eta_i = j$ such that $|\alpha_i - \beta_j|$ is minimum among all $|\alpha_i - \beta_k|$. Here there may be more than one choice of $j$, which achieves the minimum distance, in that case we take an arbitrary one. Now since every $\eta_i$ can have finitely many possible values, there exists $1 \leq k \leq d$ such that sub-sequence with indices $i$ with $\eta_i = k$ is infinite. Take this sub-sequence $\{\gamma_i^d\}$ of $\{\alpha_i^d\}$. We claim that $\lim_{i \to \infty} \gamma_i$ exists and it is equal to $\beta_k$. To show this, let $\epsilon > 0$ be some given real number. We need to show that $\exists N_0$ such that $\forall N \geq N_0$, $|\gamma_N - \beta_k| \leq \epsilon$. Since $\lim_{i \to \infty} \alpha_i^d = \lim_{i \to \infty} \gamma_i^d = \beta$, we know that $\exists M_0$ such that $\forall M \geq M_0$, $|\gamma_M^d - \beta| \leq \epsilon^d$. Now we set $M_0 = N_0$. Suppose that $|\gamma_N - \beta_k| > \epsilon$, then $f(\gamma_N) = f(\gamma_M) = \gamma_M^d - \beta = (\gamma_M - \beta_1)(\gamma_M - \beta_2)\ldots(\gamma_M - \beta_k)$. 


Since $|\gamma_N - \beta_k| = |\gamma_M - \beta_k| > \epsilon$, we get that $|\gamma_M - \beta_j| > \epsilon$ for all $1 \leq j \leq d$. Thus $|\gamma^d - \beta| > \epsilon^d$, a contradiction. Thus $\lim_{i \to \infty} \gamma_i = \beta_k = \gamma$ and $\gamma^d = \beta^d_k = \beta$. \hfill $\square$

Write $h_i = \alpha^d_{i1} x_1^d + \alpha^d_{i2} x_2^d + \ldots + \alpha^d_{in} x_n^d + b_i$, where $b_i$ is composed of the rest of monomials. Let $h = \beta_1 x_1^d + \beta_2 x_2^d + \ldots + \beta_n x_n^d + b$ for some polynomial $b$, since $\lim_{i \to \infty} h_i = h$, we know that $\{\alpha^d_{i1}\}$ converges to $\beta_1$. Thus there exists a sub-sequence $\{\gamma^d_{i1}\}$ of $\{\alpha^d_{i1}\}$ such that $\lim_{i \to \infty} \gamma^d_{i1} = \gamma_1$ and $\gamma^d = \beta_1$. Now we take the corresponding (one induced by choices of the sub-sequence $\{\gamma^d_{i1}\}$) sub-sequence of $\{h_i\}$. Now we apply the same procedure on this sub-sequence for the sequence $\{\gamma^d_{i2}\}$ and so on. After this process of selecting sub-sequences $n$ times, we get a sub-sequence $\{\alpha_i\}$ of $\{h_i\}$ and now the linear forms in $a_i$ also converge to some linear form $a$ and we get that $h = a^d$. Thus $h \in W_1$.

**Exercise 3** (10 points) Consider the action of $\mathbb{C}^{N \times N}$ on $\mathbb{C}[X_1, \ldots, X_N]$ defined in the lecture. Compute the following polynomial in the standard monomial basis:

$$
\begin{pmatrix}
2 & 3 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
(X_1 X_2^2 + X_3)
$$

**Solution**: We have

$$
\begin{pmatrix}
2 & 3 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
(X_1 X_2^2 + X_3) = 2X_1(3X_1 + X_2)^2 + X_2 + X_3
$$

$$
= 18X_1^3 + 12X_1^2 X_2 + 2X_1 X_2^2 + X_2 + X_3
$$

**Exercise 4** (10 points) Let $\text{GL}_n$ denote the group of invertible complex $n \times n$ matrices. Let $G = \text{GL}_n \times \text{GL}_n$ and let $V = \mathbb{C}^{n \times n}$. Define an action of $G$ on $V$ by

$$(g_1, g_2) v := g_1 \cdot v \cdot g_2^t,$$

where “.$" is the product of matrices. Let $v \in V$ have rank exactly $k$. Prove that

$$Gv = \{w \in V \mid \text{rk}(w) = k\}.$$

**Solution**: We use the following fact.

**Fact 2**. If $v \in \mathbb{C}^{n \times n}$ is rank $k$ matrix then there exists matrices $A, B \in \text{GL}_n$ such that $AvB = I_k \cdot 0 \cdot 0_n \cdot 0$, where $I_k$ is the $k \times k$ identity matrix.

Let $T$ be the set defined as $T \overset{\text{def}}{=} \{w \in \mathbb{C}^{n \times n} \mid \text{rk}(w) = k\}$. We need to show that $Gv = T$, where $v \in \mathbb{C}^{n \times n}$ is the given rank $k$ matrix. Since multiplying by an element of $\text{GL}_n$ does not change the rank, we get that $Gv \subseteq T$. Suppose $w \in T$ be an arbitrary rank $k$ matrix, we know that there exists matrices $A, B, C, D \in \mathbb{C}^{n \times n}$ such that
\[
AvB = \begin{bmatrix}
I_k & 0 \\
0 & 0
\end{bmatrix}_{n \times n} = CwD.
\]

Thus \( w = C^{-1}AvBD^{-1} = g_1vg_2^t \) for \( g_1 = C^{-1}A \) and \( g_2 = (BD^{-1})^t \). It is clear that \( g_1, g_2 \in \text{GL}_n \).

Thus \( w \in Gv \), which implies that \( T \subseteq Gv \).