Exercise 1 (10 points) Let $U$, $U'$, $V$, and $V'$ be finite-dimensional vector spaces. Let $t \in U \otimes V$ and let $A : U \to U'$ and $B : V \to V'$ be linear maps. Prove that $(A \otimes B)(t)$ (as defined in the lecture) is well-defined, that is, if we take two decompositions of $t$ into rank-one tensors, then we will get the same image.

**Solution:** We choose a basis $u_1, \ldots, u_m$ of $U$ and a basis $v_1, \ldots, v_n$ of $V$. Let $t = \sum_{i=1}^{r} c_i \otimes d_i = \sum_{i=1}^{s} e_i \otimes f_i$ be two rank-one decomposition of $t$.

Since $u_1, \ldots, u_m$ is a basis of $U$, there exists a matrix $C \in \mathbb{C}^{r \times m}$ such that

$$
\begin{bmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_r \\
\end{bmatrix} = C 
\begin{bmatrix}
  u_1 \\
  u_2 \\
  \vdots \\
  u_m \\
\end{bmatrix}.
$$

Similarly there exist matrices $D \in \mathbb{C}^{r \times n}$, $E \in \mathbb{C}^{s \times m}$ and $F \in \mathbb{C}^{s \times n}$ such that

$$
\begin{bmatrix}
  d_1 \\
  d_2 \\
  \vdots \\
  d_r \\
\end{bmatrix} = 
\begin{bmatrix}
  e_1 \\
  e_2 \\
  \vdots \\
  e_s \\
\end{bmatrix} = E 
\begin{bmatrix}
  u_1 \\
  u_2 \\
  \vdots \\
  u_m \\
\end{bmatrix} \quad \text{and} \quad 
\begin{bmatrix}
  f_1 \\
  f_2 \\
  \vdots \\
  f_s \\
\end{bmatrix} = D 
\begin{bmatrix}
  v_1 \\
  v_2 \\
  \vdots \\
  v_m \\
\end{bmatrix}.
$$

Thus

$$
\sum_{i=1}^{r} A(c_i) \otimes B(d_i) = \sum_{i=1}^{r} A(\sum_{j=1}^{m} C_{ij} u_j) \otimes B(\sum_{j=1}^{n} D_{ij} v_j)
$$

$$
= \sum_{k=1}^{m} \sum_{\ell=1}^{n} \lambda_{k\ell} A(u_k) \otimes B(v_{\ell}) = (A \otimes B)(\sum_{k=1}^{m} \sum_{\ell=1}^{n} \lambda_{k\ell} u_k \otimes v_{\ell})
$$

Here $\lambda_{k\ell} = \sum_{i=1}^{r} C_{ik} D_{i\ell}$. Similarly we have
\[ \sum_{i=1}^{s} A(e_i) \otimes B(f_i) = \sum_{i=1}^{s} A(\sum_{j=1}^{m} E_{ij} u_j) \otimes B(\sum_{j=1}^{n} F_{ij} v_j) \]

\[ = \sum_{k=1}^{m} \sum_{\ell=1}^{n} \gamma_{k\ell} A(u_k) \otimes B(v_\ell) = (A \otimes B)\left( \sum_{k=1}^{m} \sum_{\ell=1}^{n} \gamma_{k\ell} u_k \otimes v_\ell \right) \]

Here \( \gamma_{k\ell} = \sum_{i=1}^{s} E_{ik} F_{i\ell} \). Since \( t = \sum_{i=1}^{s} c_i \otimes d_i = \sum_{i=1}^{s} e_i \otimes f_i \), we have that \( \gamma_{k\ell} = \lambda_{k\ell} \). Thus

\[ (A \otimes B)(t) = \sum_{i=1}^{r} A(c_i) \otimes B(d_i) \]

\[ = \sum_{i=1}^{s} A(e_i) \otimes B(f_i) \]

**Exercise 2** *(5+5 points)* Let \( U, V, \) and \( W \) be finite-dimensional vector spaces. Prove that tensor product is commutative and associate, that is, there are natural isomorphisms

1. \( U \otimes V \cong V \otimes U \) and
2. \( U \otimes (V \otimes W) \cong (U \otimes V) \otimes W. \)

Try to only use the universal property to construct the isomorphisms and not the concrete construction given in the lecture.

**Solution**: 1. Let \( \phi \) be the bi-linear map \( \phi : U \times V \rightarrow U \otimes V \) and \( \psi \) be the bi-linear map \( \psi : V \times U \rightarrow V \otimes U \). Let \( f_1 : U \times V \rightarrow V \times U \) be the map which maps \((u, v)\) to \((v, u)\), \( f_2 \) is the inverse map of \( f_1 \).

Note that \( \psi \) can be converted into a bi-linear map \( \phi' : U \times V \rightarrow V \otimes U \) by simply using \( \phi' = \psi \circ f_1 \), similarly \( \psi' = \phi \circ f_2 \).

Now we apply the definition of tensor product to \( U \otimes V \) and \( \phi \) and the bi-linear map \( \phi' = \psi \circ f_1 \) and the vector space \( V \otimes U \). We get a linear map \( \ell_1 : U \otimes V \rightarrow V \otimes U \) such that \( \ell_1 \circ \phi = \phi' = \psi \). In the same way, by interchanging the roles of the two tensor products, we get a linear map \( \ell_2 : V \otimes U \rightarrow U \otimes V \) such that \( \ell_2 \circ \psi = \psi' = \phi \).

This gives us that \( \ell_2 \circ \psi \circ f_1 = \ell_2 \circ \ell_1 \circ \phi = \phi = \phi \circ f_1 \circ f_2 = \phi \).

We apply the definition of tensor product on three vector spaces. For vector spaces \( U, V, W \) and \( X \), we say that \( X \) is a tensor product of \( U, V \) and \( W \) denoted by \( U \otimes V \otimes W \), if there is a trilinear map \( \phi : U \times V \times W \rightarrow X \) such that for each trilinear map \( \psi : U \times V \times W \rightarrow Y \) there is a unique linear map \( \ell : U \otimes V \otimes W \rightarrow Y \) such that \( \psi = \ell \circ \phi \).
It can be verified by the exactly same proof as for tensor product of two vector spaces, that tensor product on three vector spaces exist and is unique up-to isomorphism.

Now we shall prove that \( U \otimes V \otimes W \cong U \otimes (V \otimes W) \). Since tensor product \( U \otimes V \otimes W \) is unique up-to isomorphism, it is enough to show that \( U \otimes (V \otimes W) \) is a tensor product on \( U, V, W \).

There is an obvious trilinear map \( \phi : U \times V \times W \to U \otimes (V \otimes W) \), a composition of two bi-linear maps \( \phi_1 : V \times W \to V \otimes W \) and \( \phi_2 : U \times (V \otimes W) \to U \otimes (V \otimes W) \) with \( \phi(u, v, w) = \phi_2(u, \phi_1(v, w)) \).

Let \( Y \) be some vector space and \( \psi : U \times V \times W \to Y \) be the given trilinear map. We need to show that there exists a unique linear map \( \ell : U \otimes (V \otimes W) \to Y \) such that \( \psi = \ell \circ \phi \).

First we define a bi-linear map \( \psi_1 : V \times W \to \text{Hom}(U, Y) \) with \( \psi_1(v, w)(u) = \psi(u, v, w) \).

By property of the tensor product we know that there exists a unique linear map \( \ell_1 : V \otimes W \to \text{Hom}(U, Y) \) such that \( \psi_1 = \ell_1 \circ \phi_1 \).

Now consider the bi-linear map \( \psi_2 : U \times (V \otimes W) \to Y \) defined by \( \psi_2(u, t) = \ell_1(t)(u) \).

By property of tensor product we know that there exists a unique linear map \( \ell_2 : U \otimes (V \otimes W) \to Y \) such that \( \psi_2 = \ell_2 \circ \phi_2 \). Now if we define \( \ell : U \otimes (V \otimes W) \to Y \) by

\[
\ell(\phi_2(u, \phi_1(v, w))) = \ell_2(\phi_2(u, \phi_1(v, w))) = \psi_2(u, \phi_1(v, w))(u) = \psi(u, v, w).
\]

Thus \( \psi = \ell \circ \phi \). Uniqueness of \( \ell \) follows from the uniqueness of \( \ell_1 \) and \( \ell_2 \).

Similarly we can show that \( U \otimes V \otimes W \cong (U \otimes V) \otimes W \). Thus \( U \otimes V \otimes W \cong U \otimes (V \otimes W) \cong (U \otimes V) \otimes W \).

Exercise 3 \((10 \text{ points})\) Let \( p(X) = \sum_{i=0}^{d} a_i X^i \) and \( q(x) = \sum_{i=0}^{d} b_i X^i \) be two univariate polynomials with indeterminates as coefficients. Define the bilinear forms \( c_0, \ldots, c_{2d} \) in \( a_0, \ldots, a_d \) and \( b_0, \ldots, b_d \) by

\[
p(X) \cdot q(X) = \sum_{i=0}^{2d} c_i X^i.
\]

Write down the corresponding tensor \( t_d \). Prove that \( R(t_d) \leq 2d + 1 \) if the underlying field has at least \( 2d + 1 \) elements. Hint: Use evaluation and interpolation. (Bonus: Can you do it with just \( 2d \) elements?)

Solution : We use the following fact.

**Fact 1.** If \( F(x) = \sum_{i=0}^{n} f_i x^i \) is a polynomial of degree \( n \) and \( x_0, x_1, x_2, \ldots, x_n \) are \( n+1 \) distinct constants then every coefficient \( f_i \) can be expressed as a linear combination of \( f(x_0), f(x_1), \ldots, f(x_n) \).

By using the above fact, we can express every \( c_i \) as a linear combination of \( p(x_0)q(x_0), p(x_1)q(x_1), \ldots, q(x_{2d+1})p(x_{2d+1}) \), where \( x_0, x_1, x_2, \ldots, x_{2d+1} \) are \( 2d + 1 \) distinct constants. Each \( p(x_i)q(x_i) \) looks like \( \ell_i(a_0, a_1, \ldots, a_d)h_i(b_0, b_1, \ldots, b_d) \), where \( \ell \) and \( h \) are linear forms. Thus \( \{c_0, c_1, \ldots, c_{2d+1}\} \in \langle \ell_1(a_0, a_1, \ldots, a_d)h_1(b_0, b_1, \ldots, b_d), \ell_2(a_0, a_1, \ldots, a_d)h_2(b_0, b_1, \ldots, b_d), \ldots, \ell_{2d+1}(a_0, a_1, \ldots, a_d)h_{2d+1}(b_0, b_1, \ldots, b_d) \rangle \).

In particular, \( R(t_d) \leq 2d + 1 \).

Exercise 4 \((5+5+10 \text{ points})\) Let \( U, V, \) and \( W \) be finite-dimensional vector spaces. Let \( t \in U \otimes V \otimes W \). Let \( k = \dim U \).
1. Prove that if \( R(t) < k \), then there is a nonzero linear form \( a \in U^* \) such that \( (a \otimes I_V \otimes I_W)(t) = 0 \). Here \( I_V \) and \( I_W \) are the identity on \( V \) and \( W \).

2. Prove that \( R(\langle r \rangle) = r \).

3. Prove that \( R(t_d) \geq 2d + 1 \), where \( t_d \) is the tensor of the previous exercise.

**Solution**: 1. Since \( R(t) < k \), we have \( t = \sum_{i=1}^\ell u_i \otimes v_i \otimes w_i \) with \( \ell < k \). Since \( U \) is of dimension \( k \), there exists a non-zero linear function \( a : U \to \mathbb{F} \) such that \( a(u_i) = 0 \) for all \( i \in [\ell] \).

Now \( (a \otimes I_V \otimes I_W)(t) = \sum_{i=1}^\ell a(u_i) \otimes I_V(v_i) \otimes I_W(w_i) = t = \sum_{i=1}^\ell 0 \otimes v_i \otimes w_i = 0 \).

2. Obviously \( R(\langle r \rangle) \leq r \) because \( \langle r \rangle = \sum_{i=1}^r e_i \otimes e_i \otimes e_i \). If \( R(\langle r \rangle) < r \) then we can find a non-zero \( a \in (\mathbb{F}^r)^* \) such that \( (a \otimes I_{\mathbb{F}^r} \otimes I_{\mathbb{F}^r})(\langle r \rangle) = 0 \). Now \( (a \otimes I_{\mathbb{F}^r} \otimes I_{\mathbb{F}^r})(\langle r \rangle) = \sum_{i=1}^r a(e_i) \otimes e_i \otimes e_i = 0 \). For this to happen, \( a(e_i) = 0 \) for \( i \in [r] \). Since \( \{e_1, e_2, \ldots, e_r\} \) is a basis of \( \mathbb{F}^r \), \( a = 0 \). Thus \( R(\langle r \rangle) = r \).

3. We have \( t_d \in \mathbb{F}^{2d+1} \otimes \mathbb{F}^{2d+1} \otimes \mathbb{F}^{2d+1} \). Each of the \( 2d + 1 \) slices of \( t_d \) is a \((d+1) \times (d+1)\) matrix. And the matrix corresponding to the \( i^{\text{th}} \) slice looks like below, if \( i \leq d + 1 \).

\[
\begin{bmatrix}
0 & \ldots & 0 & 1 & 0 & 0 \\
0 & \ldots & 1 & 0 & 0 & 0 \\
\vdots & & & \vdots & & \\
1 & 0 & & & \vdots & \\
0 & \ldots & 0 & 0 & & \vdots \\
0 & 0 & \ldots & \ldots & \ldots & \ldots & 0 & 0
\end{bmatrix}
\]

Where all the entries are zero except the entries \( (i, 1), (i - 1, 2), \ldots, (2, i - 1), (1, i) \).

If \( i > d + 1 \), then the \( i^{\text{th}} \) slice looks like below.

\[
\begin{bmatrix}
0 & \ldots & 0 & 0 & 0 & 0 \\
0 & \ldots & 0 & 0 & 0 & 0 \\
\vdots & & & \vdots & & 1 \\
0 & 0 & \vdots & \vdots & 1 & 0 \\
0 & \ldots & 0 & 1 & & \vdots \\
0 & 0 & \ldots & 1 & \ldots & 0 & 0
\end{bmatrix}
\]

Where all the entries are zero except the entries \((d+1, j), (d, j+1), \ldots, (d - j + 1, 2j - 2), (d - j + 2, 2j - 1)\) with \( j = 2d + 2 - i \).

Suppose \( R(t_d) = \ell \), then \( t_d = \sum_{i=1}^\ell u_i \otimes v_i \otimes w_i \). This means that all the \( 2d + 1 \) slices (as described above) of \( t_d \) are in \( \langle v_1 \otimes w_1, v_2 \otimes w_2, \ldots, v_\ell \otimes w_\ell \rangle \). Notice that all the slices of \( t_d \) are independent vectors of \( \mathbb{F}^{(d+1) \times (d+1)} \), thus span a subspace \( M \subseteq \mathbb{F}^{(d+1) \times (d+1)} \) of dimension \( 2d + 1 \). Thus \( \ell \geq 2d + 1 \).