Exercise 1 \((5 + 5\text{ points})\) Prove the following:

1. \(\Lambda^2 V = \langle v \otimes w - w \otimes v \mid v, w \in V \rangle\).
2. For all \(t \in \Lambda^2 V\), \((1, 2)t = -t\).

**Solution**

1. We defined

\[\Lambda^2 V = \langle v_i \otimes v_j - v_j \otimes v_i \mid 1 \leq i, j \leq n \rangle.\]

This clearly implies \(\Lambda^2 V \subseteq \langle v \otimes w - w \otimes v \mid v, w \in V \rangle\). Let \(v_1, v_2, \ldots, v_n\) be a basis of \(V\). Now \(v = \sum a_i v_i\) and \(w = \sum b_i v_i\) for some \(a_i, b_i \in \mathbb{C}\).

We have \(v \otimes w - w \otimes v = \sum_{i,j} a_i b_j v_i \otimes v_j - \sum_{i,j} a_j b_i v_j \otimes v_i = \sum_{i,j} a_i b_j (v_i \otimes v_j - v_j \otimes v_i)\). Thus \(\forall v, w \in V\) we have that \(v \otimes w - w \otimes v \in \Lambda^2 V\). Hence \(\langle v \otimes w - w \otimes v \mid v, w \in V \rangle \subseteq \Lambda^2 V\).

Therefore \(\Lambda^2 V = \langle v \otimes w - w \otimes v \mid v, w \in V \rangle\).

2. Let \(t\) be a general tensor in \(\Lambda^2 V\). Thus \(t = \sum_{i,j} \alpha_{ij} (v_i \otimes v_j - v_j \otimes v_i)\) for some \(\alpha_{ij} \in \mathbb{C}\). Now

\[
(1, 2)t = \sum_{i,j} \alpha_{ij} ((1, 2)v_i \otimes v_j - (1, 2)v_j \otimes v_i) \\
= \sum_{i,j} \alpha_{ij} (v_j \otimes v_i - v_i \otimes v_j) \\
= -t,
\]

Exercise 2 \((5 + 5\text{ points})\) Prove that there are no nontrivial linear subspaces of \(S^2 V\) and \(\Lambda^2 V\), respectively, that are invariant under the \(\text{GL}(V)\)-action.

**Solution**

Let \(n = \dim(V)\). Notice that \(S^2 V = \langle \text{GL}_n e_1 \otimes e_1 \rangle\), where the line \(\mathbb{C}(e_1 \otimes e_1)\) is a \(B_n\) stable line. Thus \(S^2 V = \langle \text{GL}_n e_1 \otimes e_1 \rangle\) is irreducible of type \((2, 0)\). Also \(\mathbb{C}(e_1 \wedge e_2)\) is a \(B_n\) stable line and \(\Lambda^2 V = \langle \text{GL}_n e_1 \wedge e_2 \rangle\). Thus \(\Lambda^2 V = \langle \text{GL}_n e_1 \wedge e_2 \rangle\) is irreducible of type \((1, 1)\).
Exercise 3 (5 + 10 points) Let $V$ be an $n$-dimensional vector space. Define the projection operators $p_2$ by

$$v_1 \otimes v_2 \otimes v_3 \rightarrow \frac{1}{2}(v_1 \otimes v_2 \otimes v_3 - v_2 \otimes v_1 \otimes v_3)$$

and $p_{13}$ by

$$v_1 \otimes v_2 \otimes v_3 \rightarrow \frac{1}{2}(v_1 \otimes v_2 \otimes v_3 + v_3 \otimes v_2 \otimes v_1)$$

Let $U := p_{13}(p_2(V))$. In the same way, let $U' := p_{12}(p_3(V))$.

1. Prove that $U$ and $U'$ are $GL(V)$-invariant.
2. Prove that $V^\otimes 3 = S^3V \oplus U \oplus U' \oplus \Lambda^3V$.

Solution: We denote the map $p_{13}(p_2(v_1 \otimes v_2 \otimes v_3))$ by $q(v_1 \otimes v_2 \otimes v_3)$. Note that

$$q(v_1 \otimes v_2 \otimes v_3) = \frac{1}{4}(v_1 \otimes v_2 \otimes v_3 + v_3 \otimes v_2 \otimes v_1 - (v_2 \otimes v_1 \otimes v_3 + v_3 \otimes v_1 \otimes v_2)) = \frac{1}{4}((v_1 \wedge v_2) \otimes v_3 - v_3 \otimes (v_1 \wedge v_2))$$

First it is easy to verify that $w = q(e_1 \otimes e_2 \otimes e_1) = (e_1 \wedge e_2) \otimes e_1 - e_1 \otimes (e_1 \wedge e_2)$ is a weight vector of weight $(2, 1)$. Now we show that $\langle GL_n w \rangle = U$.

Notice that $q(e_1 \otimes e_2 \otimes e_1) = 2e_1 \otimes e_2 \otimes e_1 - e_2 \otimes e_1 \otimes e_1 - e_1 \otimes e_1 \otimes e_2$.

One can verify that $w_1 = q((v_1 + v_3) \otimes v_2 \otimes (v_1 + v_3)) - q(v_1 \otimes v_2 \otimes v_1) - q(v_3 \otimes v_2 \otimes v_3) = \frac{1}{4}(2(v_1 \otimes v_2 \otimes v_3 + v_3 \otimes v_2 \otimes v_1) - v_2 \otimes v_1 \otimes v_3 - v_2 \otimes v_3 \otimes v_1 - v_1 \otimes v_3 \otimes v_2 - v_3 \otimes v_1 \otimes v_2)$. By switching $v_2$ and $v_3$ in above equation, we get that $w_2 = q((v_1 + v_2) \otimes v_3 \otimes (v_1 + v_2)) - q(v_1 \otimes v_3 \otimes v_1) - q(v_2 \otimes v_3 \otimes v_2) = \frac{1}{4}(2(v_1 \otimes v_3 \otimes v_2 + v_2 \otimes v_3 \otimes v_1) - v_3 \otimes v_1 \otimes v_2 - v_2 \otimes v_3 \otimes v_1 - v_1 \otimes v_2 \otimes v_3 - v_2 \otimes v_1 \otimes v_3)$.

Now consider $2w_1 + w_2 = \frac{1}{3}(3(v_1 \otimes v_2 \otimes v_3 + v_3 \otimes v_2 \otimes v_1) - 3(v_2 \otimes v_1 \otimes v_3 + v_3 \otimes v_1 \otimes v_2)) = 3q(v_1 \otimes v_2 \otimes v_3)$. Thus $q(v_1 \otimes v_2 \otimes v_3) \in \langle GL_n w \rangle$, therefore $\langle GL_n w \rangle = U$. Thus $U$ is an irreducible $GL_n$ representation of type $(2, 1)$.

Similarly $U'$ can be shown to be of type $(2, 1)$, implying that $U$ and $U'$ are isomorphic $GL_n$ representations. We saw in lecture that dimension of $U$ is number of semi-standard tableaux of shape $(2, 1)$ with entries from $[n]$. Thus $\dim U = \dim U' = 2 \left( \binom{n}{3} + \binom{n}{2} \right)$. We also know that $\dim S^3V = \binom{n+2}{3}$ and $\dim \Lambda^3V = \binom{n}{3}$. Now check that

$$\dim S^3V + \dim \Lambda^3V + \dim U + \dim U' = \binom{n+2}{3} + \binom{n}{3} + 2 \left( \binom{n}{3} + \binom{n}{2} \right) = n^3 = \dim V^\otimes 3.$$ 

Thus $V^\otimes 3 = S^3V \oplus U \oplus U' \oplus \Lambda^3V$ is the irreducible decomposition of $V^\otimes 3$ under the specified action of $GL_n$.