Exercises for Introduction to geometric complexity theory


Exercise sheet 4 Solutions Due: Wednesday, May 24, 2017

Total points: 40

Exercise 1 (10 points) Let $f$ be a polynomial computed by a multiplicatively disjoint circuit of size $s$. Prove that the degree of $f$ is bounded by $s$.

Solution: We prove it by on induction on size of the circuit. Our claim is trivially true for the circuit of size 1. Let $g$ be any intermediate gate of circuit $C$ (this $C$ is of size $s$ and computes $f$). Let $P_g$ be the polynomial which is computed at the gate $g$. We use $t$ to denote the size of the sub-circuit induced by $g$.

Let $g_1$ and $g_2$ be the children gate of $g$. Let $t_1$ and $t_2$ be the sizes of sub-circuits induced by $g_1$ and $g_2$. By induction hypothesis, we know that $\deg(P_{g_1}) \leq t_1$ and $\deg(P_{g_2}) \leq t_2$.

If $g$ is a $+$ gate then we know that $t \geq \max\{t_1, t_2\}$. Thus $\deg(P_g) \leq \max\{\deg(P_{g_1}), \deg(P_{g_2})\} \leq \max\{t_1, t_2\} \leq t$.

If $g$ is a $\times$ gate then we know that $t \geq t_1 + t_2$, since the subcircuits below $g_1$ and $g_2$ are disjoint. Thus $\deg(P_g) \leq \deg(P_{g_1}) + \deg(P_{g_2}) \leq t_1 + t_2 \leq t$.

Thus $\deg(f) \leq s$.

Exercise 2 (10 + 10 points) In this exercise, we consider algebraic branching programs with edges labeled by scalar multiples of the variables, that is, the edges have labels of the form $\alpha X_i$. They can still have labels of the form $\alpha$, too.

1. Let $f$ be a homogeneous polynomial of degree $d$ that is computed by an ABP of size $s$. Prove that there is an ABP of size polynomial in $d$ and $s$ such that at every node, a homogeneous polynomial is computed.
   (Hint: Replace every node by $d+1$ nodes.)

2. We group the nodes of equal degree together. There are now two types of edges, edges within one group and edges from degree $i$ to degree $i+1$. (Why?) Prove that there is an homogeneous ABP of size polynomial in $d$ and $s$ computing $f$ with exactly $d+1$ layers by removing all edges within the groups.
   (Hint: Sort the nodes topologically and use induction.)
Solution: 1.

For a node \( v \) in the given ABP \( A \), one can define a polynomial computed at node \( v \), which is just the sum of weights of all paths from \( s \) to \( v \). Here weight of a path is the product of the edge labels in it. We denote the polynomial computed at the node \( v \) by \( f_v \).

Now we replace every node \( v \) in \( A \) by \( d + 1 \) nodes \( v_0, v_1, \ldots, v_d \) to create a new ABP \( A' \), such that \( v_i \) will compute the degree \( i \) homogeneous part of \( f_v \). To achieve this, we add the following edges in \( A' \). If \( v \rightarrow w \) is an edge in \( A \) with a label of form \( \alpha X_j \) for some \( \alpha \in \mathbb{F} \) and some variable \( X_j \) then we add edges of weights \( \alpha X_j \) of the form \( v_i \rightarrow w_{i+1} \) for \( 0 \leq i < d \). Similarly, If \( v \rightarrow w \) is an edge in \( A \) with a label \( \alpha \) for some \( \alpha \in \mathbb{F} \) then we add edges of weights \( \alpha \) of the form \( v_i \rightarrow w_i \) for \( 0 \leq i \leq d \).

It is clear now that every node \( v_i \) in \( A' \) computes the degree \( i \) homogeneous part of \( f_v \). In particular, at every node in \( A' \), a homogeneous polynomial is computed.

2.

Now we group all the nodes which compute the homogeneous polynomials of same degree in one group. So we have \( d+1 \) groups \( V_0, V_1, \ldots, V_d \), such that nodes in \( V_i \) compute homogeneous polynomials of degree \( i \). The way we added edges in \( A' \), it is clear that only edges in \( A' \) are of form \( v \rightarrow w \) with \( v \in V_i \) and \( w \in V_{i+1} \) or \( v, w \in V_i \). We want to remove edges of kind \( v \rightarrow w \) when both \( v, w \in V_i \). We do it inductively on \( V_0, V_1, \ldots, V_d \). Suppose we have removed in-group edges in \( V_0, V_1, \ldots, V_{i-1} \) already. Now sort the nodes in \( V_i \) topologically. Let \( v \) be a node in \( V_i \) which has no internal incoming edges. Let \( v \rightarrow w \) be an edge in \( V_i \) with weight \( \alpha \), note that weights of edges in \( V_i \) can only be field constants. Let \( u_1 \rightarrow v, u_2 \rightarrow v, \ldots, u_m \rightarrow v \) be incoming edges to \( v \) with weights \( w_1, w_2, \ldots, w_m \) respectively, here \( u_j \in V_{i-1} \). Now we remove the edge \( v \rightarrow w \) and add the edges \( u_1 \rightarrow w, u_2 \rightarrow w, \ldots, u_m \rightarrow w \) with weights \( \alpha w_1, \alpha w_2, \ldots, \alpha w_m \) respectively.

It is easy to see that this process does not change polynomials computed at nodes in \( A' \). Now we do this for all outgoing edges of all nodes which have no internal incoming edges and then repeat the process until there are no edges in \( V_i \). This process will create edges which have linear forms as weights. To make edge weights of the form \( \alpha X_j \), we can split every nodes into \( n \) nodes.

Exercise 3 (10 points)

1. Write the permanent of a generic 2 × 2-matrix as the projection of a determinant. (Size of your choice.)

2. The same, but now as a projection of an iterated matrix multiplication.

Solution:

\[
\text{perm} \left( \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \right) = \text{det} \left( \begin{bmatrix} 1 & X_{22} & 0 \\ 0 & X_{21} & 1 \\ X_{11} & 0 & X_{12} \end{bmatrix} \right) = (1, 1) \text{ entry of} \left[ \begin{bmatrix} X_{11} & X_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_{22} & 0 \\ X_{21} & 0 \end{bmatrix} \right]
\]