Exercise 1 (10 points) Let $U_n$ denote the group of $n \times n$ upper triangular matrices with 1s on the main diagonal. For $1 \leq i < j \leq n$, $\alpha \in \mathbb{C}$, let $x_{ij}(\alpha) \in U_n$ denote the identity matrix with an entry $\alpha$ in row $i$ and column $j$. Prove that $U_n$ is generated as a group by the set 
\{ $x_{ij}(\alpha)$ | $1 \leq i < j \leq n$, $\alpha \in \mathbb{C}$ \}.

Solution: First let us see how does $x_{ij}(\alpha)$ act on a matrix $A \in \mathbb{C}^{n \times n}$ by the action of left multiplication. Verify that $x_{ij}(\alpha)A = A_{ij}(\alpha)$, where $A_{ij}(\alpha)$ is the matrix obtained from $A$ by adding $\alpha$-multiple $\alpha R_j$ of the $j^{th}$ row $R_j$ of $A$ to $i^{th}$ row $R_i$ of $A$. Let $B \in U_n$ be a given matrix. We want to write $B$ as a product of the matrices of the form $x_{ij}(\alpha)$. If we let $x_{1j}(\alpha)$ act on the identity matrix $I_n = x_{ij}(0)$, then we get the identity matrix whose $(1,j)$-th entry is $\alpha$. Thus if $(1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ is the first row of $B$ then $\prod_{j=2}^{n} x_{1j}(\alpha_j)I_n$ is the identity matrix whose first row is $(1, \alpha_2, \alpha_3, \ldots, \alpha_n)$. By this process we have generated matrix whose first row matches with the first row of $B$. We can repeat the same process to get the other rows of $B$. In particular,

$$B = \prod_{i=1}^{n} \prod_{j=i+1}^{n} x_{ij}(B_{ij})I_n.$$ 

Exercise 2 (10 points) Let $V := \mathbb{C}^n$. The group $\text{GL}_n$ acts on $V$ by matrix-vector multiplication. Decompose $V$ into a direct sum of irreducibles and determine their types $\lambda$.

Solution: Note that the first basis vector $e_1$ is fixed under $U_n$ and is a weight vector of weight $(1,0,\ldots,0)$. Thus $\langle \text{GL}_n e_1 \rangle$ is irreducible. But $\langle \text{GL}_n e_1 \rangle = \mathbb{C}^n$. Thus $\mathbb{C}^n$ is irreducible of type $(1,0,\ldots,0)$.

Exercise 3 (10 points) Let $V := \mathbb{C}^{n \times n}$ denote the vector space of $n \times n$ matrices. The group $\text{GL}_n$ acts on $V$ by left multiplication. Decompose $V$ into a direct sum of irreducibles and determine their types $\lambda$.

Solution: By the previous exercise, it is not hard to find a irreducible decomposition of
\( V = \mathbb{C}^{n \times n} \). Let us use the symbol \( e_{ij} \) to denote the \( n \times n \) matrix \((i, j)^{th}\) entry is 1 and all other entries are 0. Let \( V_j := \langle e_{ij} \mid i \in [n] \rangle \), i.e., \( V_j \) is the subspace of \( V = \mathbb{C}^{n \times n} \) containing the \( n \times n \) matrices whose all columns are zero except the \( j^{th} \) one. By the previous exercise, \( V_j \) is an irreducible representation of \( \text{GL}_n \). Thus \( V = \bigoplus_{j=1}^n V_j \) is an irreducible decomposition of \( V \). It also follows from the previous exercise that the type of each \( V_j \) is \( \lambda = (1, 0, \ldots, 0) \) with \( (V_j)_\lambda = \langle e_{1j} \rangle \).

**Exercise 4 (10 points)** Let \( V \) be a polynomial \( \text{GL}_n \)-representation. For \( i < j \) let \( x_{ij}(\alpha) \) be defined as in Exercise 1. The raising operator \( E_{ij} : V \to V \) is a linear map defined via

\[
E_{ij}(v) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( (x_{ij}(\epsilon)v) - v \right).
\]

Prove that \( E_{ij} \) is well-defined, i.e., that the limit exists.

**Solution** : Let \((V, \varrho)\) be the given polynomial representation of \( \text{GL}_n \). Let \( \{e_1, \ldots, e_n\} \) be the canonical basis of \( V \). Since \( \varrho \) is a polynomial representation, each coordinate function of \( x_{ij}(\alpha)v \) is a uni-variate polynomial in \( \alpha \). Thus \( x_{ij}(\alpha)v = v_0 + \alpha v_1 + \ldots + \alpha^d v_d \) for some \( d \in \mathbb{N} \) and \( v_0, v_1, \ldots, v_d \in V \). Since \( x_{ij}(0) = I_n \), we get that \( x_{ij}(0)v = I_n v = v \). Thus \( v_0 = v \). Therefore \( x_{ij}(\alpha)v = v + \alpha v_1 + \ldots + \alpha^d v_d \) for all \( \alpha \in \mathbb{C} \). In particular \( \frac{1}{\epsilon} ((x_{ij}(\epsilon)v) - v) = v_1 + \epsilon v_2 + \ldots + \epsilon^{d-1} v_d \). Thus \( E_{ij}(v) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} ((x_{ij}(\epsilon)v) - v) = v_1 \) is well defined.