Introduction to geometric complexity theory

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Abstract. Geometric complexity theory is an approach towards proving lower bounds in algebraic complexity theory via methods from algebraic geometry and representation theory. It was introduced by Mulmuley and Sohoni and has gained significant momentum over the last few years. Since deep methods from several different areas of mathematics are involved, geometric complexity theory has a steep learning curve. There are great survey articles on geometric complexity theory, but those require a significant level of mathematical maturity and often only sketch many of the proofs, see e.g. [Reg02], [Mul11], [BLMW11], [Gro12], [Ike12], [Lan13]. This survey tries to be a gentle introduction for graduate students and even advanced undergraduate students in computer science that requires almost no background knowledge except for the usual knowledge in linear algebra and some basic knowledge in analysis. All the necessary concepts from algebraic geometry and representation theory are introduced and almost all proofs are given. We focus on two questions, the permanent versus determinant problem and the border rank problem (for matrix multiplication). There have been many more results in the past few years, which we cannot cover, however, this survey should give the reader the necessary background to understand them. The survey culminates in two recent results, a negative one for the permanent versus determinant question and a positive one for the matrix multiplication problem. We present the proof that occurrence obstructions essentially cannot resolve the permanent versus determinant question. However, occurrence obstructions are the most basic tool of geometric complexity theory and it might be well possible that the more general concept of multiplicity obstructions will resolve the problem. On the other hand, as a proof of concept, we show that occurrence obstructions indeed can give lower bounds for the border rank of matrix multiplication.

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Contents

1 Boolean circuits and arithmetic circuits 5
  1.1 Introduction ................................................................. 5
  1.2 The non-uniform P ≠ NP question: NP ⊈ P/poly ....................... 5
  1.3 From boolean circuits to arithmetic circuits .......................... 7

2 Waring rank and border Waring rank 10
  2.1 Waring rank ................................................................. 10
  2.2 The discriminant ........................................................... 11
  2.3 Border Waring rank ...................................................... 11
  2.4 Closures and continuous separating functions ........................ 12
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Actions, orbits, and orbit closures</td>
<td>16</td>
</tr>
<tr>
<td>3.1</td>
<td>Monoid actions</td>
<td>16</td>
</tr>
<tr>
<td>3.2</td>
<td>Lifting the action to the function space</td>
<td>17</td>
</tr>
<tr>
<td>3.3</td>
<td>Orbits</td>
<td>19</td>
</tr>
<tr>
<td>3.4</td>
<td>Orbit closures</td>
<td>20</td>
</tr>
<tr>
<td>3.5</td>
<td>Group orbit closures</td>
<td>20</td>
</tr>
<tr>
<td>3.6</td>
<td>The orbit map</td>
<td>21</td>
</tr>
<tr>
<td>4</td>
<td>First algebraic geometry</td>
<td>23</td>
</tr>
<tr>
<td>4.1</td>
<td>Zariski-closure</td>
<td>23</td>
</tr>
<tr>
<td>4.2</td>
<td>Algebraic geometry of orbit closures</td>
<td>24</td>
</tr>
<tr>
<td>4.3</td>
<td>Cones</td>
<td>25</td>
</tr>
<tr>
<td>5</td>
<td>Algebraic complexity classes</td>
<td>27</td>
</tr>
<tr>
<td>5.1</td>
<td>VP</td>
<td>27</td>
</tr>
<tr>
<td>5.2</td>
<td>VP&lt;sub&gt;e&lt;/sub&gt;</td>
<td>28</td>
</tr>
<tr>
<td>5.3</td>
<td>Constant size iterated matrix multiplication</td>
<td>30</td>
</tr>
<tr>
<td>5.4</td>
<td>Orbit problems</td>
<td>31</td>
</tr>
<tr>
<td>6</td>
<td>VP and the determinant</td>
<td>34</td>
</tr>
<tr>
<td>6.1</td>
<td>Homogeneous circuits</td>
<td>34</td>
</tr>
<tr>
<td>6.2</td>
<td>Multiplicatively disjoint circuits</td>
<td>35</td>
</tr>
<tr>
<td>6.3</td>
<td>Combinatorial interpretation of the determinant</td>
<td>37</td>
</tr>
<tr>
<td>6.4</td>
<td>Weakly skew circuits and algebraic branching programs</td>
<td>37</td>
</tr>
<tr>
<td>7</td>
<td>The permanent</td>
<td>41</td>
</tr>
<tr>
<td>7.1</td>
<td>VNP and formulas</td>
<td>41</td>
</tr>
<tr>
<td>7.2</td>
<td>Hardness of the permanent</td>
<td>43</td>
</tr>
<tr>
<td>7.3</td>
<td>Valiant’s conjecture</td>
<td>47</td>
</tr>
<tr>
<td>8</td>
<td>Border complexity and group orbit closures</td>
<td>48</td>
</tr>
<tr>
<td>9</td>
<td>Representation Theory</td>
<td>50</td>
</tr>
<tr>
<td>9.1</td>
<td>Key example: Lifting the group action</td>
<td>50</td>
</tr>
<tr>
<td>9.2</td>
<td>General representation theory</td>
<td>51</td>
</tr>
<tr>
<td>10</td>
<td>Representation theory of finite groups and Maschke’s theorem</td>
<td>54</td>
</tr>
<tr>
<td>10.1</td>
<td>Maschke’s theorem</td>
<td>54</td>
</tr>
<tr>
<td>10.2</td>
<td>Search space restrictions</td>
<td>56</td>
</tr>
<tr>
<td>11</td>
<td>Irred. repr. of GL&lt;sub&gt;n&lt;/sub&gt;: First properties</td>
<td>57</td>
</tr>
<tr>
<td>11.1</td>
<td>Equivariant maps and isomorphisms</td>
<td>57</td>
</tr>
<tr>
<td>11.2</td>
<td>The algebraic torus and the weight decomposition</td>
<td>57</td>
</tr>
<tr>
<td>11.3</td>
<td>Highest weight vectors</td>
<td>60</td>
</tr>
<tr>
<td>11.4</td>
<td>Highest weight vector obstructions</td>
<td>63</td>
</tr>
<tr>
<td>12</td>
<td>Schur’s lemma, multipl., and isotypic decomp.</td>
<td>65</td>
</tr>
<tr>
<td>12.1</td>
<td>Schur’s lemma</td>
<td>65</td>
</tr>
<tr>
<td>12.2</td>
<td>Multiplicities</td>
<td>66</td>
</tr>
<tr>
<td>12.3</td>
<td>Isotypic components</td>
<td>67</td>
</tr>
<tr>
<td>12.4</td>
<td>Using multiplicities or occurrences as obstructions</td>
<td>68</td>
</tr>
<tr>
<td>13</td>
<td>Tensors for computer scientists</td>
<td>70</td>
</tr>
<tr>
<td>Chapter</td>
<td>Title</td>
<td>Pages</td>
</tr>
<tr>
<td>---------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>13.1</td>
<td>Bilinear forms</td>
<td>70</td>
</tr>
<tr>
<td>13.2</td>
<td>Universal property</td>
<td>71</td>
</tr>
<tr>
<td>13.3</td>
<td>Tensor rank</td>
<td>73</td>
</tr>
<tr>
<td>13.4</td>
<td>Actions on tensor products</td>
<td>73</td>
</tr>
<tr>
<td>14</td>
<td>Complexity of bilinear maps</td>
<td>75</td>
</tr>
<tr>
<td>14.1</td>
<td>Strassen’s algorithm</td>
<td>75</td>
</tr>
<tr>
<td>14.2</td>
<td>Relation to tensor rank</td>
<td>76</td>
</tr>
<tr>
<td>14.3</td>
<td>The exponent of matrix multiplication</td>
<td>76</td>
</tr>
<tr>
<td>14.4</td>
<td>Rank and restrictions</td>
<td>78</td>
</tr>
<tr>
<td>14.5</td>
<td>Permutations of matrix multiplication tensors</td>
<td>78</td>
</tr>
<tr>
<td>14.6</td>
<td>Products of matrix multiplication tensors</td>
<td>79</td>
</tr>
<tr>
<td>15</td>
<td>Border rank</td>
<td>81</td>
</tr>
<tr>
<td>15.1</td>
<td>Approximate computations</td>
<td>81</td>
</tr>
<tr>
<td>15.2</td>
<td>Properties of border rank</td>
<td>82</td>
</tr>
<tr>
<td>15.3</td>
<td>From approximate to exact computations</td>
<td>83</td>
</tr>
<tr>
<td>15.4</td>
<td>Degeneration</td>
<td>84</td>
</tr>
<tr>
<td>16</td>
<td>Symmetric and alternating tensors</td>
<td>85</td>
</tr>
<tr>
<td>16.1</td>
<td>$S^2V$ and $A^2V$</td>
<td>85</td>
</tr>
<tr>
<td>16.2</td>
<td>Symmetric tensors</td>
<td>86</td>
</tr>
<tr>
<td>16.3</td>
<td>Alternating tensors</td>
<td>86</td>
</tr>
<tr>
<td>17</td>
<td>Constr. of the irred. repr. of $GL_n$</td>
<td>89</td>
</tr>
<tr>
<td>17.1</td>
<td>Young tableaux</td>
<td>89</td>
</tr>
<tr>
<td>17.2</td>
<td>Construction as a quotient space</td>
<td>90</td>
</tr>
<tr>
<td>17.3</td>
<td>A more explicit quotient space</td>
<td>90</td>
</tr>
<tr>
<td>17.4</td>
<td>Sylvester’s lemma</td>
<td>91</td>
</tr>
<tr>
<td>17.5</td>
<td>An explicit basis of the Schur module</td>
<td>93</td>
</tr>
<tr>
<td>17.6</td>
<td>Highest weight vectors</td>
<td>94</td>
</tr>
<tr>
<td>18</td>
<td>The algebraic Peter-Weyl theorem</td>
<td>95</td>
</tr>
<tr>
<td>18.1</td>
<td>Regular functions</td>
<td>95</td>
</tr>
<tr>
<td>18.2</td>
<td>Invariants under the stabilizer</td>
<td>96</td>
</tr>
<tr>
<td>18.3</td>
<td>Algebraic Peter-Weyl theorem</td>
<td>97</td>
</tr>
<tr>
<td>18.4</td>
<td>The determinant and rectangular Kronecker coefficients</td>
<td>98</td>
</tr>
<tr>
<td>19</td>
<td>Explicit HWV constructions via Schur-Weyl duality</td>
<td>99</td>
</tr>
<tr>
<td>19.1</td>
<td>Specht modules</td>
<td>99</td>
</tr>
<tr>
<td>19.2</td>
<td>Explicit Schur-Weyl duality</td>
<td>100</td>
</tr>
<tr>
<td>19.3</td>
<td>Polynomials as symmetric tensors: Plethysms</td>
<td>101</td>
</tr>
<tr>
<td>20</td>
<td>Tensor contraction</td>
<td>108</td>
</tr>
<tr>
<td>20.1</td>
<td>Contracting highest weight vectors in plethysms with rank one tensors</td>
<td>108</td>
</tr>
<tr>
<td>20.2</td>
<td>Applications: Waring rank and a proof of Weintraub’s conjecture</td>
<td>109</td>
</tr>
<tr>
<td>20.3</td>
<td>Application: Obstructions require long first rows</td>
<td>111</td>
</tr>
<tr>
<td>21</td>
<td>Good occ. obstructions for det vs per do not exist</td>
<td>112</td>
</tr>
<tr>
<td>21.1</td>
<td>The degree lower bound</td>
<td>112</td>
</tr>
<tr>
<td>21.2</td>
<td>No occurrence obstructions</td>
<td>115</td>
</tr>
<tr>
<td>22</td>
<td>Occurrence obstructions for matrix multiplication</td>
<td>123</td>
</tr>
</tbody>
</table>
Chapter 1

Boolean circuits and arithmetic circuits

1.1 Introduction

Computational complexity theory is concerned with the study of the inherent complexity of computational problems. Its flagship conjecture is the famous $P \neq NP$ conjecture, which is one of the seven Millenium Problems of the Clay Mathematics Institute [Coo00], ranking this conjecture as the most important one at the intersection of mathematics and theoretical computer science. To this day several thousands of computational problems are classified as NP-complete, i.e., they have a polynomial time algorithm if $P = NP$. The practical importance of the $P \neq NP$ conjecture is at least twofold: First of all, many NP-complete problems are of high practical relevance and have to be solved every day in commercial and scientific applications, for example the traveling salesman problem, integer programming, facility location, subset sum, knapsack, longest path, multiprocessor scheduling, tensor rank, to name a few. Secondly, all current security notions in cryptography heavily rely on $P \neq NP$. Indeed, $P = NP$ would break all existing cryptographic ciphers. A lot of effort by many researchers has been put into resolving the $P \neq NP$ conjecture, but progress has been slow, see for example [For09] for a survey.

To attack the $P \neq NP$ conjecture with algebraic methods Valiant [Val79] introduced an intriguing algebraization of the boolean complexity model called algebraic complexity theory. On top of this, Mulmuley and Sohoni built what is now called geometric complexity theory. Geometric complexity theory is an approach towards computational complexity lower bounds questions via methods from algebraic geometry and representation theory. It has gained significant momentum over the last few years, but it has a steep learning curve which is a result of the many different areas of mathematics involved. This course tries to be a gentle introduction that requires almost no background knowledge. There are great survey articles on geometric complexity theory, but those require a significant level of mathematical maturity and often only sketch many of the proofs, see e.g. [Reg02], [Mul11], [BLMW11], [Gro12], [Ike12], [Lan13].

1.2 The non-uniform $P \neq NP$ question: $NP \not\subseteq P/poly$

To make our lifes easier we will not directly discuss the $P$ vs NP question, but its so-called non-uniform analogue, i.e., its circuit complexity version. If the non-uniform analogue is true, then also $P \neq NP$.

We start with the basic definition of a circuit.
1.2.1 Definition (Circuit). Fix a set $\mathbb{F}$ (in our case $\mathbb{F} = \{0, 1\}$ or the set $\mathbb{C}$ of complex numbers) and a set $S = \{s_i\}$ of functions $s_i$ of arbitrary arity $a_i$, i.e., $s_i$ maps from $\mathbb{F}^{a_i}$ to $\mathbb{F}$, where each $a_i \in \mathbb{N}_{\geq 1}$. (For example, for boolean circuits, choose $\mathbb{F} = \mathbb{F}_2$ and $S = \{\text{and, or, not}\}$). A circuit $C$ is a directed graph (abbreviated digraph) that contains no directed cycle such that the following properties hold (see Figure 1.1):

- A subset of the vertices with indegree 0 is labeled by indeterminates. These vertices are called the input gates. The other vertices with indegree 0 are labeled with elements of $\mathbb{F}$ and are called constant gates. All other vertices are called computation gates. A computation gate with outdegree 0 is called an output gate.

- Each computation gate $g$ is labeled with a function $s_i \in S$ with arity $a_i$ coinciding with the indegree of $g$.

![Figure 1.1](image_url)

Let $m$ be the number of input gates of a circuit $C$. Since by definition circuits contain no directed cycle, for each output gate $g$ we can define a function $C^g : \mathbb{F}^m \rightarrow \mathbb{F}$ in the natural way by induction over the structure of the digraph as follows: For each input gate $g$ labeled with a constant $\alpha$ we define $C^g$ to be the constant function $\alpha$. For each input gate $g$ labeled with a variable $X_j$ we define $C^g(x_1, \ldots, x_m) = x_j$. For a computation gate $g$ with label $s$ and parents $g_1^1, \ldots, g^s$ we define $C^g(x_1, \ldots, x_m) = s(g_1^1(x_1, \ldots, x_m), \ldots, g^s(x_1, \ldots, x_m))$.

We say that the functions $C^g$ on the output gates $g$ of $C$ are computed by $C$. We call a circuit a single-output circuit, if it has only one output gate and in this case $C : \mathbb{F}^m \rightarrow \mathbb{F}$ denotes the function of the output gate. The size $|C|$ of a circuit $C$ is defined to be the number of its vertices.

1.2.2 Definition. The circuit complexity $c_{\mathbb{F}, S}(h)$ of a function $h : \mathbb{F}^m \rightarrow \mathbb{F}$ is the minimal size of a circuit $C$ computing $h$.

1.2.3 Remark. In the literature, sometimes the number of edges or the number of computation gates is used as the definition of circuit complexity. In most contexts this does not make a significant difference.

A boolean circuit is defined to be a circuit with $\mathbb{F} = \mathbb{F}_2$ and $S = \{\text{and, or, not}\}$. When we speak of a univariate polynomial, we mean a polynomial in one variable with real coefficients. The polynomials which we discuss in later chapters will be multivariate and will have complex coefficients. Those serve a completely different purpose and need to be distinguished from their univariate namesakes.

1.2.4 Definition. A sequence $(n_m)_{m \in \mathbb{N}}$ of natural numbers is called polynomially bounded if there exists a univariate polynomial $q$ such that for all $m \in \mathbb{N}$ we have $n_m \leq q(m)$. 
For a sequence of functions \((h_m)\) we obtain a sequence of natural numbers \(c_{F,S}(h_m)\). Formally a family of objects is the same as a sequence of objects. We use the word family when we are interested in the sequence of complexity values.

1.2.5 Definition. Fix \(F := \mathbb{F}_2\) and \(S = \{\text{and, or, not}\}\). The class \(P/poly\) consists of all function families \((h_m)\) with \(h_m : F^m \rightarrow \mathbb{F}\) whose complexity sequence \(c_{F,S}(h_m)\) is polynomially bounded.

1.2.6 Example. Let \(h_m : \{0,1\}^m \rightarrow \{0,1\}\) denote the palindrome function: \(h_m(w) = 1\) iff \(w_i = w_{m+1-i}\) for all \(1 \leq i \leq m\). It is easy to construct a boolean circuit that computes \(h_m\) whose size is polynomially bounded in \(m\).

More generally, for computer scientists, take any language \(L \subseteq \{0,1\}^*\) in \(P\). Then define the function \(h_m : \{0,1\}^m \rightarrow \{0,1\}\) to be the indicator function of \(L\) restricted to input words of length exactly \(m\). Then \((h_m) \in P/poly\). In other words \(P \subseteq P/poly\). This result is a bit technical and we will not discuss it any further.

1.2(i) SAT and \(NP\)

We do not define the class \(NP\) here, but we define the \(NP \not\subseteq P/poly\) conjecture via the satisfiability function. The technical details in this subsection are only used locally.

A boolean formula is a finite character string consisting of variables \(x(1), x(2), x(3), \ldots\) and parantheses symbols \((, )\), as well as the classical logical junctors \(\text{and, or, not}\). For example:

\[(x(1) \text{ and not } x(3)) \text{ or not}(x(1) \text{ or } x(4)) \text{ or } x(2)\]

is a boolean formula. A boolean formula is called satisfiable if we can replace every variable \(x(i)\) by either \(\text{true}\) or \(\text{false}\) such that the resulting statement is true. In our example, one of these assignments would be \(x(1) = \text{true}, x(2) = \text{true}, x(3) = \text{true}, x(4) = \text{false}\), and hence the boolean formula is satisfiable. We fix any reasonable way of encoding boolean formulas as finite bit strings, so, for example, we could choose 0000 = 0, 0001 = 1, 0010 = 2, ..., 1001 = 9, 1010 = \(x\), 1011 = (, 1100 = ), 1101 = \text{and}, 1110 = \text{or}, 1111 = \text{not}. For example the boolean formula \(x(2)\text{ and } x(3)\) is represented by 10101011001101101011010110011100. Using this encoding we can interpret the set of satisfiable boolean formulas as a subset of the set of finite length bit strings. Now we are ready to define the satisfiability problem. Let \((h_m)\) be the sequence of functions \(h_m : \{0,1\}^m \rightarrow \{0,1\}\) defined by the property

\[h_m(w) = 1\] iff \(w \in \{0,1\}^m\) encodes a satisfiable boolean formula.

The \(NP \not\subseteq P/poly\) conjecture can be stated as \((h_m) \not\in P/poly\), or equivalently as

the sequence of boolean circuit complexities \(c_{F_S}(h_m)\) is not polynomially bounded.

1.3 From boolean circuits to arithmetic circuits

If we interpret the set \(F_2 = \{0,1\}\) as the set of cosets modulo 2, then we see that besides the boolean operations the set \(F_2\) is also a ring (even a field) and hence has an addition and a multiplication operation. The operation tables look as follows:

\[
\begin{array}{c|cc}
+ & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cc}
* & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 1 \\
\end{array}
\]

The following rephrased version of Definition 1.2.5 has a more algebraic flavor.
1.3.1 Proposition (Arithmetic characterization of $\text{P/poly}$). Let $F := \mathbb{F}_2$ and let $S := \{+, \ast\}$, where “+” and “\ast” have arity 2 and represent the addition and multiplication. The class $\text{P/poly}$ consists of all families $(h_m)$ of functions $h_m : F^m \rightarrow F$ whose circuit complexity sequence $c_{F,S}(h_m)$ is polynomially bounded.

Proof. For $F = \mathbb{F}_2$, a circuit using $S = \{\text{and, or, not}\}$ can be converted into a circuit using $S = \{+, \ast\}$ and vice versa by replacing gates with subcircuits of constant size, as follows:

- $X$ and $Y = X \ast Y$
- $\text{not } X = X + 1$
- $X \text{ or } Y = X \ast Y + X + Y$
- $X + Y = X \text{ xor } Y = (X \text{ or } Y) \text{ and not}(X \text{ and } Y)$

Circuits where $F$ is a field and $S$ is the set $\{+, \ast\}$ of arithmetic operations are called arithmetic circuits over $F$. Computation gates labeled with $+$ are called addition gates and computation gates labeled with $\ast$ are called multiplication gates.

Given the characterization from Proposition 1.3.1, it is straightforward to work over other rings than $F_2$. Infinite fields for example have a big advantage, as we will see in Lemma 1.3.2. For a fixed ring $F$, single-output arithmetic circuits with $m$ input gates not only naturally compute a function $F^m \rightarrow F$, but they also compute a polynomial in the polynomial ring $F[X_1, \ldots, X_m]$ in $m$ variables by induction on the circuit structure, see Figure 1.2. Two polynomials are considered to be equal if for each monomial their corresponding coefficients coincide. Single-output arithmetic circuits over $F_2$ that compute different polynomials can compute the same function, as the following small example shows: Let $h_1(X,Y) = X^2Y$ and $h_2(X,Y) = XY^2$. Clearly $h_1$ and $h_2$ are different polynomials, but as functions they coincide:

$$\forall x \in (F_2)^2 : h_1(x) = h_2(x).$$

This is a cumbersome subtlety which does not arise over infinite fields.

1.3.2 Lemma. Let $F$ be an infinite field. Then for two polynomials $h_1, h_2 \in F[X_1, X_2, \ldots, X_m]$ we have

$$h_1 = h_2 \text{ as polynomials} \iff \text{for all } x \in F^m \text{ we have } h_1(x) = h_2(x).$$

![Figure 1.2: A circuit computing the polynomial $X^3 + X^2Y - XY^2 - Y^3$. Here $F = \mathbb{C}$ and $S = \{\ast, +\}$.](image)
Proof. We show by induction that a polynomial that vanishes on the whole \( F^m \) is the zero polynomial. For \( m = 1 \) the result follows easily from successive polynomial division by linear factors: A nonzero degree \( d \) polynomial cannot have more than \( d \) zeros. For \( m > 1 \) we can decompose every \( h \) that vanishes on \( F^m \) as

\[ h = \sum_{i=0}^{\deg h} g_i X_{m}^{i} \in F[X_1, \ldots, X_{m-1}][X_m]. \]

Fix a point \( (x_1, \ldots, x_{m-1}) \in F^{m-1} \). Define \( p(y) := h(x_1, \ldots, x_{m-1}, y) = \sum_{i=0}^{\deg h} g_i(x_1, \ldots, x_{m-1}) y^i \in F[y] \). Note that \( p(y) \) vanishes on \( F \) and hence \( p \) is the zero polynomial. But the coefficients of \( p \) are the \( g_i(x_1, \ldots, x_{m-1}) \). Thus equating coefficients of \( p \) yields that for all \( i \) we have \( g_i(x_1, \ldots, x_{m-1}) = 0 \). Since the point \( (x_1, \ldots, x_{m-1}) \) was chosen arbitrarily, all \( g_i \) vanish on the whole \( F^{m-1} \). By induction hypothesis each \( g_i \) is the zero polynomial. Therefore \( h \) is the zero polynomial.

In the light of Lemma 1.3.2 we see that if we are working over an infinite field we can focus on the polynomials computed by arithmetic circuits instead of the functions computed by them.

Our field of choice will be the complex numbers \( \mathbb{C} \) from now on.

1.3.3 Definition. The arithmetic complexity \( L(h) \) of a polynomial \( h \) is the size of the smallest single-output arithmetic circuit computing \( h \).

1.3.4 Example. \( L(\prod_{i=1}^{m} X_i) = \mathcal{O}(m) \).

1.3.5 Example. \( L(\sum_{i=1}^{m} (X_i)^m) = \mathcal{O}(m \log_2(m)) \) using the repeated squaring algorithm.

Let \( \mathfrak{S}_m \) denote the symmetric group on \( m \) letters, i.e., the group of bijective maps \( \{1, \ldots, m\} \to \{1, \ldots, m\} \). We define the permanent polynomial as follows:

\[
\text{per}_m := \sum_{\pi \in \mathfrak{S}_m} \prod_{i=1}^{m} X_{i, \pi(i)} \in \mathbb{C}[X_{1,1}, X_{1,2}, \ldots, X_{m,m}]
\]

Notice the striking similarity to the determinant:

\[
\text{det}_m := \sum_{\pi \in \mathfrak{S}_m} \text{sgn}(\pi) \prod_{i=1}^{m} X_{i, \pi(i)} \in \mathbb{C}[X_{1,1}, X_{1,2}, \ldots, X_{m,m}],
\]

where \( \text{sgn}(\pi) \in \{-1, 1\} \) denotes the sign of the permutation \( \pi \).

Computing the permanent of a matrix is NP-hard, while determinants of matrices can be efficiently computed using Gaussian elimination. We postpone the definition of the Valiant’s complexity classes VP and VNP, but Valiant’s famous VP \( \neq \) VNP conjecture can be stated as

the sequence \( L(\text{per}_m) \) is not polynomially bounded.

A simplification via algebra

Proving circuit complexity lower bounds for boolean functions is difficult. Replacing the base field \( \mathbb{F}_2 \) with \( \mathbb{C} \) lets us study polynomials instead. Valiant’s famous conjecture, a conjecture similar to \( P \neq NP \), says that \( L(\text{per}_m) \) is not polynomially bounded.

This is a first step towards a rich set of algebraic tools that will become available in later chapters.
Chapter 2

Waring rank and border

2.1 Waring rank

We start our study of arithmetic circuits with a very special case of circuits. For this we use a new gate in our arithmetic circuits: raising a polynomial to some fixed power $d$. We call these gates degree $d$ powering gates. A circuit is layered if we can assign to each gate a natural number (its layer) so that edges from gates in layer $i$ only go to gates in layer $i + 1$. Also for our addition gates we allow arbitrarily high arities.

2.1.1 Definition. A layered arithmetic circuit $C$ is called a $\Sigma \Lambda^d \Sigma_{\text{hom}}$-circuit if $C$ is a tree of depth 4 whose leafs are variables and constants, the next layer are multiplication gates whose parents are exactly one variable and one constant, the next layer are addition gates with arbitrary arity, the next layer are degree $d$ powering gates, the last layer is a single summation gate of arbitrary arity. The size of $C$ is defined as the number of powering gates.

A polynomial in many variables is called multivariate. To each monomial we assign a degree, which is the sum of its exponents. For example $\deg(XY^2) = 3$ and $\deg(X^2Y^3Z) = 6$. If all monomials of a polynomial $h$ have the same degree $d$, then we say that $h$ is homogeneous of degree $d$. For example, $XY + 3X^2$ is homogeneous (of degree 2), but $XY + X + 1$ is not homogeneous. Constants are homogeneous of degree 0. The zero polynomial is homogeneous of all degrees. The permanent $\text{per}_m$ is homogeneous of degree $m$.

Homogeneous polynomials of degree $d$ are sometimes called forms. In particular homogeneous degree 1 polynomials are called linear forms, but it is less ambiguous to say homogeneous linear form.

For any fixed set of variables, the set of homogeneous degree $d$ polynomials forms a vector space that we denote by $\mathbb{C}[X_1, \ldots, X_m]_d$. Moreover, the degree function makes the polynomial ring $\mathbb{C}[X_1, \ldots, X_m]$ a graded algebra. We observe that each $\Sigma \Lambda^d \Sigma_{\text{hom}}$-circuit computes a homogeneous degree $d$ polynomial.

2.1.2 Claim. Equivalently, we can allow the leafs of the circuits in Definition 2.1.1 to be labeled with homogeneous linear forms, followed by degree $d$ powering gates and then a single addition gate. Explicitly writing these linear forms as sums of scalar multiples of variables does not change the size of the circuit.

Proof. The complexity is defined as the number of powering gates, which does not change when replacing the computation of the homogeneous linear forms with just a leaf whose label is the homogeneous linear form or vice versa.

10
2.1.3 Definition. For a homogeneous degree $d$ polynomial $h$, the Waring rank is defined as the smallest size of a $\Sigma\Lambda^{d}\Sigma^{\text{hom}}$-circuit computing $h$. Alternatively, the Waring rank of $h$ is the smallest number of summands such that $h$ can be expressed as a sum of $d$-th powers of homogeneous linear forms.

The Waring rank is an important quantity in classical algebraic geometry, also called symmetric rank.

2.1.4 Example. $XY = (\frac{X}{2} + \frac{Y}{2})^2 + (i\frac{X}{2} - i\frac{Y}{2})^2$, therefore the Waring rank of $XY$ is at most 2.

2.1.5 Example (taken from a presentation by Luke Oeding in 2017). Let $h := X^2Y \in \mathbb{C}[X,Y]_3$. Then

$$6h = (X + Y)^3 + (-X + Y)^3 + (\sqrt[3]{2} Y)^3.$$ 

There is no better way: The Waring rank of $6h$ is 3. The following lemma shows that indeed the Waring rank of $h$ is 3.

2.1.6 Lemma. Waring rank is invariant under nonzero rescaling: $h$ and $\alpha h$ have the same Waring rank for $\alpha \neq 0$.

Proof. Let $h$ be of degree $d$ with Waring rank $n$ and let $C$ be the smallest $\Sigma\Lambda^{d}\Sigma^{\text{hom}}$-circuit computing $h$. Let $0 \neq \alpha \in \mathbb{C}$. We rescale all the homogeneous linear forms at the leafs with $\sqrt[d]{\alpha}$ to obtain a circuit of the same size computing $\alpha h$. For the other direction we apply the same argument, but we rescale by $\sqrt[d]{\alpha^{-1}}$.

Here we see again how convenient the choice of $\mathbb{C}$ as a base field is: $d$th roots of numbers are guaranteed to exist.

2.2 The discriminant

How can we prove a Waring rank complexity lower bound for specific $h$? Consider the case of two variables $X$ and $Y$. We study homogeneous degree 2 polynomials. The set of these polynomials is denoted by $\mathbb{C}[X,Y]_2$. Every $h \in \mathbb{C}[X,Y]_2$ can be written as

$$h = aX^2 + bXY + cY^2.$$ 

In high school we studied the case when $Y = 1$ and we learned that $aX^2 + bX + c$ has a double root iff $b^2 - 4ac = 0$. Note that $aX^2 + bX + c$ has a double root $\alpha$ iff $aX^2 + bX + c = (X - \alpha)^2$.

The same holds here:

There exist scalars $\alpha, \beta \in \mathbb{C}$ with $aX^2 + bXY + cY^2 = (\alpha X + \beta Y)^2$ iff $b^2 - 4ac = 0$.

This is a first example of a general method to prove complexity lower bounds.

For instance, expressing $XY = aX^2 + bXY + cY^2$ we obtain $a = c = 0$ and $b = 1$, so that $b^2 - 4ac = 1 \neq 0$. Therefore the Waring rank of $XY$ is at least 2. This means that the $\Sigma\Lambda^{d}\Sigma^{\text{hom}}$-circuit in Example 2.1.4 is optimal.

2.3 Border Waring rank

As in Example 2.1.5 let $h := X^2Y \in \mathbb{C}[X,Y]_3$. The Waring rank of $h$ is 3, but:

$$\lim_{\varepsilon \to 0} \left( \frac{1}{3\varepsilon} X^3 + \frac{1}{3\varepsilon} (\varepsilon Y - X)^3 \right) = \lim_{\varepsilon \to 0} \left( X^2Y - \varepsilon XY^2 + \frac{1}{3\varepsilon^2} Y^3 \right) = X^2Y.$$ (2.3.1)
CHAPTER 2. WARING RANK AND BORDER WARING RANK

So there is a curve of polynomials of Waring rank \( \leq 2 \) that converges to \( h \).

Let us formally define what this means. Let \( A := \mathbb{C}[X_1, \ldots, X_N] \). This is a finite dimensional vector space with \( \dim A = \binom{N+r-1}{n} \). Every element \( h \in A \) can be written as

\[
h = \sum_{\lambda \in \mathbb{N}^N \mid |\lambda| = n} \alpha_{\lambda} X_1^{\lambda_1} \cdots X_N^{\lambda_N}
\]

for some constants \( \alpha_{\lambda} \). We define the norm or length of a polynomial \( h \)

\[
|h| := \sum_{\lambda \in \mathbb{N}^N \mid |\lambda| = n} |\alpha_{\lambda}|
\]

It is easy to check that this satisfies the axioms of a norm \((|h| \geq 0, |h| = 0 \iff h = 0, |\alpha h| = |\alpha| \cdot |h|, |h_1 + h_2| \leq |h_1| + |h_2|)\).

The distance between \( h_1 \in A \) and \( h_2 \in A \) is defined as \( \text{dist}(h_1, h_2) := |h_1 - h_2| \). This satisfies the axioms of a metric.

For example,

\[
\text{dist}(2X_1^2, 2X_1^2 + \frac{1}{100}X_1X_2) = |2 - 2| + |0 - \frac{1}{100}| = \frac{1}{100}.
\]

The triangle inequality can be written as:

\[
\text{dist}(h_1, h_3) \leq \text{dist}(h_1, h_2) + \text{dist}(h_2, h_3).
\]

2.3.5 Definition. Let \( A \) be a finite dimensional complex vector space and let \( h \in A \). We say that a sequence \((h_i)_i\) with all \( h_i \in A \) converges to \( h \in A \), if

\[
\forall \varepsilon \in \mathbb{R}_{>0} \exists i_0 \in \mathbb{N} \forall i > i_0 : \text{dist}(h - h_i) < \varepsilon.
\]

In this case we write \( \lim_{i \to \infty} h_i = h \) and say that \( h \) is the limit of the sequence \((h_i)_i\). A sequence for which a limit exists is called convergent.

(2.3.1) is an example of a convergent sequence if we set \( \varepsilon := \frac{1}{100} \). It is easy to see that every convergent sequence has a unique limit.

2.3.6 Remark. Choosing different norms in (2.3.3) has no effect on the convergence behaviour of sequences. The limits stay the same.

2.3.7 Definition. For a homogeneous degree \( d \) polynomial \( h \) the border Waring rank is defined as the smallest \( n \) such that \( h \) is the limit of a sequence of polynomials of Waring rank \( \leq n \).

In Example 2.3.2 we see that the border Waring rank of \( X^2Y \) is at most 2. Clearly, the border Waring rank of \( h \) cannot exceed the Waring rank of \( h \), because for all \( h \) the constant sequence \((h, h, \ldots)\) converges to \( h \).

2.4 Closures and continuous separating functions

We want to show complexity lower bounds using functions like the discriminant. But in this section we see that continuous functions cannot distinguish between Waring rank and border Waring rank. On the other hand we discover that if we search for functions that prove lower bounds on border Waring rank, then we can restrict our search to continuous functions only. We will see later that we can restrict our search space significantly further using algebraic geometry and representation theory.

We will use a very simple definition of continuity:
2.4.1 Definition. A function \( f: \mathbb{A} \to \mathbb{A}' \) between two finite dimensional metric spaces is called continuous if for every convergent sequence \( (h_i)_i \) in \( \mathbb{A} \), the sequence \( f(h_i) \) converges in \( \mathbb{A}' \) to \( f(\lim_{i \to \infty} h_i) \).

2.4.2 Claim. Functions defined by multivariate polynomials are continuous. (“Multivariate polynomials are continuous”).

Proof sketch. It is easy to see that for all \( 1 \leq k \leq N \) the coordinate function \( f_k: \mathbb{C}^N \to \mathbb{C} \), \( (T_1, \ldots, T_N) \mapsto T_k \) is continuous. Moreover, it is not hard to derive the facts that finite products and finite sums of continuous functions are continuous (here the proof for products is only slightly more involved). It follows by induction that all multivariate polynomials \( f \in \mathbb{C}[T_1, \ldots, T_N] \) are continuous.

2.4.3 Example. Claim 2.4.2 implies that the discriminant \( b^2 - 4ac \) is continuous.

We will now see that if we use continuous functions to prove Waring rank lower bounds, then we actually prove border Waring rank lower bounds. Moreover, if we want to prove border Waring rank lower bounds, then we can restrict our search for separating functions to continuous functions. As a first step we reformulate border Waring rank in the language of \( \mathbb{C} \)-closures.

2.4.4 Definition. Given a (not necessarily linear) subset \( W \subseteq \mathbb{A} \), the \( \mathbb{C} \)-closure \( \overline{W} \) in \( \mathbb{A} \) is defined as the set of the limits of all convergent sequences whose elements are taken from \( W \).

2.4.5 Example. Consider \( \mathbb{C} \setminus \{0\} \subseteq \mathbb{A} \). Then \( \overline{\mathbb{C} \setminus \{0\}} = \mathbb{C} \).

Clearly, \( W \subseteq \overline{W} \), because for all \( h \in W \), the constant sequence \( (h, h, \ldots) \) converges to \( h \). Moreover, if \( V \subseteq W \), then \( \overline{V} \subseteq \overline{W} \), because every sequence with elements from \( V \) is also a sequence with elements from \( W \). Using the definition above, we see that the set of border Waring rank \( \leq n \) polynomials is the \( \mathbb{C} \)-closure of the set of Waring rank \( \leq n \) polynomials, that is,

\[
\{ h \in \mathbb{C}[X_1, \ldots, X_M]_m \mid \text{border Waring rank}(h) \leq n \} = \{ h \in \mathbb{C}[X_1, \ldots, X_M]_m \mid \text{Waring rank}(h) \leq n \}.
\]

2.4.6 Definition. A subset \( W \subseteq \mathbb{A} \) is called \( \mathbb{C} \)-closed in \( \mathbb{A} \), if \( \overline{W} = W \), i.e., the limit of every convergent sequence \( (h_i)_i \), with \( h_i \in W \) is contained in \( W \).

2.4.7 Lemma. Let \( W \subseteq \mathbb{A} \) be any subset. After taking the \( \mathbb{C} \)-closure in \( \mathbb{A} \) once, taking the \( \mathbb{C} \)-closure in \( \mathbb{A} \) again has no additional effect: \( \overline{\overline{W}} = \overline{W} \). In particular, \( \mathbb{C} \)-closures in \( \mathbb{A} \) are \( \mathbb{C} \)-closed in \( \mathbb{A} \).

Proof. Clearly \( \overline{W} \subseteq \overline{\overline{W}} \). Let \( h \in \overline{W} \) be arbitrary and let \( (h_i)_i \) denote a sequence converging to \( h \) with \( h_i \in \overline{W} \), i.e., for each \( i \) there exists a sequence \( (h_{i,j})_j \) such that \( \lim_{j \to \infty} h_{i,j} = h_i \) and the \( h_{i,j} \) are elements of \( W \). Let \( h_i(\varepsilon) \) denote the first entry in \( (h_{i,j})_j \) such that the distance between \( h_{i,j} \) and \( h_i \) is less than \( \varepsilon \). Taking \( \varepsilon = \frac{1}{i} \), it follows that the sequence \( (h_i(\frac{1}{i}))_i \), converges to \( h \) and all elements \( h_i(\frac{1}{i}) \) are taken from \( W \). Therefore \( h \in \overline{W} \).

The next Proposition 2.4.8 shows that continuous functions cannot distinguish between a set and its \( \mathbb{C} \)-closure. We will see in Lemma 2.4.9 that continuous functions are exactly those functions that can be used to distinguish points from \( \mathbb{C} \)-closed sets. Note that this is exactly what we need for proving border Waring rank lower bounds.

2.4.8 Proposition. A continuous function \( f: \mathbb{A} \to \mathbb{C} \) vanishes on a set \( W \subseteq \mathbb{A} \) iff \( f \) vanishes on the \( \mathbb{C} \)-closure \( \overline{W} \).

Proof. Since \( W \subseteq \overline{W} \), one direction is clear. Let \( h \in \overline{W} \setminus W \). Let \( h_i \) be a sequence in \( W \) with \( \lim_{i \to \infty} h_i = h \). Then \( f(h_i) = 0 \) and since \( f \) is continuous we have \( f(\lim_{i \to \infty} h_i) = \lim_{i \to \infty} f(h_i) = \lim_{i \to \infty} 0 = 0 \).\( \square \)
2.4.9 Lemma. Let $\mathbb{W} \subseteq \mathbb{A}$ be a $\mathbb{C}$-closed set. Let $h \in \mathbb{A}$. We have $h \notin \mathbb{W}$ iff there exists a continuous function $f: \mathbb{A} \to \mathbb{C}$ such that $f$ vanishes on $\mathbb{W}$ and $f(h) \neq 0$.

2.4.10 Remark. Note that the statement of Lemma 2.4.9 is trivial if we drop the requirement of $f$ being continuous. However, it is also absolutely useless then.

Proof. For the easy direction, we see that the existence of an $f$ that vanishes on $\mathbb{W}$ with $f(h) \neq 0$ clearly implies $h \notin \mathbb{W}$. For the other direction we need to construct an obstruction $f$ against $h \in \mathbb{W}$. Intuitively $f$ is the distance function to $\mathbb{W}$. We define the distance $f$ of $h$ to $\mathbb{W}$ to be the infimum

$$ f(h) := \inf \{ \text{dist}(h, h_1) \mid h_1 \in \mathbb{W} \}, $$

which is the largest $\alpha \in \mathbb{R}$ such that for all $h_1 \in \mathbb{W}$ we have $\text{dist}(h, h_1) \geq \alpha$. Clearly if $h \in \mathbb{W}$, then $f(h) = 0$. Therefore if $f(h) \neq 0$ we have $h \notin \mathbb{W}$. For the other direction we have to show that every $h$ with $f(h) = 0$ lies in $\mathbb{W}$. Let $h$ satisfy $f(h) = 0$. This means that there exists a sequence $(h_i)$ of elements of $\mathbb{W}$ such that the distance sequence $\text{dist}(h, h_i)$ converges to 0. Therefore $\lim_{i \to \infty} h_i = h$, but since $\mathbb{W}$ is $\mathbb{C}$-closed in $\mathbb{A}$ it follows that $h \in \mathbb{W}$.

It remains to show that $f$ is continuous. It is sufficient to show that for all $h$ and $h_1$, we have

$$ |f(h) - f(h_1)| \leq \text{dist}(h, h_1), $$

because for a sequence $h_i$ converging to $h$ this implies $\lim_{i \to \infty} f(h_i) = f(h)$.

Let $h_2 \in \mathbb{W}$ be arbitrary. Then by (2.3.4) we have $\text{dist}(h, h_2) \leq \text{dist}(h, h_1) + \text{dist}(h_1, h_2)$ and therefore $f(h) \leq \text{dist}(h, h_1) + \text{dist}(h_1, h_2)$ or in other words $\text{dist}(h, h_1) \geq f(h) - \text{dist}(h_1, h_2)$. Since $h_2$ was arbitrary we obtain $\text{dist}(h, h_1) \geq f(h) - f(h_1)$. Note that we here used the fact that if a non-strict inequality holds for a subset of the real numbers, then it also holds for its infimum. Reversing the roles of $h$ and $h_1$ we obtain $\text{dist}(h, h_1) \geq f(h_1) - f(h)$ and therefore $\text{dist}(h, h_1) \geq |f(h) - f(h_1)|$. □

If we want to prove border Waring rank lower bounds, then we can restrict our search for separating functions to continuous functions. The proofs did not involve anything specific about Waring rank, so this holds in far higher generality. Using some algebraic geometry we will see later that we can restrict our search further to homogeneous polynomials (for example the discriminant is a homogeneous polynomial). Using some representation theory we will restrict the search space even further to homogeneous polynomials in irreducible representations.

A general approach to lower bounds

Proving lower bounds means that we have a set of functions $W$ (the “easy” functions) and we want to prove that some function $h$ is not in $W$ (a “hard” function). One approach is to find a function $f$—defined on the space of functions that we consider—that vanishes on $W$ but $f(h) \neq 0$. While finding any such $f$ is as hard as showing that $h \notin W$, we can restrict our search to “nice” functions $f$. Here “nice” means continuous. You should be aware that with this approach, we can only prove that $h \notin W$. If we are unlucky, $h \in \mathbb{W} \setminus W$ and we will never be able to prove this with this approach.

1Think of an obstruction as a function that disproves $h \in \mathbb{W}$.
Example: Waring rank

The Waring rank of a homogeneous polynomial $h$ of degree $d$ is the smallest number of summands such that $h$ can be expressed as a sum of $d$-th powers of homogeneous linear forms.

The smallest $r$ such that $h$ is the limit of a sequence of polynomials of Waring rank $\leq r$ is the border Waring rank.

The discriminant $b^2 - 4ac$ is a polynomial that vanishes on all polynomials $aX^2 + bXY + cY^2 \in \mathbb{C}[X,Y]_2$ of border Waring rank 1.
Chapter 3

Actions, orbits, and orbit closures

Complexity lower bounds are about separating points $h$ from $C$-closures $W$ by using functions $f$ that vanish on $W$ but not on $h$. In the previous chapter we saw that since $W$ is $C$-closed, we can restrict our search to continuous functions $f$ only. In this chapter we find more properties of $W$ that will help to reduce the search space for $f$ even further: Our sets $W$ are group orbit closures.

As our main example we consider the Waring rank problem: We express the Waring rank problem as a monoid orbit problem and the border Waring rank problem as an orbit closure problem.

3.1 Monoid actions

For a set $V$, let $V \rightarrow V$ denote the set of maps from $V$ to $V$. If $V$ is a vector space, then let $End(V)$ denote its monoid of endomorphisms, i.e., the monoid of linear maps $V \rightarrow V$. For $V = \mathbb{C}^N$ we can identify $End(V) = \mathbb{C}^{N \times N}$, i.e., the space of $N \times N$ complex matrices. We use the shorthand $End_N := End(\mathbb{C}^N)$. Here, we are interested in $V = \mathbb{C}[X_1, \ldots, X_n]_d$, the space of homogeneous polynomials of degree $d$, as we will see in Section 3.2.

3.1.1 Definition. Let $G$ be a monoid and $V$ be a set.

1. An action of $G$ on $V$ is a monoid homomorphism $\varrho : G \rightarrow (V \rightarrow V)$.

2. If $V$ is a vector space, then a linear action of $G$ on $V$ defined as a monoid homomorphism $\varrho : G \rightarrow End(V)$.

We say that $G$ acts on $V$ and we write $gv$ as a shorthand for $(\varrho(g))(v)$, $g \in G, v \in V$.

Recall that the axiom for a monoid homomorphism in this case is $\varrho(g \cdot \tilde{g}) = \varrho(g) \circ \varrho(\tilde{g})$, where “$\cdot$” is the monoid operation and “$\circ$” is the composition of maps. You can think of the monoid elements moving the points of $V$ around. The identity element fixes all points.

Caveat: The monoids that we are mainly interested in are also endomorphism spaces, so $G = End_n$ for some $n$. This might be a source of confusion.

3.1.2 Example. Let $G = End_n$, let $V = \mathbb{C}^n$, and let $\varrho(g) = g$. In this case $gv$ can be interpreted as the usual matrix-vector product.

A more interesting example is obtained by lifting the action to the function space, as we explain in the following section.
3.2 Lifting the action to the function space

Using this action we define the action of \( \text{End}_n \) on \( V = \mathbb{C}[X_1, \ldots, X_n]_d \). For a polynomial \( h \in \mathbb{C}[X_1, \ldots, X_n]_d, \ g \in \text{End}_n, \ x \in \mathbb{C}^n \), define

\[
(gh)(x) := h(g^T x),
\]

where \( g^T \) is the transpose and \( g^T x \) is the monoid action of \( \text{End}_n \) on \( x \in \mathbb{C}^n \), i.e., the matrix-vector multiplication (see Example 3.1.2). You can also think of \( g \) as \( (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \) and let \( g \) act as \( \mathbb{C}^n \to \mathbb{C}^n \) via the transpose since \( g \cdot \cdot \cdot g \) is the transpose and \( T \) is the standard basis of \( \mathbb{C}^n \). Thus, another way to define this action is to use another definition to use. We verify that the definition in (3.2.1) satisfies \( (gh)h = g(gh) \) as follows:

\[
(g(\tilde{g} \cdot h))(x) = (g \cdot h)(g^T x) = (g \cdot \cdot \cdot g)^T (g^T x) = g((\cdot \cdot \cdot g^T \cdot x))(3.2.1) \equiv (g \cdot \cdot \cdot g)(x).
\]

Note that we take the transpose since \( G \) acts “from the left” on \( V \) and transposing reverses the order of the two monoid elements. Another way to define this action is to use \( g_{-1} \) instead of \( g^T \), but that only works if \( G \) is a group. On the other hand, if \( G \) is an arbitrary group and \( g \in G \), then it is unclear what \( g^T \) means. In all the cases we encounter it is just a matter of style which definition to use.

3.2.2 Example. We have

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\]

\( (X_1 X_2 \cdots X_n) = X_1^n. \)

Calculation: Let \( e_1, \ldots, e_n \) be the standard basis of \( \mathbb{C}^n \). Let \( x := \alpha_1 e_1 + \cdots + \alpha_n e_n \) with \( \alpha_i \in \mathbb{C} \). We have \( (g(X_1 X_2 \cdots X_n))(x) := (X_1 X_2 \cdots X_n)(g^T x) \). But \( g^T x = \alpha_1 e_1 + \alpha_1 e_2 + \cdots + \alpha_1 e_n \). Now \( (X_1 X_2 \cdots X_n)(\alpha_1 e_1 + \alpha_1 e_2 + \cdots + \alpha_1 e_n) = (\alpha_1)^n = X_1^n(x). \)

3.2.3 Example. Let \( g \in \mathbb{C}^{n \times n} \) and let \( \ell = (\ell_1, \ldots, \ell_n) \in \mathbb{C}^n \) be the first column of \( g \).

\[
g(X_1^d) = (\ell_1 X_1 + \ell_2 X_2 + \cdots + \ell_n X_n)^d.
\]

Calculation: Let \( x := \alpha_1 e_1 + \cdots + \alpha_n e_n \). We have \( (g(X_1^d))(x) := (X_1^d)(g^T x). \) But

\[
g^T x = (\ell_1 \alpha_1 + \ell_2 \alpha_2 + \cdots + \ell_n \alpha_n) e_1 + \beta_2 e_2 + \cdots + \beta_n e_n
\]

for some \( \beta_2, \ldots, \beta_n \in \mathbb{C} \). Thus,

\[
(X_1^d)(g^T x) = (\ell_1 \alpha_1 + \ell_2 \alpha_2 + \cdots + \ell_n \alpha_n)^d = ((\ell_1 X_1 + \ell_2 X_2 + \cdots + \ell_n X_n)^d)(x).
\]

More generally, Example 3.2.3 shows that

\[
g(X_1^d) = (\ell_1' X_1 + \ell_2' X_2 + \cdots + \ell_n' X_n)^d,
\]

where \( (\ell_1', \ldots, \ell_n') \) is the \( i \)th column of \( g \).

We will combine these insights with the following two structural properties. This lifted action is linear and an algebra homomorphism, as the following two lemmas show. Let \( \mathbb{A} = \mathbb{C}^n \) and define \( \mathbb{C}[\mathbb{A}] := \mathbb{C}[X_1, \ldots, X_N]. \)
CHAPTER 3. ACTIONS, ORBITS, AND ORBIT CLOSURES

3.2.5 Lemma. Let $h, h' \in \mathbb{C}[A]$ and let $g \in G$. For all complex numbers $\alpha, \alpha'$ we have

$$g(\alpha h + \alpha' h') = \alpha(gh) + \alpha'(gh').$$

Proof. Let $x \in A$ be arbitrary. We calculate

$$(g(\alpha h + \alpha' h'))(x) \overset{(3.2.1)}{=} (\alpha h + \alpha' h')(g^T x) \overset{(3.2.1)}{=} \alpha(gh(x)) + \alpha'((gh')(x)) \overset{(*)}{=} (\alpha(gh) + \alpha'(gh'))(x),$$

where $(*)$ uses the fact that $\mathbb{C}[A]$ is a vector space.

More generally, Lemma 3.2.5 holds by induction for arbitrary finite linear combinations of functions in $\mathbb{C}[A]$.

3.2.6 Lemma. Let $h, h' \in \mathbb{C}[A]$ and let $g \in G$. Then

$$g(h \cdot h') = (gh) \cdot (gh').$$

Proof. Let $x \in A$ be arbitrary. We calculate

$$(g(h \cdot h'))(x) \overset{(3.2.1)}{=} (h \cdot h')(g^T x) \overset{(3.2.1)}{=} h(g^T x) \cdot h'(g^T x) \overset{(3.2.1)}{=} (gh)(x) \cdot (gh')(x) \overset{(*)}{=} ((gh) \cdot (gh'))(x),$$

where $(*)$ follows from the definition of the product of two functions.

Note that Lemma 3.2.5 and Lemma 3.2.6 imply that $gh$ can be calculated by taking products and linear combinations of $gX_i$, but $gX_i$ is just the homogeneous linear form given by the $i$th column of $g$. For example

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} (X_1 X_2 + X_1^2) = (X_1 + X_2)(2X_1) + (X_1 + X_2)^2 = 2X_1^2 + 2X_1X_2 + X_1^2 + 2X_1X_2 + X_2^2 = 3X_1^2 + 4X_1X_2 + X_2^2.$$

3.2.7 Corollary. Let $g \in \mathbb{C}^{n \times n}$ and let $\ell^i \in \mathbb{C}^n$ be the $i$th column of $g$. Let $L^i := \ell^i_1 X_1 + \cdots + \ell^i_n X_n \in \mathbb{C}[X_1, \ldots, X_n]$.

$$g(X^d_1 + \cdots + X^d_n) = (L^1)^d + \cdots + (L^n)^d.$$

Proof. Combine (3.2.4) with Lemma 3.2.5.

3.2.8 Example. We can rewrite Example (2.3.1) as

$$\lim_{\varepsilon \to 0} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -\varepsilon & 1 \end{pmatrix} \left( X^3 + Y^3 \right) = X^2Y.$$
3.3 Orbits

In this section we express the Waring rank problem as a problem on orbits of monoids. This is the first step towards phrasing the border Waring rank problem as a problem of group orbit closures.

3.3.1 Definition. For a monoid $G$ acting on a set $V$ define $Gh := \{gh \mid g \in G\}$ for $h \in V$. We call $Gh$ the orbit of $h$.

3.3.2 Example. For $G = \mathbb{C}^{N \times N}$ the orbit $Gx^d_1 \subseteq \mathbb{C}[X_1, \ldots, X_N]_d$ is the set of Waring rank 1 homogeneous degree $d$ polynomials in $N$ variables.

Proof. For $g \in G$ let $\ell := g.l_1X_1 + g.l_2X_2 + \cdots + g.l_NX_N$. Then $gx^d_1 = \ell^d$ by Example 3.2.3.

For the other direction, let $\ell \in \mathbb{C}^N$. Let $g \in \mathbb{C}^{N \times N}$ with first column $\ell$. Then by Example 3.2.3 we have $gx^d_1 = \ell^d$. \hfill $\square$

We can now phrase the Waring rank problem in the language of monoid orbits as follows. For $n \leq N$ we embed $\mathbb{C}[X_1, \ldots, X_n]_d \subseteq \mathbb{C}[X_1, \ldots, X_N]_d$ in the natural way.

3.3.3 Proposition. Let $N \geq n$ and $N \geq m$. Let $G := \text{End}_N$.

$$\{h \in \mathbb{C}[X_1, \ldots, X_n]_d \mid \text{Waring rank of } h \text{ is at most } n\} = G(X^d_1 + \cdots + X^d_n) \cap \mathbb{C}[X_1, \ldots, X_m]_d.$$  

Proof. If $h = g(X^d_1 + \cdots + X^d_n)$ for some $g \in \mathbb{C}^{N \times N}$, then $h = (g.l_1X_1 + g.l_2X_2 + \cdots + g.l_NX_N)^d + \cdots + (g.l_nX_1 + g.l_{n+1}X_2 + \cdots + g.l_NX_N)^d$ by Cor. 3.2.7 and thus the Waring rank of $h$ is at most $n$. (Note that we could also have set all $g.l_j$ to zero for which $j > n$ or $i > m$.) Conversely, if the Waring rank of $h \in \mathbb{C}[X_1, \ldots, X_n]$ is at most $n$, then $h = (g.l_1X_1 + g.l_2X_2 + \cdots + g.l_nX_n)^d + \cdots + (g.l_nX_1 + g.l_{n+1}X_2 + \cdots + g.l_NX_N)^d$.

Since $N \geq n$ and $N \geq m$, we can construct a matrix $g \in \mathbb{C}^{N \times N}$ by filling the remaining cells with zeros. Then $h = g(X^d_1 + \cdots + X^d_n)$ by Cor. 3.2.7. \hfill $\square$

In the proposition above, we have to intersect the orbit on the right-hand side by $\mathbb{C}[X_1, \ldots, X_m]_d$, since the $G$-action can introduce variables with index $> m$, which cannot occur on the left-hand side.

We now want to go one step further and look at the inclusion of monoid orbits than of just point membership. In this way we could use properties of the point $h$ to show $h \notin W$.

3.3.4 Lemma. The orbit $Gh$ is the smallest set that contains $h$ and is closed under the monoid action.

Proof. If a set contains $h$ and is closed under the monoid action, then it contains all $gh$ with $g \in G$, so by definition it contains $Gh$.

Moreover, $Gh$ is closed under the monoid action: Let $g \in G$ be arbitrary and let $h' \in Gh$ be arbitrary. By definition there exist $g' \in G$ such that $h' = g'h$. Thus $gh' = g(g'h) \supseteq (gg')h \in Gh$, where $\supseteq$ follows from the axioms of a monoid action. \hfill $\square$

3.3.5 Corollary. Let $N \geq n$ and $N \geq m$. Let $G := \text{End}_N$. Let $h \in \mathbb{C}[X_1, \ldots, X_m]_d$. The Waring rank of $h$ is at most $n$ iﬀ $Gh \subseteq G(X^d_1 + \cdots + X^d_n)$.

Proof. $h \in G(X^d_1 + \cdots + X^d_n)$ iﬀ $Gh \subseteq G(X^d_1 + \cdots + X^d_n)$ by Lemma 3.3.4. Prop. 3.3.3 says that the Waring rank of $h$ is at most $n$ iﬀ $h \in G(X^d_1 + \cdots + X^d_n)$, which finishes the proof. \hfill $\square$
3.4 Orbit closures

In this section we express the border Waring rank problem as a problem on monoid orbit closures. The vector space $\text{End}_N = \mathbb{C}^{N \times N}$ is endowed with the standard metric

$$\text{dist}(g,g') = \sum_{i,j=1}^{N} |g_{i,j} - g'_{i,j}|,$$

g, g' \in \text{End}_N.

Consider the metric space $\text{End}_N \times \mathbb{C}[X_1, \ldots, X_m]$ via $\text{dist}((g,h),(g',h')) := \text{dist}(g,g') + \text{dist}(h,h')$. We postpone the proof of the following simple technical lemma to Section 3.6.

3.4.1 Lemma. The map $\text{End}_N \times \mathbb{C}[X_1, \ldots, X_m] \to \mathbb{C}[X_1, \ldots, X_N]$, $(g,h) \mapsto gh$ given by the action in Section 3.2 is continuous.

3.4.2 Corollary.

1. For a fixed $h \in \mathbb{C}[X_1, \ldots, X_m]$ the map $\text{End}_N \to \mathbb{C}[X_1, \ldots, X_N]$, $g \mapsto gh$ is continuous.

2. For a fixed $g \in \text{End}_N$ the map $\mathbb{C}[X_1, \ldots, X_m] \to \mathbb{C}[X_1, \ldots, X_N]$, $h \mapsto gh$ is continuous.

Proof. Both maps are restrictions of the continuous map in Lemma 3.4.1. \qed

3.4.3 Lemma. The monoid orbit closure $\overline{Gh}$ is the smallest set that contains $h$, is closed under the monoid action, and $\mathbb{C}$-closed.

Proof. Let $X$ be a set that contains $h$, is closed under the monoid action, and is $\mathbb{C}$-closed. We have

$$h \in X \iff Gh \subseteq X \iff \overline{Gh} \subseteq X.$$

For the last equivalence we used that if $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$ and that $\overline{B} = B$ for $\mathbb{C}$-closed sets $B$. One subtlety remains: A priori it is unclear that $\overline{Gh}$ is closed under the monoid action. We prove this as follows. Let $h' \in \overline{Gh}$, $h' = \lim_{i \to \infty} g_i h$ with $g_i \in G$. Let $g \in G$ be arbitrary. Then $gh' = g \lim_{i \to \infty} g_i h = \lim_{i \to \infty} g(g_i h) = \lim_{i \to \infty} (gg_i) h \in \overline{Gh}$, because the map $h \mapsto gh$ is continuous for every $g \in G$ (Cor. 3.4.2). \qed

3.4.4 Corollary. Let $N \geq n$ and $N \geq m$. Let $G := \text{End}_N$. Let $h \in \mathbb{C}[X_1, \ldots, X_m]$. The border Waring rank of $h$ is at most $n$ iff $\overline{Gh} \subseteq G(X_1^d + \cdots + X_n^d)$.

Proof. Using Prop. 3.3.3 we see that the border Waring rank of $h$ is at most $n$ iff $h \in G(X_1^d + \cdots + X_n^d)$. But since $G(X_1^d + \cdots + X_n^d)$ is $\mathbb{C}$-closed and closed under the action of $G$, from Lemma 3.4.3 it follows that $h \in G(X_1^d + \cdots + X_n^d)$ iff $\overline{Gh} \subseteq G(X_1^d + \cdots + X_n^d)$. \qed

3.5 Group orbit closures

It is more common to talk about group orbit closures instead of monoid orbit closures. The reason is that we can replace $\text{End}_N$ in Cor. 3.4.4 by the general linear group $\text{GL}_N := \{ g \in \text{End}_N \mid \det(g) \neq 0 \}$.

3.5.1 Lemma (Density of $\text{GL}_N \subseteq \text{End}_N$). For every $g \in \text{End}_N$ there exists a sequence $(g_i)$ with each $g_i \in \text{GL}_N$ such that $\lim_{i \to \infty} g_i = g$.

Proof. Consider $\det(g + \varepsilon \text{Id}_N)$, which is a nonzero univariate polynomial in $\varepsilon$ of degree $\leq N$. Thus it has at most $N$ zeros. From the sequence $(g_i)$, $g_i := g + \frac{i}{N} \text{Id}_N$ we remove those $g_i$ with zero determinant (these are at most $N$ many). Then $(g_i)$ converges to $g$ with all $g_i \in \text{GL}_N$. \qed
3.5.2 Proposition. Let $N \geq n$ and $N \geq m$. Let $G := \mathcal{GL}_N$. Let $h \in \mathbb{C}[X_1, \ldots, X_m]$. The border Waring rank of $h$ is at most $n$ iff $\overline{Gh} \subseteq \overline{G(X_1^d + \cdots + X_n^d)}$.

Proof. We prove that in general, $\mathcal{GL}_N h = \mathcal{End}_N h$. According to Cor. 3.4.2 the map $\varphi : g \mapsto gh$ is continuous. In general, for any continuous map $\varphi$ and any set $G$ we have $\varphi(G) \subseteq \varphi(G)$. The proof of this fact is short: let $g \in \overline{G}$ with $g = \lim_{i \to \infty} g_i$. Then $\varphi(g) = \varphi((\lim_{i \to \infty} g_i)) = \lim_{i \to \infty} \varphi(g_i) \in \varphi(G)$.

Using $\varphi(G) \subseteq \varphi(G)$, we take closures on both sides: $\overline{\varphi(G)} \subseteq \overline{\varphi(G)}$ (Lemma 2.4.7). Since clearly $\varphi(G) \subseteq \varphi(G)$, we have $\varphi(G) = \varphi(G)$. Setting $G = \mathcal{GL}_N$ and using that $\overline{G} = \mathcal{End}_N$ by Lemma 3.5.1, the statement follows.

3.6 The orbit map

In this section we prove Lemma 3.4.1. Indeed, we prove the following more general statement.

3.6.1 Proposition. Let $\varphi : \mathcal{End}_N \times \mathbb{C}[X_1, \ldots, X_N] \to \mathbb{C}[X_1, \ldots, X_N]$, $\varphi(g, h) = gh$. Let $\eta := \dim \mathbb{C}[X_1, \ldots, X_N] = (N + d - 1)$. For $1 \leq i \leq \eta$ define $\varphi_i$ to be the $i$th coordinate function of $\varphi$. Then each $\varphi_i$ is given by a polynomial in the $N^2 + \eta$ coordinate variables of $\mathcal{End}_N \times \mathbb{C}[X_1, \ldots, X_N]$.

Since polynomials are continuous and combining continuous coordinate functions gives a continuous function, Proposition 3.6.1 implies Lemma 3.4.1.

Proof of Prop. 3.6.1. Let the entry in row $i$ and column $j$ of $g$ be denoted by $g_{ij}$.

For a list $(i_1, i_2, \ldots, i_d)$ of numbers let $S(i_1, i_2, \ldots, i_d)$ denote the set of all lists that have the same entries as $(i_1, i_2, \ldots, i_d)$, but where the positions are permuted. Let $s_{i_1, i_2, \ldots, i_d}^{i_1, i_2, \ldots, i_d}$ denote the sum

$$s_{i_1, i_2, \ldots, i_d}^{i_1, i_2, \ldots, i_d} := \sum_{(j_1, \ldots, j_d) \in S(i_1, i_2, \ldots, i_d)} g_{j_1}^{i_1} g_{j_2}^{i_2} \cdots g_{j_d}^{i_d}.$$ 

For $1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq N$ we have

$$g(X_{i_1} \cdots X_{i_d}) = (g_{i_1}^{i_1} X_{i_1} + \cdots + g_{i_d}^N X_N) \cdots (g_{i_1}^1 X_{i_1} + \cdots + g_{i_d}^N X_N)$$

$$= \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq N} s_{i_1, i_2, \ldots, i_d}^{i_1, i_2, \ldots, i_d} X_{i_1} \cdots X_{i_d}$$

and thus

$$g(\sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq N} \alpha_{i_1, \ldots, i_d} X_{i_1} \cdots X_{i_d})$$

$$= \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq N} \left( \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq N} \alpha_{i_1, \ldots, i_d} s_{i_1, i_2, \ldots, i_d}^{i_1, i_2, \ldots, i_d} \right) X_{i_1} \cdots X_{i_d}.$$ 

The term in parentheses is homogeneous of degree $d + 1$ in the $\eta$ variables $\alpha_{i_1, \ldots, i_d}$ and the $N^2$ variables $g_{ij}$.
Orbits and orbit closures

Recall from the previous section that we want to prove that a particular polynomial $h$ is not contained in a set of polynomials $W$. The monoid $G = \text{End}_n$ acts on $\mathbb{C}[X_1, \ldots, X_n]$ or $\mathbb{C}[X_1, \ldots, X_n]$ by replacing the variables by homogeneous linear forms. Instead of showing that $h \notin W$, we try to prove that $h$ is not contained in a certain $G$-orbit.

In the case of Waring rank, this works particularly well, since the set of all polynomials of Waring $\leq r$ has a complete polynomial, namely $X_1^d + \cdots + X_n^d$.

If we want to prove that $h \notin W$, then we replace the orbit by the corresponding orbit closure. This has the nice effect that we can replace $\text{End}_n$ by $\text{GL}_n$, which is a group (and very well understood).
Chapter 4

First algebraic geometry

We explain the crucial link of our observations so far to algebraic geometry, namely that our orbit closures are actually Zariski-closed, so that we can restrict our search for obstructions $f$ to separating polynomials $f$, which we call polynomial obstructions. This is formally stated in Definition 4.2.9. The proof of this insight requires a good amount of algebraic geometry and thus we do not give all the details.

4.1 Zariski-closure

Recall Lemma 2.4.9: If a set $W$ is $C$-closed, then there is a continuous function vanishing precisely on $W$. Moreover, in the other direction Prop. 2.4.8 shows that if a continuous function vanishes precisely on $W$, then $W$ is $C$-closed. It follows that we could define $C$-closed sets to be exactly those sets which can be separated from arbitrary points by continuous functions vanishing on the sets. In Definition 4.1.1 we use exactly this approach to define what a Zariski-closed set is: Those are the sets that can be separated from arbitrary points by multivariate polynomials vanishing on the sets.

Let $A = \mathbb{C}[X_1, \ldots, X_m]_d$ and let $T_1, \ldots, T_n$ be a basis of $A$. For example, for the discriminant $b^2 - 4ac$ we have $T_1 = a$, $T_2 = b$, $T_3 = c$.

In the following, we will have two polynomials $c, h \in A$. $(c$ is not the variable appearing in the discriminant above!) $h$ is the polynomial for which we search complexity lower bounds, for example, we want to prove a lower bound on the border Waring rank. The variable name $c$ stands for complexity or complete polynomial. In the case of the Waring rank, think of $c$ being the power sum $c = X_1^d + \cdots + X_m^d$.

4.1.1 Definition. A subset $W \subseteq A$ (think of $W = \overline{G_c}$) is called Zariski-closed in $A$ if there exists a natural number $r \in \mathbb{N}$ and polynomials $f_1, \ldots, f_r \in \mathbb{C}[T_1, \ldots, T_n]$ such that

$$h \in W \iff f_1(h) = f_2(h) = \cdots = f_r(h) = 0.$$  

We say that the polynomials $f_1, \ldots, f_r$ cut out $W$.

4.1.2 Example. Let $A = \mathbb{C}[X, Y]_2$ with basis $T_1 = a, T_2 = b, T_3 = c$. Then $b^2 - 4ac$ cuts out the (border) Waring rank 1 polynomials.

4.1.3 Example. Consider $A = \mathbb{C}^2$ and let the Zariski-closed set $W \subseteq A$ be cut out by the polynomial $(T_1)^2 + (T_2)^2 - 1$. Those points in $W$ that have real coordinates form a circle with radius 1 in $\mathbb{R}^2 \subseteq \mathbb{C}^2$.

4.1.4 Example. If $A = \mathbb{C}^{n \times n}$, then the set of $n \times n$ matrices that have rank at most $n - 2$ is Zariski-closed. It is cut out by the determinants of all $(n - 1) \times (n - 1)$ submatrices.
In order to prove complexity lower bounds we would like to know a set of polynomials cutting out the set $G_c$, but unfortunately this kind of analysis is only feasible for some very small cases.

From now on we use the short notation

$$\mathbb{C}[\mathbb{A}] := \mathbb{C}[T_1, \ldots, T_\eta]$$

and call $\mathbb{C}[\mathbb{A}]$ the coordinate ring of the ambient space. Note that this replaces the clumsy $\mathbb{C}[\mathbb{C}[X_1, \ldots, X_m]]$.

We will see in Theorem 4.2.8 that our orbit closures $G_c$ are Zariski-closed. This is perfect for our purposes, because for Zariski-closed sets non-membership of a point is equivalent to nonvanishing of a single polynomial $f$ as the following straightforward lemma highlights. This is much stronger than separation by merely continuous functions.

4.1.5 Lemma. Let $W$ be Zariski-closed in $\mathbb{A}$. For $h \in \mathbb{A}$ we have $h \notin W$ iff there exists a polynomial $f \in \mathbb{C}[A]$ such that $f$ vanishes on $W$ and $f(h) \neq 0$.

Proof. Let $W$ be cut out by $f_1, \ldots, f_r$, i.e., $h \in W \iff f_1(h) = \cdots = f_r(h) = 0$. Then $h \notin W$ iff $\exists 1 \leq i \leq r : f_i(h) \neq 0$.

The following lemma shows a first property of Zariski-closed sets, namely that they are $\mathbb{C}$-closed.

4.1.6 Lemma. If $W \subseteq \mathbb{A}$ is Zariski-closed in $\mathbb{A}$, then $W$ is $\mathbb{C}$-closed in $\mathbb{A}$.

Proof. Let $W$ be cut out by the polynomials $f_1, \ldots, f_r$. Let $(h_i)_i$ be a sequence in $W$ that converges to some $h \in \mathbb{A}$. Then $f_j(h_i) = 0$ for all $i, j$. Since the $f_j$ are continuous, it follows that $f_j(h) = 0$. Therefore $h \in W$. We conclude that $W$ is $\mathbb{C}$-closed.

4.2 Algebraic geometry of orbit closures

Lemma 4.1.6 says that Zariski-closed sets are $\mathbb{C}$-closed. The crucial point is that in our case the converse of Lemma 4.1.6 holds: Orbit closures $\mathbb{GL}_N^c$ are not only $\mathbb{C}$-closed but also Zariski-closed.

4.2.1 Definition. A map $\mathbb{C}^a \to \mathbb{C}^b$ is called a polynomial map if all its $b$ coordinate functions are multivariate polynomials in the $a$ standard basis vectors.

4.2.2 Example. Using Prop. 3.6.1 we see that for a fixed $c \in \mathbb{A}$, the map $\mathbb{GL}_N \to \mathbb{A}$, $g \mapsto gc$ is a polynomial map. Its image is the orbit $\mathbb{GL}_N^c$.

4.2.3 Definition. We use the following definitions with respect to the Zariski topology.

1. A subset $W \subseteq \mathbb{A}$ to be open if its complement $\mathbb{A} \setminus W$ is closed.

2. A subset $W \subseteq \mathbb{A}$ is called locally closed if $W$ is an intersection of an open and a closed set. Equivalently (for those who know a little bit of topology) $W$ is locally closed iff $W$ is open in its closure.

3. A subset $W \subseteq \mathbb{A}$ is called constructible if $W$ is a finite union of locally closed sets. (Equivalently, the set of constructible sets is the smallest set that contains all closed sets and is closed under taking complements and finite unions.)

4.2.4 Example. The set $\{ A \in \mathbb{End}_N \mid \det(A) = 0 \}$ is Zariski-closed in $\mathbb{End}_N$. Thus $\mathbb{GL}_N \subseteq \mathbb{End}_N$ is open and hence locally closed (and hence constructible).

We state the following theorem without proof.

4.2.5 Theorem (Chevalley’s Theorem, see e.g. [Kra85, AI.3.3 Folgerung 2] or [TY05, Prop. 15.4.3]). The image of a constructible set under a polynomial map is again constructible.
4.2.6 Corollary. For any $c \in A$ the orbit $GL_N c$ is constructible.

Proof. Combine Example 4.2.2 and Example 4.2.4. \qed

4.2.7 Remark. One can even prove that $GL_N c$ is locally closed, see e.g. [Kra85, II.2.2 c].

We state the following result without proof.

4.2.8 Theorem ([Kra85, AI.7.2 Folgerung]). For constructible sets, Zariski closure and $C$-closure coincide.

We can use Theorem 4.2.8 and Lemma 4.1.5 to draw the following immediate crucial conclusion that states that polynomials can always be used to separate points in $A$ from $GL_N c \subseteq A$. This greatly reduces the search space for obstructions and is one of the key ideas in geometric complexity theory.

4.2.9 Definition (Polynomial Obstruction). We call the polynomials $f$ that separate $h$ from $GL_N c$ by satisfying $f(GL_N c) = \{0\}$ and $f(h) \neq 0$ polynomial obstructions.

From our previous discussions we see that polynomial obstructions are guaranteed to exist if $h \notin GL_N c$. (The hard task is to find them.)

4.3 Cones

We will now use the additional structure of $GL_N c$ being a cone to restrict our search for obstructions to homogeneous polynomials only, see the upcoming Corollary 4.3.5. Note that for example the discriminant $b^2 - 4ac$ is homogeneous.

4.3.1 Definition. Recall that $A = \mathbb{C}^g$ is a complex vector space and hence is endowed with a scalar multiplication. For a vector $c \in A$ and $\alpha \in \mathbb{C}$ let $\alpha c$ denote the scalar multiple of $c$. A subset $W \subseteq A$ is called a cone if it is closed under scalar multiplication, i.e.,

$$\forall \alpha \in \mathbb{C}, c \in W : \alpha c \in W.$$  

4.3.2 Lemma. For any polynomial $c \in \mathbb{C}[X_1,\ldots,X_N]_d$ the orbit closure $GL_N c$ is a cone.

Proof. Let $h \in GL_N c$ and let $\alpha \in \mathbb{C}$ be arbitrary. Let $(h_i)_i$ be a sequence in $GL_N c$ that converges to $h$. Let $g_i \in GL_N$ such that $h_i = g_i c$. Choose $\beta \in \mathbb{C}$ such that $\beta^d = \alpha$ and let $\beta g_i$ denote the product of the scalar $\beta$ and the matrix $g_i$, i.e., the matrix $g_i$ in which all entries are scaled with $\beta$. We observe that $(\beta g_i)c = \alpha (g_i c)$. Since scaling with $\alpha$ is continuous, the sequence $((\beta g_i)c)_i$ converges to $\alpha h$ and hence $\alpha h \in GL_N c$. \qed

4.3.3 Remark. Usually in algebraic geometry one makes the transition to projective geometry whenever cones are encountered, but here, it is not necessary to do so.

4.3.4 Proposition. Let $W \subseteq A$ be a cone. If a polynomial $f \in \mathbb{C}[A]$ vanishes on $W$, then all its homogeneous parts vanish on $W$.

Proof. The statement is clear for $W = \emptyset$. Let $h \in W$ be arbitrary. Let $f_i$ be the $i$th homogeneous part of $f$, i.e., $f_i(\alpha h) = \alpha^i f_i(h)$. We interpret $f(\alpha h)$ as a univariate polynomial $\tilde{f}$ in $\alpha$. We have

$$\tilde{f}(\alpha) = f(\alpha h) = \sum_{i=0}^d f_i(\alpha h) = \sum_{i=0}^d \alpha^i f_i(h).$$

The coefficient of the monomial $\alpha^i$ in $\tilde{f}$ is $f_i(h) \in \mathbb{C}$. Since $f$ vanishes on $W$ and $W$ is a cone we have that $\tilde{f}$ vanishes everywhere on $\mathbb{C}$, so $\tilde{f} = 0$. Therefore all coefficients $f_i(h)$ of $\tilde{f}$ are zero. Since $h \in W$ was arbitrary, we get that $f_i$ vanishes on $W$. \qed
The cone structure of $GL_N$ and Proposition 4.3.4 allow us to reduce our search for polynomial obstructions to homogeneous polynomials in $C[A]$, as can be seen in the next corollary.

4.3.5 Corollary. For $f \in C[A]$, if $f(\overline{GL_N}) = \{0\}$ and $f(h) \neq 0$, then there exists a homogeneous polynomial $f_{\text{hom}} \in C[A]$ such that $f_{\text{hom}}(\overline{GL_N}) = \{0\}$ and $f_{\text{hom}}(h) \neq 0$.

Proof. Let $S \subseteq \mathbb{N}_{\geq 0}$ denote the finite set of degrees $i$ such that the homogeneous degree $i$ part of $f$ is nonzero. Since $f(h) \neq 0$ we have that $f \neq 0$ and therefore $S \neq \emptyset$. Decompose $f$ into its nonzero homogeneous parts $f = \sum_{i \in S} f_i$. Since $GL_N$ is a cone and $f(\overline{GL_N}) = \{0\}$, using Proposition 4.3.4 we see that all $f_i$ vanish on $GL_N$. Since $0 \neq f(h) = \sum_{i \in S} f_i(h)$, it follows that there exists $i \in S$ such that $f_i(h) \neq 0$. Choose $f_{\text{hom}}$ to be such an $f_i$.

For a subset $W \subseteq A$ let

$$I(W) := \{ f \in C[A] \mid f(w) = 0 \quad \forall w \in W \}$$

denote the vanishing ideal of $W$. This is clearly a complex vector space and is closed under multiplication with arbitrary polynomials, thus $I(W)$ is an ideal in the ring $C[A]$. Proposition 4.3.4 implies that if $W$ is a cone, then $I(W)$ is a graded $C$-algebra, i.e., every element can be decomposed into a unique sum of homogeneous parts, where each part is in $I(W)$. We denote by $I(W)_i$, the $i$th homogeneous part of $I(W)$. We have $I(W)_i \cdot I(W)_{i'} \subseteq I(W)_{i+i'}$. (This is the crucial property of a graded algebra.)

<table>
<thead>
<tr>
<th>Search space reduction via algebraic geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>• For orbit closures Zariski closure equals $C$-closure.</td>
</tr>
<tr>
<td>Consequence: If $h \notin \overline{GL_N}$, then it is separated by polynomial (instead of simply a continuous function).</td>
</tr>
<tr>
<td>• Orbit closures are cones</td>
</tr>
<tr>
<td>Consequence: If $h \notin \overline{GL_N}$, then it is even separated by a homogenous polynomial.</td>
</tr>
</tbody>
</table>
Chapter 5

Algebraic complexity classes

While the Waring rank is a very instructive and important example, we also want to define complexity measures that are more powerful and closer to actual computations. These are the so-called Valiant’s classes.¹ Our observations in this and some following chapters work over other fields \( \mathbb{F} \) than the complex numbers. We will go back to the complex numbers when rephrasing Valiant’s classes in terms of orbit closures.

We denoted by \( f \) a separating function and by \( g \) a group element, but traditionally in the field of algebraic complexity theory both \( f \) and \( g \) denote polynomials. We stick with this classical notation.

5.1 VP

Let \( X = (X_1, X_2, \ldots) \) be an infinite family of indeterminates over some field \( \mathbb{F} \).

5.1.1 Definition. A sequence of polynomials \((f_n) \in \mathbb{F}[X] \) is called a p-family if for all \( n \),

1. \( f_n \in \mathbb{F}[X_1, \ldots, X_{p(n)}] \) for some polynomially bounded function \( p \) and
2. \( \deg f_n \leq q(n) \) for some polynomially bounded function \( q \).

Recall Definition 1.3.3.

5.1.2 Definition. The class \( \text{VP} \) consists of all p-families \((f_n) \) such that \( L(f_n) \) is polynomially bounded.

5.1.3 Example. Let \( \det_n = \sum_{\pi \in S_n} \text{sgn}(\pi) X_1,\pi(1) \ldots X_n,\pi(n) \). We will see soon that \( \det_n \) has polynomial-sized arithmetic circuits. Therefore, \((\det_n) \in \text{VP} \).

In the above example, the indeterminates have two indices instead of one. Of course we could write \( \det_n \) as a polynomial in \( X_1, X_2, \ldots \) by using a bijection between \( \mathbb{N}^2 \) and \( \mathbb{N} \). However, we prefer the natural naming of the variables (and will do so with other polynomials).

Let \( f \in \mathbb{F}[X] \) be a polynomial and \( s : X \to \mathbb{F}[X] \) be a mapping that maps indeterminates to polynomials. \( s \) can be extended in a unique way to an algebra endomorphism \( \mathbb{F}[X] \to \mathbb{F}[X] \). We call \( s \) a substitution. (Think of the variables replaced by polynomials.)

5.1.4 Definition. 1. Let \( f, g \in \mathbb{F}[X] \). \( f \) is called a projection of \( g \) if there is a substitution \( r : X \to X \cup \mathbb{F} \) such that \( f = r(g) \). We write \( f \leq_p g \) in this case. (Since \( g \) is a polynomial, it only depends on a finite number of indeterminates. Therefore, we only need to specify a finite part of \( r \).)

¹They are usually all called Valiant’s classes, although the more recent ones like \( \text{VP}_{ws} \) were not defined by Valiant.
2. Let \((f_n)\) and \((g_n)\) be p-families. \((f_n)\) is a p-projection of \((g_n)\) if there is a polynomially bounded function \(q : \mathbb{N} \to \mathbb{N}\) such that \(f_n \leq_p g_{q(n)}\). We write \((f_n) \leq_p (g_n)\).

Projections are very simple reductions. Therefore, we can also use them to define hardness for “small” complexity classes like VP. Projections fulfill the usual requirements of a reductions:

5.1.5 Lemma. (1) If \((f_n) \leq_p (g_n)\) and \((g_n) \in \text{VP}\), then \((f_n) \in \text{VP}\).

(2) \(\leq_p\) is a transitive relation.

Proof. 1. Let \(q\) be a polynomially bounded function and \(s_n\) be a projection such that \(f_n = s_n(g_{q(n)})\) for all \(n\). Let \(C_m\) be a circuit computing \(g_n\). We get a circuit computing \(f_n\) by replacing every variable \(X_i\) in \(C_{q(n)}\) by \(s_n(X_i)\). This circuit has the same size as \(C_{q(n)}\).

2. The composition of two polynomially bounded functions is polynomially bounded and the composition of two substitutions is a substitution again.

5.1.6 Definition. 1. A p-family \((f_n)\) is called VP-hard (under p-projections) if \((g_n) \leq_p (f_n)\) for all \((g_n) \in \text{VP}\).

2. It is called VP-complete if in addition \((f_n) \in \text{VP}\).

5.1.7 Lemma. If \((f_n)\) is VP-hard and \((f_n) \leq_p (g_n)\), then \((g_n)\) is VP-hard, too.

Proof. Let \((h_n) \in \text{VP}\) be arbitrary. Since \((f_n)\) is VP-hard, \((h_n) \leq_p (f_n)\). By transitivity, \((h_n) \leq_p (g_n)\). Since \((h_n)\) was arbitrary, the VP-hardness of \((g_n)\) follows.

5.1.8 Example. Let \(X^{(t)}_{i,j}\), \(1 \leq i, j, \ell \leq n\), be indeterminates and let \(M_{\ell} = (X^{(t)}_{i,j})_{1 \leq i, j \leq n}\) for \(1 \leq \ell \leq n\). The polynomial \(\text{imm}_n\) is a polynomial in \(n^3\) variables and is the \((1, 1)\) entry of the matrix product \(M_1 \cdots M_n\). The p-family \(\text{imm} = (\text{imm}_n)\) is the called iterated matrix multiplication.

By using the trivial algorithm for matrix multiplication, it is easy to see that \(\text{imm} \in \text{VP}\). We will see in the next chapter that \(\text{imm}\) and \(\text{det}\) are equivalent under p-projections. We do not know whether the determinant (or the iterated matrix multiplication polynomial) is VP-complete. However, there are generic problems that are VP-complete. But also more natural complete problems are known, see [DMM+14].

5.1.9 Question. Is \(\text{det}\) VP-complete?

5.1.10 Remark. When we replace polynomial upper bounds by quasipolynomial upper bounds (of the form \(O(n^{\log^c(n)})\) for constant \(c\)) in the definition of VP and p-projections, then the determinant is complete for this class usually called VQP, see [Bür00] and [Blä01] for more complete families. Here, “QP” stands for “quasi-polynomial”.

5.2 VP\(_e\)

We call an arithmetic circuit a formula if the underlying graph structure is a tree. In this case, every computation gate has fanout one, that is, every intermediate result can only be used once.

5.2.1 Definition. A p-family \((f_n)\) is contained in the class VP\(_e\) if there is a family of formulas \((F_n)\) such that \(F_n\) has polynomial size in \(n\) and computes \(f_n\).

The “\(e\)” in the subscript stands for expression, another word for formula. Since every formula is a circuit, we have VP\(_e\) \(\subseteq\) VP. It is not known that whether this inclusion is strict, but most researchers believe it is.
5.2.2 Question. Is $\text{VP}_e$ a strict subset of $\text{VP}$?

5.2.3 Corollary. If there exists a $\text{VP}$-complete function in $\text{VP}_e$, then $\text{VP}_e = \text{VP}$.

Proof. This follows from the transitivity of $\leq_p$ (Lemma 5.1.5(2)) and the fact that if $(g_n) \in \text{VP}_e$ and $(f_n) \preceq_p (g_n)$, then $(f_n) \in \text{VP}_e$, analogously to Lemma 5.1.5(1).

5.2.4 Definition. 1. A $p$-family $(f_n)$ is in the class $\text{VNC}_i$ if there is a family of circuits $(C_n)$ such that the size of $C_n$ is polynomially bounded in $n$ and the depth of $C_n$ is bounded by $O(\log^i n)$.

2. $\text{VNC} := \bigcup_{i \in \mathbb{N}} \text{VNC}_i$.

It turns out that $\text{VP}_e = \text{VNC}_1$, that is, every $p$-family that is computable by formulas of polynomial size has efficient parallel algorithms.

5.2.5 Lemma. Let $T$ be a binary tree with $n$ nodes. Then there is an edge $e$ in $T$ such that removing $e$ separates $T$ into two trees both having between $n/3$ and $2n/3$ nodes.

Proof. We construct a path $u_1, \ldots, u_m$ starting from the root as follows: We set $u_1$ to be the root of the tree. Let $u_i$ be the current end node of the path and let $w$ and $w'$ be its children. If the subtree with root $w$ is larger than the subtree with root $w'$, then $u_{i+1} := w$, otherwise $u_{i+1} := w'$. We stop when the size of the subtree with root $u_i$ is $< 2/3n$ and set $m := i$. The edge $e$ is the edge $(u_{m-1}, u_m)$. The subtree with root $u_m$ has size $< 2/3n$ by construction. The subtree with root $u_{m-1}$ has size $\geq 2/3n$. Since $u_m$ is the root of the larger subtree, the subtree with root $u_m$ has size at least $n/3$. The size of the remaining tree is between $n - 2/3n = n/3$ and $n - n/3 = 2n/3$.

5.2.6 Theorem (Brent [Bre74]). Let $F$ be a formula of size $s$. Then there is a formula $F'$ of size $\text{poly}(s)$ and depth $O(\log s)$ computing the same polynomial as $F$.

Proof. By Lemma 5.2.5, there is an edge in $F$ such that when removing $e$, we get two parts, each of size between $s/3$ and $2s/3$. The part not containing the output gate of $F$ is again a formula, which we call $H$. The part containing the output gate is not a formula, since one of the gates has fanin one after removal of $e$. We add a new child to this gate, which is labeled with a new input variable $Y$. Call the resulting formula $G$. $G$ computes a linear form $ay + b$, since $Y$ appears only once in $G$. ($a$ and $b$ are polynomials in the original input variables.) If we substitute the polynomial $h$ computed by $H$ for $Y$, then we get the polynomial $f$ computed by $F$. If we substitute 1 for $Y$, then we get $a + b$, and if we substitute 0 for $Y$, then we get $b$. Therefore, there are formulas of size $\leq 2s/3$ computing $a + b, b$ and $h$. With these formulas, we can proceed recursively. We get formulas $G'_1, G'_a$, and $H'$ computing $a + b, b$, and $h$, respectively. We can combine them to a formula computing $ah + b = f$ as depicted in Figure 5.1: For the size $\sigma(s)$ of this new formula, we get the recursion

$$\sigma(s) = 4 \cdot \sigma(2s/3) + 3$$

and for the depth $d(s)$, we get the recursion

$$d(s) = d(2s/3) + 3.$$ 

It is a routine check that $\sigma(s) = \text{poly}(s)$ and $d(s) = O(\log s)$.

5.2.7 Corollary. $\text{VP}_e = \text{VNC}_1$. 

Figure 5.1: The formula $F$ with the edge $(u_{m-1}, u_m)$, the two formulas $G$ and $H$, and the new formula computing $f$

5.3 Constant size iterated matrix multiplication

For some $c \in \mathbb{N}$, we define the family $(\text{imm}_n^{(c)})$ like the family $(\text{imm}_n)$, except that every polynomial is an iterated matrix product of $c \times c$-matrices (instead of $n \times n$-matrices), so $\text{imm}_n^{(c)}$ is a polynomial in $c^2 n$ variables.

**5.3.1 Theorem** (Ben-Or & Cleve [BC92]). Let $F$ be a formula of depth $d$ computing a polynomial $f$, then $f$ is a projection of $\text{imm}_n^{(3)}$.

**Proof.** We will prove by induction on $d$ that we can find $4^d$ many $3 \times 3$-matrices whose entries are either indeterminates or constants such that the product of these matrices is

\[
\begin{pmatrix}
1 & f & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

This is obviously true for depth zero formulas, since these formulas compute constants or single variables.

If the depth $d$ is larger than zero, we either have $f = g + h$ or $f = gh$ and $g$ and $h$ are both computed by formulas of depth $\leq d - 1$. By the induction hypothesis, there are two sets of $4^{d-1}$ $3 \times 3$-matrices each such that their products are

\[
\begin{pmatrix}
1 & g & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
1 & h & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

respectively. In the case of an addition gate we have

\[
\begin{pmatrix}
1 & g & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & h & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & g + h & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Therefore we can write $f$ as a projection of a $3 \times 3$-iterated matrix multiplication of length $2 \cdot 4^{d-1} \leq 4^d$.

In the case of a multiplication gate, we have

\[
\begin{pmatrix}
1 & g & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & h \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & g \cdot h & 0 \\
0 & 1 & h \\
0 & 0 & 1
\end{pmatrix}.
\]
Note that $h$ is standing in the “wrong” position. But we can easily fix this by applying permutation matrices from the left and the right. This just corresponds to exchanging the rows or columns of the first and last matrix of the corresponding matrix product, respectively. We proceed with

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -h \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & g & gh \\ 0 & 1 & h \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -g & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & gh \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that we now have a $-g$ and $-h$ instead of a $g$ and $h$. But this is easily achieved by multiplying the second row and column by $-1$. This can again be achieved by doing this with the first and last matrix of the $4^{d-1}$ matrices. Altogether, we get that $f$ is a projection of a product of $4 \cdot 4^{d-1} = 4^d$ matrices.

5.3.2 Corollary. imm$^{(3)}$ is VP$_e$-complete.

Proof. Let $f = (f_n) \in$ VP$_e$. Let $F_n$ be a formula of polynomial size computing $f_n$. By Theorem 5.2.6, there is an equivalent formula of polynomial size and depth $O(\log n)$. By Theorem 5.3.1, $f_n$ is a projection of imm$_{\text{poly}(n)}^{(3)}$. This proves the hardness.

To construct a formula of polynomial size for imm$^{(3)}_n$, we divide the product into two products of size $n/2$ each. We can assume that $n$ is a power of 2, since imm$^{(3)}_{n'} \leq$ imm$^{(3)}_n$ if $n' \leq n$. The entries of the result of the two products can be computed by 18 instances of imm$^{(3)}_{n/2}$, one for each entry of the two resulting matrices. From these two results, we can compute imm$^{(3)}_n$ by a constant size formula. Since each entry of the two resulting matrices is used three times, we need three distinct copies of the formulas for each entry. Therefore, we get the following recursion for the size $s(n)$ of the formula:

$$s(n) = 54s(n/2) + O(1).$$

Therefore, $s(n) = \text{poly}(n)$. $\square$

Obviously, imm$^{(c)}$ is VP$_e$-complete for any $c \geq 3$. On the other hand, imm$^{(1)}$ is not, since it only computes a single monomial. Allender and Wang [AW16] prove that imm$^{(2)}$ is also not VP$_e$-complete by exhibiting a polynomial that is not the projection of imm$^{(2)}_n$ for any $n$!

5.4 Orbit problems

Historically, $p$-projections have been the reduction of choice in algebraic complexity theory, because they are very simple reductions yet sufficiently powerful to prove hardness results (as we will see in the next two chapters).

Let $(f_n)$ be a $p$-family. Then there is a polynomially bounded function $p$ such that $f_n \in \mathbb{F}[X_1, \ldots, X_{p(n)}]$. We saw how to let End$_m$ act on $\mathbb{F}[X_1, \ldots, X_m]$. We let endomorphisms act on the sequence $(f_n)$ by letting End$_{p(n)}$ act on $f_n$. An element $g \in$ End$_{p(n)}$ replaces each variable by a homogeneous linear combination of the variables. In particular, it preserves the degree, that is, $\deg gf_n = \deg f_n$. $p$-projections, on the other hand, do not preserve the degree. We can use the following “trick” called padding. We only need it for homogeneous polynomials, since all our polynomials under consideration will be homogeneous, but it also works for non-homogeneous polynomials in the obvious way. Assume that $a \preceq_p b$ for two homogeneous polynomials $a$ and $b$ and let $s$ be the corresponding substitution. We define a new substitution $\tilde{s}$ that whenever $s(X_i)$ is a constant $\alpha$, then we set $\tilde{s}(X_i) = \alpha T$ for some new indeterminate $T$ instead. When $s(X_i)$ is a variable, then $\tilde{s}(X_i) = s(X_i)$. We have $\tilde{s}(b) = T^{\deg b - \deg a} \cdot \alpha$. Note that $\tilde{s}$ is a very special endomorphism, replacing every variable by a scalar multiple of some other variable. Assume $b \in \mathbb{F}[X_1, \ldots, X_n]$, then we will now consider it as a polynomial in $\mathbb{F}[T, X_1, \ldots, X_n]$ and let End$_{n+1}$ act on it. Since $T$ does not appear in $b$, we can restrict ourselves to endomorphisms that leave $T$ fixed.
5.4.1 Definition. Let \((f_n)\) and \((h_n)\) be homogeneous \(p\)-families. Let \(p(n)\) be minimal such that that \(h_n \in \mathbb{F}[X_1, \ldots, X_{p(n)}]\). We write \((f_n) \leq_{\text{end}} (h_n)\) if there is a polynomially bounded function \(q\) such that

\[
T^{\deg h_{q(n)} - \deg f_n} \cdot f_n \in \mathbb{E}nd_{p(q(n)) + 1} h_{q(n)}.
\]

Note that since \(p\) is minimal, it is polynomially bounded by the definition of \(p\)-family. We have chosen a fresh variable \(T\) for padding. In the literature, \(X_1\) or \(X_{p(q(n))}\) has frequently been used for padding. Taking a new variable turns out to be simpler, in particular, \(\leq_{\text{end}}\) will be transitive. We will see below that in our situation, it actually does not matter which variable we will take.

5.4.2 Lemma. Let \((f_n)\) and \((h_n)\) be homogeneous \(p\)-families.

1. If \((f_n) \leq_p (h_n)\), then \((f_n) \leq_{\text{end}} (h_n)\).
2. If \((f_n) \leq_{\text{end}} (h_n)\) and \((h_n) \in \text{VP}\), then \((f_n) \in \text{VP}\). The same statement is true if \(\text{VP}\) is replaced by \(\text{VP}_e\).
3. \(\leq_{\text{end}}\) is transitive.

Proof. Let \(p\) and \(q\) be defined as in Definition 5.4.1.

1. Let \(s_n\) is the substitution mapping \(f_n\) to \(h_{q(n)}\). As above, we define \(\hat{s}_n\) to be the substitution that whenever \(s_n(X_i) = \alpha\), then \(\hat{s}_n(X_i) = \alpha T\). Then \(\hat{s}_n(h_n) = T^{\deg h_{q(n)} - \deg f_n} \cdot f_n\). The substitution obviously defines an endomorphism.

2. It is very easy to see that for \(g_n \in \text{End}_p\), the sequence \((g_{p(n)} + 1)h_n\) is in \(\text{VP}\) or \(\text{VP}_e\), respectively, since \(g_{p(n)} + 1\) is just a linear transformation of the variables, which can be implemented by formulas of polynomial size. Therefore, the sequence \((T^{\deg h_{q(n)} - \deg f_n} \cdot f_n)\) is in \(\text{VP}\) or \(\text{VP}_e\), respectively. Since \(T\) does not appear in \(f_n\), we can set \(T = 1\).

3. Let \((f_n) \leq_{\text{end}} (h_n)\) and \((h_n) \leq_{\text{end}} (a_n)\). There is an endomorphism \(g_{p(q(n)) + 1}\) such that

\[
T^{\deg h_{q(n)} - \deg f_n} \cdot f_n = g_{p(q(n)) + 1} h_{q(n)}.
\]

Furthermore, there are polynomially bounded functions \(p'\) and \(q'\) such that

\[
T^{\deg a_{q'(m)} - \deg h_m} \cdot h_m \in \text{End}_{p'(q'(m)) + 1} a_{q'(m)}.
\]

Thus, there is an endomorphism \(g'_{p'(q'(m)) + 1}\) such that

\[
T^{\deg a_{q'(m)} - \deg h_m} \cdot h_m = g'_{p'(q'(m)) + 1} a_{q'(m)}.
\]

We set \(m = q(n)\), apply \(g_{p(q(n)) + 1}\) to (5.4.4), and plug in (5.4.3), where we interpret \(g_{p(q(n)) + 1}\) as an endomorphism in \(\text{End}_{p'(q'(q(n))) + 1}\) (setting further variables to zero for instance). Therefore,

\[
T^{\deg a_{q'(q(n)) - \deg f_n}} \cdot f_n = g_{p(q(n)) + 1} g'_{p'(q'(q(n))) + 1} a_{q'(q(n))},
\]

since we can assume that \(g_{p(q(n)) + 1}(T) = T\).

The rest of this section is rather technical. It can be skipped at a first reading. We assume \(\mathbb{F}\) to be large enough.

The next lemma shows that it does not matter too much, whether we take a fresh variable \(T\) for padding or an existing one. We generalize ideas by Ikenmeyer and Panova [IP16].

5.4.5 Lemma. Let \(f, h \in \mathbb{F}[X_1, \ldots, X_m]\) be homogeneous polynomials.
1. If \( T^{\deg h - \deg f} f \in \text{End}_{m+1}h \), then \( X_i^{\deg h - \deg f} f \in \text{End}_m h \) (where we interpret \( h \) in the first equation as a polynomial in \( F[T, X_1, \ldots, X_m] \)).

2. If \( X_i^{\deg h - \deg f} f \in \text{End}_m h \) for some \( i \), then there is a circuit of size polynomial in \( L(h), \deg h, \) and \( m \) computing \( f \).

Proof. 1. Since \( T \) does not appear in \( h \), we can simply replace \( T \) by \( X_i \).

2. Write \( f = \sum_{j=0}^{\deg f} X_i^j f_j \). As above, there is a circuit of size polynomial in \( L(h) \) and \( m \) computing \( X_i^{\deg h - \deg f} f = \sum_{j=\deg h - \deg f}^{\deg h} X_i^j f_j \). We now take \( \deg h + 1 \) copies of this circuit, and in each of them, we plug in a different value from \( F \) for \( X_i \). From the results, we can obtain the polynomials \( f_0, \ldots, f_{\deg f} \) by interpolation (see e.g. [BCS96]). Once we have these polynomials, we can easily compute \( f \).

Let \( C \) be a class of \( p \)-families. We call \( C \) closed under interpolation if for every \( (f_n) \in C, f_n = \sum_{j=0}^{\deg f_n} X_i^j f_{n,j} \) with \( f_{n,j} \in F[X_2, X_3, \ldots] \), the family \( (\sum_{j=0}^{\deg f_n} Y_j f_{n,j}) \in C \), where \( Y_0, Y_1, Y_2, \ldots \) are new variables. So essentially, given \( f_n \) and considering it as a univariate polynomial in \( X_1 \), we can compute the coefficients of \( f_n \). The new variables \( Y_i \) are introduced for book-keeping purposes to have again only one polynomial.

We call the class \( C \) closed under substitutions, if for two \( p \)-families \( (f_n), (h_n) \in C \), the family obtained by substituting some of the variables \( X_i \) in \( f_n \) by \( h_{n,(i)} \) for some \( p \)-bounded function \( j \) is again in \( C \).

The next lemma is a strengthening of Lemma 5.4.5, part (2).

5.4.6 Lemma. Let \( C \) be a class of \( p \)-families that is closed under interpolation and substitutions.

Let \( (f_n), (h_n) \in C \) such that

1. \( X_i^{\deg h_{q(n)} - \deg f_n} f_n \in \text{End}_{p(q(n))} h_{q(n)} \) for some \( i \) (here \( p \) and \( q \) are as in Definition 5.4.1),

2. every sequence \( (m_i) \) of monomials of \( p \)-bounded degree is in \( C \), and

3. \( (h_n) \) is \( C \)-hard (under \( p \)-projections).

Then there is a polynomially bounded function \( q' \) such that \( T^{\deg h_{q'(n)} - \deg f_n} f_n \in \text{End}_{p(q'(n)) + 1} h_{q'(n)} \).

Proof. Write \( X_i^{\deg h_{q(n)} - \deg f_n} f_n = \sum_{j=0}^{\deg f_n} X_i^{\deg h_{q(n)} - \deg f_n + j} f_{n,j} \), where \( f_{n,j} \) does not depend on \( X_i \). Since \( C \) is closed under interpolation, the family \( (\sum_{j=\deg f_n}^{\deg h_{q(n)} - \deg f_n} Y_j f_{n,j}) \in C \), too. Renaming the indices we see that also \( (\sum_{j=0}^{\deg f_n} Y_j f_{n,j}) \in C \). Now consider the family \( (T^{\deg h_{q(n)} - \deg f_n} X_i^j) \) and substitute it for the \( Y \)-variables into the former family. We get that \( (T^{\deg h_{q(n)} - \deg f_n} f_n) \in C \). Now since \( (h_n) \) is \( C \)-complete, the statement of the lemma follows.

Valiant’s classes

Objects: families of polynomials of polynomial degree
Computational model: Arithmetic circuits
Reductions: \( p \)-projections

- \( \text{VP} \) is characterized by circuits of polynomial size.
- \( \text{VP}_e \) is characterized by formulas of polynomial size.
- \( \text{imm}^{(3)} \) is complete for \( \text{VP}_e \).
Chapter 6

VP and the determinant

Constant size iterated matrix multiplication \( \text{imm}^{(3)} \) is complete for \( \text{VP}_e \). For VP, we do not know a nice complete polynomial, but first natural families have been found in [DMM+14]. For instance, it is not known whether general iterated matrix multiplication or the determinant are complete for VP. We first prove some normal forms for circuits for VP. Then we look for a subclass of VP such that the determinant and iterated matrix multiplication are complete for it.

6.1 Homogeneous circuits

Recall that a polynomial is homogeneous if all its monomials have the same total degree. A circuit is called homogeneous if at every gate it computes a homogeneous polynomial. Of course, nonhomogeneous polynomials cannot be computed by homogeneous circuits. However, we have the following result.

6.1.1 Lemma. If \( f \) is a polynomial of degree \( d \) that is computed by a circuit of size \( s \), then there is a homogeneous circuit of size \( O(d^2s) \) computing the homogeneous parts of \( f \). Furthermore, at every gate we only compute a polynomial of degree at most \( d \).

Proof. We replace every gate \( g \) by \( d + 1 \) gates. If \( g \) computes a polynomial \( f \), then the new gates will compute the homogeneous components of \( f \). We do this in a bottom up fashion. If \( g \) is an input gate, then there is nothing to do. We just have to add \( d \) dummy gates computing the zero polynomial. Let \( g \) be a gate with children \( h_1 \) and \( h_2 \) in the original circuit. Assume that \( h_1 \) and \( h_2 \) have been replaced by gates \( h_{1,0}, \ldots, h_{1,d} \) and \( h_{2,0}, \ldots, h_{2,d} \) computing polynomials \( p_{1,0}, \ldots, p_{1,d} \) and \( p_{2,0}, \ldots, p_{2,d} \), respectively. If \( g \) is an addition gate, then we will introduce new gates \( g_{0}, \ldots, g_{d} \) computing \( p_{1,i} + p_{2,i} \). If \( g \) is a multiplication gate, then \( g_{i} \) computes \( \sum_{j=0}^{d} p_{1,j}p_{2,i-j} \).

6.1.2 Corollary. If \( f \) is a polynomial of degree \( d \) that is computed by an arithmetic circuit of size \( s \), then there is a circuit \( C \) of size \( \text{poly}(s,d) \) computing \( f \) such that every node in \( C \) computes a polynomial of degree at most \( d \). Furthermore, for every multiplication gate, at least one of the inputs is not a constant.

Proof. We homogenize the given circuit as above. This immediately gives the upper bound on the degree. When two constants are multiplied, then either two degree zero components are multiplied or one of the higher degree homogeneous parts became zero. In the first case, we can replace the multiplication gate by an input gate labeled with the product of the two constants. (Remember that we can use every constant from \( F \).) In the second case, we simply can remove the gate that outputs 0. (Note that we do not have to construct the circuit, we just need to prove it existence.)
6.2 Multiplicatively disjoint circuits

6.2.1 Definition. An arithmetic circuit is multiplicatively disjoint if for all multiplication gates, the subcircuits induced by its two children are disjoint.

Multiplicatively disjoint circuits are between circuits and formulas. In a formula, also the subcircuits of addition gates are disjoint. Note that in a multiplicative disjoint circuit, only the induced subcircuits are disjoint. Nodes of these subcircuits can be connected to arbitrary nodes outside these circuits.

6.2.2 Definition. Let C be an arithmetic circuit. The formal degree of a gate g is defined inductively: A leaf has formal degree 1. If g is a multiplication gate, then its formal degree is the sum of the formal degrees of its two children. If g is an addition gate, then the formal degree of g is the maximum of the formal degrees of its children. The formal degree of C is the formal degree of its output gate.

The formal degree of a circuit disregards that the degree at gate might drop when there are cancellations. Multiplications with constants might also increase the formal degree.

6.2.3 Lemma. If a circuit has size s and formal degree d, then there is a multiplicatively disjoint circuit C′ of size ≤ sd computing the same polynomial.

Proof. Each gate g of formal degree e ≤ d will be replaced by d + 1 − e copies g₁, ..., gₙ. Let gᵢ be one of these copies. We call i the index of the copy. We will make sure that all gates of the subcircuit with output gᵢ are copies with an index lying between i and i + e − 1. In this way we ensure that we will get multiplicatively disjoint circuits.

Inductively, we construct a circuit Cᵩ with the following property: For each gate g of formal degree f ≤ e in C, there are copies of the gates g₁, ..., gᵢ₋₁ − f in Cᵩ computing the same function as g and all the gates of the subcircuit with root gᵢ have indices lying between i and i + f − 1.

The nodes of formal degree one are all input nodes and sums of formal degree one nodes. C₁ consists of d copies of the formal degree 1 nodes. Since C is acyclic, we can order the addition gates in such a way, that whenever we deal with a gate g, all its predecessors have been processed. For each addition gate g of formal degree one, we add copies g₁, ..., gᵢ. Let gᵢ’, gᵢ’’ be the children of g in C with formal degrees one. We connect gᵢ with the copy gᵢ’ and gᵢ’’. The restriction on the ranges is fulfilled by construction.

Assume that we constructed Cᵩ₋₁ (induction hypothesis). To obtain Cᵩ, we now add copies of all gates g of formal degree e in C. Let gᵢ’ and gᵢ’’ be the children of such a gate g of formal degrees e’ and e” respectively.

We start with the multiplication gates. In this case e = e’ + e” with e’, e” < e. This means that the copies g₁, ..., gᵢ₋₁ − e’, gᵢ’, ..., gᵢ₋₁ − e” were constructed in a previous step. We add the copies g₁, ..., gᵢ₋₁ − e and connect gᵢ with gᵢ’ and gᵢ’’ e’ and e”. These copies exist, since i ≤ d + 1 − e ≤ d + 1 − e’ and i + e’ ≤ d + 1 − e + e’ = d + 1 − e”. The indices of the copies of the subcircuit with root gᵢ’ lie between i and i + e’ − 1, the indices of the copies of the subcircuit with root gᵢ’’ lie between i + e’ and i + e’ + e” − 1 = i + e − 1. The two subcircuits of gᵢ are disjoint, because they contain gates with indices from two disjoint intervals. Therefore the condition on the indices of the subcircuits is fulfilled.

Next come the addition gates of formal degree e. Note that an addition gate of formal degree e might have a predecessor of formal degree e. As in the base case, we can order the addition gates in such a way, that whenever we deal with a gate g, all its predecessors have been processed. For each addition gate g of formal degree e, we add copies g₁, ..., gᵢ₋₁. Let gᵢ’ and gᵢ’’ be the children of g in C with formal degrees e’ ≤ e and e” ≤ e, respectively. We connect gᵢ with the copy gᵢ’ and gᵢ’’. The indices of the copies in these subcircuits lie in the range from i to i + e’ − 1 ≤ i + e − 1 and i + e” − 1 ≤ i + e − 1, respectively.
The circuit $C_d$ is the circuit we are looking for. It contains a copy of the output gate of $C$. The circuit is multiplicatively disjoint by the way we chose the indices when connecting the copies of the children to the multiplication gate.

\textbf{6.2.4 Lemma.} Let $f$ be a polynomial of degree $d$ computed by a circuit $C$ of size $s$. Then there is a circuit of size polynomial in $d$ and $s$ computing $f$ such that its formal degree is bounded by $sd + 1$.

\textit{Proof.} Let $C$ be the given circuit and $C'$ be the circuit constructed in Corollary 6.1.2. Recall that the circuit $C'$ is a simulation of the circuit $C$. Every node is replaced by $d + 1$ nodes, one for each homogeneous component. Then every operation in $C$ is simulated by several operations in $C'$. Let the depth of a gate in $C$ be the length of a longest path from any leaf to this gate. The depth of the nodes in $C'$ that compute the homogeneous components is defined as the depth of their corresponding node in $C$. We do not define depth for the other nodes in $C'$. We will now prove by induction on the depth that the formal degree of any gate $g$ of $C'$ of depth $\delta$ computing a homogeneous component of degree $i$ is bounded by $\delta \cdot i + 1$. For the base case note that every leaf has formal degree 1. Now let $g$ be a gate in $C'$ of depth $\delta$ computing a homogeneous component of degree $i$. If $i = 0$, then note that $g$ has formal degree 1 by construction. So we assume that $i \geq 1$. We first treat the case when $g$ corresponds to an addition gate in $C$. In $C'$, $g$ is an addition gate, its two inputs are gates $g'$ and $g''$ both computing homogeneous polynomials of degree $i$. The formal degree of these two gates is bounded by $\delta' \cdot i + 1$ and $\delta'' \cdot i + 1$ where $\delta'$ and $\delta''$ are the depth of $g'$ and $g''$, respectively. The formal degree of $g$ is $\max\{\delta' \cdot i + 1, \delta'' \cdot i + 1\} \leq \delta \cdot i + 1$.

If $g$ is a multiplication gate, then

$$g = \sum_{j=0}^{i} g'_j g''_{i-j}$$

where $g'_j$ and $g''_{i-j}$ are the homogeneous components of the predecessors of $g$. By the induction hypothesis, the formal degrees of $g'_j$ and $g''_{i-j}$ are bounded by $\delta'j + 1$ and $\delta''(i-j) + 1$, respectively. The formal degree of $g'_j g''_{i-j}$ is bounded by $\delta' j + 1 + \delta''(i-j) + 1 \leq \delta i + 1$, when $0 < i < j$. Note for the upper bound that $\delta > \delta', \delta''$, and $i \geq 1$. The formal degree of $g'_0 g''_i$ is bounded by $1 + \delta'' i + 1 \leq \delta i + 1$. The same argument works for $g'_j g''_0$. This concludes the inductive step.

From the claim the bound on the formal degree of the new circuit follows immediately.

\textbf{6.2.5 Theorem (Malod & Portier [MP08]).} A $p$-family $(g_n)$ is in VP if and only if there is a family of polynomial size multiplicative disjoint circuits $(C_n)$ computing $(g_n)$.

\textit{Proof.} If $(g_n) \in$ VP, then by Lemma 6.2.4, there is a sequence of circuits $(C_n)$ of size poly$(n)$ computing $(g_n)$ such that the formal degree of $C_n$ is polynomially bounded. Now we can apply Lemma 6.2.3.

For the other direction, note that it can be easily proven by induction that the degree of a multiplicatively disjoint circuit of size $s$ is bounded by $s$.

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**Circuits for VP**

The following models characterise VP:

- arithmetic circuits of polynomial size (and the degree of the family is polynomially bounded)
- multiplicative disjoint circuits of polynomial size
- homogeneous circuits of polynomial size (when the family is homogeneous of polynomial degree)
6.3 Combinatorial interpretation of the determinant

Let $M = (m_{i,j})$ be an $n \times n$ matrix. We can interpret $M$ as the weighted adjacency matrix of some directed graph over the node set $\{1, \ldots, n\}$. For every $(i,j)$, there is an edge $(i,j)$ of weight $m_{i,j}$. A cycle cover in a directed graph is a collection of node-disjoint directed cycles such that every node is contained in exactly one cycle. Permutations in $S_n$ stand in a one-to-one correspondence with cycle covers. Every permutation $\sigma$ yields a cycle cover consisting of the edges $(i, \sigma(i))$. On the other hand, the edges of a cycle cover encode a permutation of the nodes with the interpretation that an edge $(i,j)$ means that $i$ is mapped to $j$. Note that this is nothing but the cycle decomposition of a permutation. The sign of the permutation is $-1$ if the number of even-length cycles is odd, and $1$ if it is even. The weight $w(C)$ of a cycle cover $C$ is the product of the weights of the edges in it. Therefore,

$$\det M = \sum_{\text{cycle covers } C} (-1)^{n+\text{number of cycles in } C} w(C).$$

Conceptually, it is often easier to think of an edge of weight zero as not being present in the graph. Since the weight of a cycle cover is the product of its edge weights, this does not make any difference in the above equation for $\det M$.

Instead of interpreting $M$ as the adjacency matrix of some directed graph, we can also interpret it as the adjacency matrix of some bipartite graph. We have nodes $\{1, \ldots, n\}$ on the lefthand side and “copies” $\{1', \ldots, n'\}$ on the other side. For every $(i,j)$, there is an edge $(i,j')$ with weight $m_{i,j}$. A matching $N$ in a graph is a set of edges such that every node is incident with at most one edge from $N$. It is called perfect, if every node node is incident with exactly one edge from $N$. Permutations in $S_n$ stand in a one-to-one correspondence with perfect matchings in bipartite graphs: Every permutation $\sigma$ yields a perfect matching consisting of the edges $(i, \sigma(i))$. This construction can be reversed. If we set the sign of a perfect matching in a bipartite graph to be the sign of the corresponding permutation, we get the following expression:

$$\det M = \sum_{\text{perfect matchings } N} \text{sgn}(N)w(N).$$

The weight $w(N)$ is the product of the weights of the edges in $N$.

6.4 Weakly skew circuits and algebraic branching programs

6.4.1 Definition. A circuit is called weakly skew if every multiplication gate $g$ has at least one child $g'$ such that after removing the edge $(g', g)$, the graph consists of two weakly connected components.

In a formula, this is true for every child of a gate. In a formula, no intermediate result is reusable, that is, the output of every gate can only be used as the input of exactly one other gate. In a weakly skew circuit, one child of every gate can be reused, but not both. Weakly skew is however stronger than multiplicatively disjoint, since in the later case, while the subcircuits need to be disjoint, they can be connected to the rest of the circuit.

6.4.2 Definition. Let $F$ be a field and $X_1, \ldots, X_n$ be indeterminates.

1. An algebraic branching program $A$ is an acyclic graph with two distinguished nodes $s$ and $t$ and an edge labeling with labels from $F \cup \{X_1, \ldots, X_n\}$.\(^1\)

2. The weight $w(P)$ of a path $P$ from $s$ to $t$ is the product of the labels of the edges in the path.

\(^1\)Some authors allow affine linear forms, but this will not make any difference.
3. The polynomial computed by $A$ is
   \[ \sum_{s \to v \text{ path } P} w(P). \]

4. The size of an algebraic branching program is the number of edges in it.

5. $A$ is called layered if for every node $v$ in $A$, all $s$-$v$ paths have the same length.

If $A$ is layered, then we can think of the nodes of $A$ being grouped into layers: two nodes $u$ and $v$ are in the same layer $i$ if the lengths of all paths from $s$ to $u$ and from $s$ to $v$ is $i$. In a layered branching program, edges only go from one layer to the next.

6.4.3 Lemma. Let $A$ be a branching program of size $s$. Then there is a layered branching program of size $O(s^2)$ computing the same function.

Proof. For a node $v$ in the branching program, let $d(v)$ be the length of a longest path from $s$ to $v$. The node $v$ will be put into layer $d(v)$. By construction, for every edge $\{u, v\}$, we have $d(u) < d(v)$. Therefore, we only have edges from layers with smaller index to larger index. If there is an edge $e$ from layer $i$ going to layer $j$ with $i + 1 < j$, then we replace this edge by a path of length $j - i$ and put the nodes of this path into the layers in between. One (arbitrary) edge of the path gets the weight of $e$ and all other edges get the weight 1.

We formalize the notion of being reusable. Intuitively, a gate in a weakly skew circuit is reusable if it is not in the subcircuit of a multiplication gate that is not connected to the rest of the circuit.

6.4.4 Definition. Let $C$ be a weakly skew arithmetic circuit. The set of reusable gates in $C$ is inductively defined as follows: Every gate of outdegree zero is reusable. (We consider circuits with multiple output gates to simplify some proofs in the following.) We remove every gate $g$ of outdegree zero from $C$ and for each such multiplication gate, we also remove the subcircuit of that child $g'$ that is only connected to the rest of the circuit via the edge $(g', g)$. Let $C'$ be the resulting circuit. Every gate that is reusable in $C'$ is reusable in $C$, too.

6.4.5 Theorem. Let $f \in F[X_1, \ldots, X_n]$ with $\deg f = \text{poly}(n)$. The following statements are equivalent:

1. $f$ is computed by a weakly skew circuit of size $\text{poly}(n)$.
2. $f$ is computed by an algebraic branching program of size $\text{poly}(n)$.
3. $f$ is a projection of $\text{imm}_{\text{poly}(n)}$ for some polynomially bounded function $p$.
4. $f$ is a projection of $\text{det}_{\text{poly}(n)}$ for some polynomially bounded function $p$.

Proof. (1) \implies (2): Assume that $f$ is computed by a weakly skew circuit $C$ of size $m$. We now prove by induction on $m$ that there is a algebraic branching program computing $A$ of size $\leq 2m$ such that for every reusable gate $g$ in $C$ there is a node $v_g$ such that the sum of the weights of all paths from $s$ to $v_g$ is the same polynomial as computed at $g$.

Let $g$ be some output node. If $g$ is also an input node, then $A$ consists of a single edge. (This is the induction basis.)

For the induction hypothesis, assume that $g$ is not an input gate. If $g$ is an addition gate, then we remove $g$ from $C$, let $C'$ be the resulting circuit. By the induction hypothesis, there is an algebraic branching program $A'$ such that for every gate $g'$ that is reusable in $C'$, there is a node $v_{g'}$ in $C'$ such that the sum of the weights of all path from $s$ to $v_{g'}$ equals the polynomial computed at $g'$. Let $h$ and $h'$ be the children of $g$. They are both reusable. We add a new node $v_g$ and connect the nodes $v_h$ and $v_{g'}$ to it. Both edges get weight one. If $h = h'$, then we add only one edge with weight two. By construction, the sum of the weights of all paths from $s$ to $v_g$ is the sum of the polynomials computed at $h$ and $h'$. The resulting algebraic branching program has two more edges than $A'$. For all reusable nodes $g'$ of $C'$, the node $v_{g'}$ is still present in $A$ and the sum of the weights of all path from $s$ to $v_{g'}$ equals the polynomial computed at $g'$. 

If \( g \) is a multiplication gate, then after removal of \( g \), we get two separate circuits \( C_1 \) and \( C_2 \).
Let \( g_1 \) and \( g_2 \) be the children of \( g \). Only the gates of one of them, say \( C_2 \), can be reusable in \( C \).
Let \( m_1 \) and \( m_2 \) be the sizes of \( C_1 \) and \( C_2 \). From the induction hypotheses, we get corresponding algebraic branching programs \( A_1 \) and \( A_2 \) with sources \( s_1 \) and \( s_2 \). In \( A_1 \), there are vertices \( s_1 \) and \( v_{g_1} \) such that the sum of the weights of all path from \( s_1 \) to \( v_{g_1} \) equals the polynomial computed at \( g_1 \). We connect the node \( v_{g_2} \) in \( A_2 \) with the node \( s_1 \) of \( A_1 \) by an edge of weight 1. Then the sum of the weights of all path from \( s_2 \) to \( v_{g_2} \) is the product computed at \( g \). For all gates \( h \) in \( C_2 \), the sum of the weights of all paths from \( s_2 \) to \( v_h \) paths equals the polynomial computed at \( h \).
The size of the new branching program is \( 2m_1 + 2m_2 \leq 2m \).\(^2\)

(2) \( \Rightarrow \) (3): Let \( A \) be an algebraic branching program computing \( f \). By Lemma 6.4.3 we can assume that \( A \) is layered. Let \( \ell \) be the maximum size of a layer and let \( m \) be the number of layers. We order the nodes in each layer arbitrarily. We will inductively construct \( \ell \times \ell \)-matrices \( M_1, \ldots, M_m \) with entries from \( \mathbb{F} \cup \{ X_1, \ldots, X_n \} \) such that the first row of \( M_1 \cdots M_i \) are the polynomials computed at the nodes in the \( i \)th layer, that is, the sum of the weights of all path from \( s \) to each node in this layer. \( M_1 \) has a 1 in position \((1,1)\) and zeros everywhere else. This single 1 corresponds the the source node \( s \). Assume we constructed \( M_1, \ldots, M_{i-1} \). Let \((a_1, \ldots, a_{\ell}) \) be the first row of \( M_1 \cdots M_i \). A node \( v \) in the \((i+1)\)th layer receives edges from the nodes of the \( i \)th layer. Let \((b_1, \ldots, b_{\ell}) \) be the labels of these edges (if an edge is not present, the corresponding \( b_j = 0 \)). The polynomial computed at \( v \) is given by

\[
(a_1, \ldots, a_{\ell}) \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_{\ell} \end{pmatrix}.
\]

The matrix \( M_{i+1} \) simply consists of the corresponding columns \((b_1, \ldots, b_{\ell})^T \). If the \((i+1)\)th layer has less than \( \ell \) nodes, we append zero rows to \( M_{i+1} \).

Since we can embed a product of \( m \ \ell \times \ell \)-matrices into a product of \( d \ d \times d \)-matrices with \( d = \max\{m, \ell\} \), we get that \( f \) is a projection of \( \text{imm}_{\text{poly}(n)} \).

(3) \( \Rightarrow \) (4): Note that an iterated matrix product can be easily computed by a layered algebraic branching program, you just have to “reverse” the construction of the previous step. Therefore it suffices to prove that every polynomial that is computed by a layered algebraic branching program \( A \) is a projection of a determinant of polynomial size. We modify \( A \) as follows: add an edge of weight one from \( t \) to \( s \) and add a self loop of weight one to every node except \( s \) and \( t \). Let \( M \) be the weighted adjacency matrix of this modified program \( A'. \det M \) is the sum of the weights of all cycle covers in \( A' \). All cycle covers in \( A' \) consist of one big cycle through \( s \) and \( t \) and the remaining nodes are covered by self-loops. Since the program is layered, all cycle covers have the same number of cycles and therefore the same sign. The weight of a cycle cover equals the weight of the corresponding path from \( s \) to \( t \), potentially with an opposite sign (but this sign is the same for all cycle covers). Therefore, \( f \) is a projection of a determinant of a polynomially large matrix.

(4) \( \Rightarrow \) (1): One way to evaluate the determinant by a weakly skew circuit is known as Csanky’s algorithm [Csa76]. Another one is due to Mahajan and Vinay [MV97].

6.4.6 Remark. For a polynomial \( f \), the smallest \( n \) such that \( f \) can be written as a projection of \( \det_n \) is called the determinantal complexity \( \text{dc}(f) \) of \( f \).

6.4.7 Definition. A \( p \)-family \((f_n)\) is in VP\(_{\text{ws}} \) if it is computed by weakly skew circuits of polynomial size.
Theorem 6.4.5 gives us further, equivalent definitions of VP<sub>ws</sub>. In particular, a p-family (f<sub>n</sub>) is in VP<sub>ws</sub> if it is a p-projection of the determinant family. Note that imm can be computed by very restricted weakly skew circuits, namely for every multiplication gate, one of the inputs is a variable or a constant. We call such circuits skew. This is achieved by sequentially multiplying the matrices using the trivial methods. Since by Theorem 6.4.5, every polynomial that is computed by a weakly skew circuits of polynomial size is a p-projection of imm, we get the following corollary.

**6.4.8 Corollary.** If a polynomial is computed by a weakly skew circuit of size <i>s</i>, then it is computed by a skew circuit of size poly(<i>s</i>).

### The determinant

VP<sub>ws</sub> describes the complexity of the determinant. Equivalent models are: algebraic branching programs and projections of iterated matrix multiplication.
Chapter 7

The permanent

In this chapter we define the class VNP and prove the VNP-completeness of the permanent polynomial.

7.1 VNP and formulas

A language \( L \) is in NP if there is a deterministic polynomial time relation \( R \) such that for all \( x, x \in L \) iff there is a polynomially long bit string \( y \) such that \( R(x, y) = 1 \). Think of \( x \) being a formula in 3-CNF and \( y \) being an assignment. \( R(x, y) = 1 \) means that \( y \) satisfies \( x \). The class \( \#P \) is the class of functions which assign to each \( x \) the number of \( y \) such that \( R(x, y) = 1 \), that is, we compute

\[
\sum_y [R(x, y) = 1].
\]

Here, the bracket is Iverson bracket, which is one if the Boolean expression is true. So in our example, we want to count the number of satisfying assignments.

7.1.1 Definition. 1. A p-family \((f_n)\) is in \( \text{VNP} \), if there are polynomially bounded functions \( p \) and \( q \) and a sequence \((g_n) \in \text{VP}\) of polynomials \( g_n \in \mathbb{F}[X_1, \ldots, X_{p(n)}, Y_1, \ldots, Y_{q(n)}] \) such that

\[
f_n = \sum_{e \in \{0, 1\}^{q(n)}} g_n(X_1, \ldots, X_{p(n)}, e_1, \ldots, e_{q(n)}).\]

2. A family of polynomials \( f_n \) is in \( \text{VNP}_e \) if in the definition of \( \text{VNP} \), the family \((g_n)\) is in \( \text{VP}_e \).

You can think of the \( X \)-variables representing the input and the \( Y \)-variables the witness. With this interpretation, \( \text{VNP} \) is more like \( \#P \). In particular, we will see that the permanent polynomial

\[
\text{per}_n = \sum_{\sigma \in \mathfrak{S}_n} X_{1, \sigma(1)} \cdots X_{n, \sigma(n)}
\]

is complete for \( \text{VNP} \).

With the help of so-called parse-trees we will now show \( \text{VNP} = \text{VNP}_e \).

7.1.2 Definition. Let \( C \) be an arithmetic circuit.

1. A parse tree of \( C \) is defined recursively as follows: Every circuit consisting of one node is a parse tree. If the size of \( C \) is larger than one, let \( g \) be the output gate and \( g_1 \) and \( g_2 \) be its children. Let \( C_1 \) and \( C_2 \) be the subcircuits with output gates \( g_1 \) and \( g_2 \). If \( g \) is an addition gate, then we get the set of all parse trees by either taking a parse tree of \( C_1 \) or a path tree of \( C_2 \) and connecting it to \( g \). If \( g \) is a multiplication gate, then we get the set of all parse trees by taking a parse of \( C_1 \) and a parse tree of \( C_2 \) and connecting both to \( g \).
2. The set of all parse trees of $C$ is denoted by $\text{pt}(C)$.

3. The weight $w(T)$ of a parse tree $T$ is the product of the labels of its leaves.

For every multiplication gate, we have to include both children in the parse tree, for every addition gate we have to choose one of them. Note that a gate may occur several times in a parse tree, since it is reused in the circuit several times. For each occurrence in the parse tree, we introduce a new copy. (Otherwise, it would not be a tree.)

**7.1.3 Exercise.** Let $C$ be a circuit and $p$ be the polynomial computed by $C$. Prove (for instance by structural induction) that

$$p = \sum_{T \in \text{pt}(C)} w(T).$$

**7.1.4 Lemma.** A circuit $C$ is multiplicatively disjoint if every parse tree of $C$ is a subcircuit of $C$.

Proof. Assume that $C$ is not multiplicatively disjoint. Then there is a node $v$ in $C$ such that there are two node disjoint paths to some multiplication gate $g$. Since $g$ is a multiplication gate, these two paths can be extended to a parse tree.

Conversely, if there is a parse tree $T$ that is not a subcircuit of $C$, then there are gates $g$ and $h$ in $C$ such that there are two node disjoint paths from $g$ to $h$. Since $T$ is a parse tree, $h$ is a multiplication gate. Thus, $C$ is not multiplicatively disjoint. \qed

**7.1.5 Lemma.** Let $C$ be a multiplicatively disjoint circuit with edge set $E$. For each edge $e \in E$, let $X_e$ be an indeterminate. There is a formula $F$ in the $X_e$'s of size polynomial in the size of $C$ such that for every $a \in \{0, 1\}^{|E|}$, $F(a)$ is the weight of the parse tree, if the edges “selected” by the vector $a$ form a parse tree in $C$, and zero otherwise.

Proof. Since by Lemma 7.1.4, every parse tree is a subcircuit of $C$, it is sufficient to consider subtrees of the given circuit. For every node $v$ in $C$, we introduce an additional variable $Y_v$. Note that for $\{0, 1\}$ valued variables $X$ and $Y$, we can simulate Boolean AND by $XY$ and Boolean NOT by $1 - X$. We can write the fact that a given vector encodes a parse tree by the following Boolean expressions:

$$\bigwedge_{(i, j) \in E} X_{(i, j)} \Rightarrow Y_i \land Y_j$$

ensures that whenever an edge is selected, its end points are selected, too. Let $g$ be the output gate of $C$. Then

$$Y_g$$

ensures that the output gate is selected. For a gate $g$, let $\ell(g)$ and $r(g)$ be its children. The following expression ensures that for every selected multiplication gate $g$ that is selected, both incoming edges are selected, too.

$$\bigwedge_{\text{multiplication gate } g} Y_g \Rightarrow X_{\ell(g), g} \land X_{r(g), g}.$$
We can eliminate all occurrences of the newly introduced variables by replacing $Y_v$ by the expression
\[
\bigwedge_{(v,u) \in E} X_{(v,u)}
\]
and $Y_g$ by 1. The Boolean AND of these expressions is a Boolean formula that is true iff the vector $a$ encodes a parse tree. By the considerations above, it can be replaced by an arithmetic formula. If $a$ encodes a parser tree, we can get the corresponding weight by the following expression:
\[
\prod_{v \in V} (Y_v \cdot w_v + 1 - Y_v).
\]
Here $w_v$ is the label of $v$ if it is an input gate and 1 otherwise. Again, we can eliminate the $Y_v$'s as above. The product of the two expressions, one for checking whether $a$ is a parse tree and one for computing its weight, is the formula $F$.

**7.1.6 Corollary.** Let $f$ be a polynomial computed by an arithmetic circuit of size $s$. Then there is an arithmetic formula $F$ of size polynomial in $s$ and a polynomially bounded $p$ such that
\[
f(X) = \sum_{a \in \{0,1\}^{s(n)}} F(X, a).
\]

**Proof.** This follows from combining Theorem 6.2.5, Exercise 7.1.3, and Lemma 7.1.5.

**7.1.7 Theorem.** $\text{VNP} = \text{VNP}_e$.

**Proof.** Let $(f_n)$ be in VNP and $(g_n) \in \text{VP}$ such that
\[
f(X) = \sum_{e \in \{0,1\}^{n(\epsilon)}} g_n(X, e).
\]
Using Corollary 7.1.6, there is a formula $F_n$ of polynomial size such that
\[
g_n(X, Y) = \sum_{a \in \{0,1\}^{s(n)}} F_n(X, Y, a).
\]
Therefore,
\[
f(X) = \sum_{e \in \{0,1\}^{n(\epsilon)}, a \in \{0,1\}^{s(n)}} F_n(X, e, a).
\]

While the statement of the theorem sounds astonishing at a first glance, it just uses the fact that we can write the result of a polynomially large circuit by an exponential sum over a polynomially large formula and then combines the two exponential sums into one.

### 7.2 Hardness of the permanent

Let $G = (V, E)$ be an edge weighted graph. Recall that a cycle cover $C$ of $G$ is a selection of node disjoint directed cycles such that every node is contained in exactly one cycle. The weight $w(C)$ of $C$ is the product of the weight of the edges in $C$. Cycle covers can be viewed as the graph of a permutation. The cycles in the cycle cover correspond to the cycles in the cycle decomposition of a permutation. If we also write $G$ for the weighted adjacency matrix of $G$ (by abuse of notation), then
\[
\text{per}(G) = \sum_{\text{cycle cover } C \text{ of } G} w(C).
\]
Figure 7.1: The equality gadget. The pair of edges \((u, v)\) and \((u', v')\) of the left-hand side is connected as shown on the right-hand side.

Figure 7.2: First row: The one possible configuration if both edges are taken. Second row: The six possible configurations if none of the edges is taken.

Let \(G\) be a graph and \(e = (u, v)\) and \(e' = (u', v')\) be two edges in \(G\). As a first step, we want to replace \(G\) by a graph \(\hat{G}\) such that \(\text{per}(\hat{G})\) is the sum over all \(w(C)\) such that \(C\) is a cycle cover of \(G\) that either contains both \(e\) and \(e'\) or none of them. This is achieved by subdividing the edges and connecting them by an equality gadget as depicted in Figure 7.1.

Let \(C\) be a cycle cover of \(G\) that takes both edges. Then there is one way to extend this to a cycle cover of \(\hat{G}\). The weight of this new cycle cover is \(2 \cdot w(C)\), see Figure 7.2. When \(C\) does not take any of the two edges, then there are six ways to extend \(C\). These six ways sum up to weight \(2 \cdot w(C)\).

If \(C\) is a cycle cover of \(G\) that takes only one edge of \(e\) and \(e'\), say \(e\), then there are two ways to extend \(C\) to \(\hat{G}\), see Figure 7.3. The weight of these covers is the same, but they differ in sign, therefore the contributions of these two covers cancel each other.

Finally, there are inconsistent ways to cover the equality gadget in \(\hat{G}\), that is, covers of \(\hat{G}\) that do not correspond to any cover in \(G\), see Figure 7.3. Again, we can form pairs of these covers such that the contribution of these covers cancel each other.

This construction proves the following lemma.

7.2.1 Lemma. Let \(\mathbb{F}\) be a field of characteristic distinct from 2. Let \(G\) be a graph and \(e\) and \(e'\) be edges in \(G\). Then there is a graph \(\hat{G}\) such that

\[
\frac{1}{2} \text{per}(\hat{G}) = \sum_{C} w(C),
\]

where the sum is taken over all cycle covers \(C\) of \(G\) that either use both of \(e\) and \(e'\) or none of them.
Let \((f_n) \in \text{VNP}\) and let \((g_n) \in \text{VP}\) such that
\[
f_n(X_1, \ldots, X_{p(n)}) = \sum_{e \in \{0, 1\}^{q(n)}} g_n(X_1, \ldots, X_{p(n)}, e_1, \ldots, e_{q(n)}).
\]
By Theorem 7.1.7 we may assume that \((g_n) \in \text{VP}_e\). We proved that every polynomial that is computed by a formula of size \(s\) is a projection of a determinant of polynomial size. The same proof yields that it is also a projection of a polynomially large permanent, since the cycle covers of the arithmetic branching program occurring in the proof all had the same sign. It follows that we can write \(f_n\) as an exponential sums of permanents. The permanent itself is an exponential sum. So we are done if we can “squeeze” the outer exponential sum into the inner one.

The rosette graph of size \(t\) consists of a directed cycle of size \(t\). The edges \(c_1, \ldots, c_t\) of this cycle are called connector edges. The head and the tail of each connector edge are connected by a path of length two. Every node has a self-loop. All edges have weight one in the rosette graph. The following fact is easily verified:

7.2.2 Lemma. Let \(S\) be a subset of the connector edges.

1. If \(S\) is nonempty, then there is exactly one cycle cover of the rosette graph containing the edges in \(S\) and no other connector edges.

2. There are two cycle covers containing no connector edges.

\(g_n\) is a projection of a polynomially large permanent. This means that there is an edge weighted graph \(G\) (with the weights being field elements and variables) such that
\[
g_n(X_1, \ldots, X_{p(n)}, Y_1, \ldots, Y_{q(n)}) = \sum_{\text{cycle cover } C} w(C).
\]
Assume that the variable \(Y_i\) occurs \(\ell_i\) times in \(G\). We add a rosette graph of size \(\ell_i\) and connect every edge labeled with \(Y_i\) with one of the connector edges of the rosette using an equality gadget. All edges inherit their weights from the corresponding subgraphs, that is, the edges from \(G\) get the weights they have in \(G\), the edges in the equality gadgets keep their weights, and the edges in the rosette graph all have weight one. The only exception are the edges carrying a weight \(Y_i\) in \(G\), they get the weight 1 instead. We do this for each \(i\). Assume, we introduced \(t\) equality gadgets.
altogether. We will add one isolated self loop with weight $1/2^t$ to compensate for the 2 that is introduced by every equality gadget. (The characteristic of $k$ should be distinct from 2 for this!) Let $H$ be the resulting graph.

Let $C$ be a cycle cover of $G$. $w(C)$ is a monomial $m(X_1,\ldots,X_n,Y_1,\ldots,Y_{q(n)})$. Let $I$ be the set of indices such that $Y_i$ appears in $w(C)$. What is the contribution of $C$ in

$$\sum e_gn\left(X_1,\ldots,X_{p(n)},e_1,\ldots,e_{q(n)}\right)?$$

If $Y_i$ appears in $w(C)$, then we have to set $e_i = 1$, otherwise, the contribution to the exponential sum will be zero. If $Y_i$ does not appear in $w(C)$, then we can set $e_i$ to 0 or 1. Therefore, the contribution of $C$ is

$$2^{q(n)-|I|}m(X_1,\ldots,X_{p(n)},1,\ldots,1).$$

We call a cycle cover $D$ of $H$ consistent if for every equality gadget, either both edges it connects are chosen or none of them is chosen. A cycle cover $C$ of $G$ can be extended to a consistent cycle cover of $H$. If an edge with label $Y_i$ appears in $C$, then we can extend it in one possible way in the corresponding rosette. If no such edge appears in $C$ then there are two ways. In total, there are $2^{q(n)-|I|}$ extensions. By Lemma 7.2.1, we know that

$$\text{per}_H = \sum_{\text{consistent } D} w(D).$$

Therefore,

$$\text{per}_H = \sum_e g_n(X_1,\ldots,X_{p(n)},e_1,\ldots,e_{q(n)}).$$

7.2.3 Theorem. Over fields of characteristic distinct from 2, per is VNP-complete.

Proof. It remains to show that per $\in$ VNP. It is quite easy to write a Boolean expression $E(Y)$ of polynomial size which checks whether a given matrix $Y \in \{0,1\}^{n\times n}$ is a permutation matrix. As done before, we can write this as an equivalent arithmetic formula $\hat{E}(Y)$. Now it is easy to check that

$$\text{per}_X = \sum_{Y \in \{0,1\}^{n\times n}} \hat{E}(Y) \prod_{i,j}(X_{i,j}Y_{i,j} + 1 - Y_{i,j}).$$

Over fields of characteristic 2, the permanent can only be VNP-hard, if VNP = VP, since it coincides with the determinant in this case. But there are other VNP-complete polynomials that are also hard over fields of characteristic two.
7.3 Valiant’s conjecture

Valiant’s conjecture is the algebraic counterpart of the P versus NP conjecture.

7.3.1 Conjecture (Valiant). VP ⊊ VNP.

Since the permanent is VNP-complete, we can rephrase this conjecture as

\[ \text{per} \notin \text{VP}. \]

Since VP ws ⊊ VP, we can formulate a weaker (or stronger, depending on your point of view) version of Valiant’s conjecture, namely, VP ws ⊊ VNP. Since VP ws has a nice complete family, this version can be reformulated as

\[ \text{per} \leq_{p} \text{det}. \]

It is easy to check that VP ws is closed under interpolation and substitutions. Therefore, by Lemma 5.4.6, the conjecture VP ws ⊊ VP can also be restated as

\[ \text{per} \leq_{end} \text{det}. \]

Here, geometric complexity theory starts. As in the case of Waring rank, we will replace arbitrary endomorphisms by invertible ones.

### Valiant’s conjecture

- VNP = VNP c
- per is complete for VNP over fields of characteristic \( \neq 2 \).
- Valiant’s conjecture: VP ⊊ VNP
- weaker variant: VP ws ⊊ VNP (equivalent per \( \leq_{end} \text{det} \))
Chapter 8

Border complexity and group orbit closures

In this chapter we explain why in Section 3.5 we went from monoid orbit closures to group orbit closures: We phrase the questions from algebraic complexity theory in terms of group orbit closures.

Recall that on the space of polynomials $A = \mathbb{C}[X_1, \ldots, X_M]$ we have seen several ways to measure complexity:

- minimal size of an arithmetic formula in which the leaves are constants or variables,
- minimal size of an arithmetic formula in which the leaves are affine linear forms,
- $\min \{ n \mid \ell^n - d_h \in \text{End}_{3n^2} \text{imm}_{n^3}(3) \}$, where $\ell$ denotes the padding variable (which could be one of the existing) variables.

For a p-family either all or none of these measures are polynomially bounded, that is, all measures can be used to characterize the class $\text{VP}_e$. Recall that $\text{VP}_e$ consists of all p-families where these measures are polynomially bounded.

The class $\text{VP}_{ws}$ is characterized by the following measures:

- minimal weakly skew circuit size,
- minimal skew circuit size,
- minimal algebraic branching program size,
- $\min \{ n \mid \ell^n - d_h \in \text{End}_{n^2} \text{imm}_n \}$,
- $\min \{ n \mid \ell^n - d_h \in \text{End}_{n^2} \text{det}_n \}$ (i.e., determinantal complexity),

For a p-family either all or none of these measures are polynomially bounded. The class $\text{VP}_{ws}$ consists of the p-families where these measures are polynomially bounded.

Both classes have a characterization in terms of an endomorphism orbit. Since group orbits are much easier to handle than monoid orbits, we replace each orbit by its closure. Then we can replace $\text{End}_m$ by $\text{GL}_m$ (like we did for Waring rank). We obtain new ways of measuring the complexity of polynomials.

We define $\text{VP}_e$ to be the class of p-families where $\min \{ n \mid \ell^n - d_h \in \text{GL}_{3n^2} \text{imm}_{n^3}(3) \}$ is polynomially bounded. We define $\text{VP}_{ws}$ to be the class of p-families where $\min \{ n \mid \ell^n - d_h \in \text{GL}_{n^2} \text{det}_n \}$ is polynomially bounded. It is easy to see that taking the closure in the other endomorphism description yields the same class.

In general for a set $C$ of sequences of polynomials we define its closure $\overline{C}$ as follows: The sequence $(f_n)_n$ is in $\overline{C}$ iff there exist polynomials $f_{n,i}$ such that
• for all $i$, the sequences $(f_{n,i})_{n \in \mathbb{N}}$ are in $C$

• for all $n$, the sequences $(f_{n,i})_{i \in \mathbb{N}}$ converge to $f_n$.

Clearly $C \subseteq \overline{C}$, in particular $\text{VP}_e \subseteq \overline{\text{VP}}_e$, $\text{VP}_{ws} \subseteq \overline{\text{VP}}_{ws}$, and $\text{VNP} \subseteq \overline{\text{VNP}}$. But the relationship between $C$ and $\overline{C}$ is unknown in most cases. It is not even known whether $\overline{\text{VP}}_e \subseteq \text{VNP}$. In particular we could have that $\text{VP}_e \neq \text{VNP}$ but their closures are the same. In this case, the geometric complexity theory would fail. It is a challenging problem to understand the closures of algebraic complexity classes.

The classical group orbit closure studied in geometric complexity theory is $\overline{\text{GL}_{n^2} \text{det}_n}$. For fixed $m$ and $n$ we search for ways to prove $\ell^{n-m} \text{per}_m \not\in \overline{\text{GL}_{n^2} \text{det}_n}$. Since $(\text{per}_m)$ is VNP-complete, proving superpolynomial lower bounds is equivalent so separating $\text{VNP} \not\subseteq \overline{\text{VP}_{ws}}$.

### Border complexity and group orbit closures

The lower bound questions in algebraic complexity theory can be stated in terms of border complexity.

Proving Border complexity lower bounds is a special case of the problem of separating a point from a group orbit closure.
Chapter 9

Representation Theory

In this chapter we study group actions until in Section 10.2 we obtain new significant search space restrictions for obstructions that come from representation theory. Our ground field is the complex numbers.

9.1 Key example: Lifting the group action

Let $\mathbb{A} = \mathbb{C}[X_1, \ldots, X_N]_d$. Recall that $\mathbb{C}[\mathbb{A}]_\delta$ is the vector space of homogeneous degree $\delta$ polynomials on $\mathbb{A}$. Let $G := \text{GL}_N$. Since $G$ acts on $\mathbb{A}$, we know that $G$ also acts linearly on $\mathbb{C}[\mathbb{A}]$ as follows: for every $g \in G$ and every $f \in \mathbb{C}[\mathbb{A}]$ we define the polynomial $gf \in \mathbb{C}[\mathbb{A}]$ via:

$$ (gf)(h) := f(g^T h). $$  \hspace{1cm} (9.1.1)

Completely analogously to Lemmas 3.2.5 and 3.2.6 we prove the following two lemmas.

9.1.2 Lemma. Let $f, f' \in \mathbb{C}[\mathbb{A}]$ and let $g \in G$. For all complex numbers $\alpha, \alpha'$ we have

$$ g(\alpha f + \alpha' f') = \alpha(gf) + \alpha'(gf'). $$

By induction Lemma 9.1.2 holds for arbitrary finite linear combinations.

9.1.3 Lemma. Let $f, f' \in \mathbb{C}[\mathbb{A}]$ and let $g \in G$. Then

$$ g(f \cdot f') = (gf) \cdot (gf'). $$

Again, as in Chapter 3, this means that we only need to understand the action on single variables.

9.1.4 Example. Let $N = 2$, $d = 2$, $\delta = 2$, $\mathbb{A} = \mathbb{C}[X,Y]_2$, $\mathbb{C}[\mathbb{A}]_2 = \langle T_1, T_2, T_3 \rangle$. For the sake of readability, we here reuse the letter $d$ differently (with a different meaning): Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $g^T X = aX + bY$ and $g^T Y = cX + dY$. Thus,

$$ g^T X^2 = (aX + bY)^2 = a^2 X^2 + 2abXY + b^2 Y^2, $$
$$ g^T XY = acX^2 + (bc + ad)XY + bdY^2, $$
$$ g^T Y^2 = c^2 X^2 + 2cdXY + d^2 Y^2. $$
For every polynomial $f \in \mathbb{C}[\Lambda]$, $f = \alpha T_1 + \beta T_2 + \gamma T_3$, we have $f(X^2) = \alpha$, $f(Y^2) = \beta$, and $f(Y^2) = \gamma$. Thus $f = f(X^2) \cdot T_1 + f(Y^2) \cdot T_2 + f(Y^2) \cdot T_3$. Using $(gT_1)(h) = T_1(gT_1 h)$ we obtain:

\[
\begin{align*}
gT_1 &= (gT_1)(X^2) \cdot T_1 + (gT_1)(XY) \cdot T_2 + (gT_1)(Y^2) \cdot T_3 = a^2 T_1 + a c T_2 + c^2 T_3, \\
gT_2 &= 2abT_1 + (bc + ad)T_2 + 2cdT_3, \\
gT_3 &= b^2 T_1 + bdT_2 + d^2 T_3, \\
gT_1^2 &= 4a^2b^2T_1^2 + 4ab(bc + ad)T_1T_2 + 8abcdT_1T_3 + (bc + ad)^2 T_2^2 + 4(bc + ad)cT_2T_3 + 4c^2d^2 T_3^2, \\
gT_2T_3 &= a^2b^2T_1^2 + (a^2bd + ab^2c)T_1T_2 + (a^2d^2 + b^2c^2)T_1T_3 + abcdT_2^2 + (acd^2 + bc^2d)T_2T_3 + c^2d^2 T_3^2.
\end{align*}
\]

For the discriminant, we obtain

\[
\begin{align*}
g(T_2^2 - 4T_1T_3) &= gT_2^2 - 4gT_1T_3 \\
&= 8abcdT_1T_3 + (bc + ad)^2 T_2^2 - 4((a^2d^2 + b^2c^2)T_1T_3 + abcdT_2^2) \\
&= 8abcdT_1T_3 + (b^2c^2 + 2abcd + a^2d^2)T_2^2 - (4a^2d^2 + 4b^2c^2)T_1T_3 - 4abcdT_2^2 \\
&= (8abcd - 4a^2d^2 - 4b^2c^2)T_1T_3 + (b^2c^2 + 2abcd + a^2d^2 - 4abcd)T_2^2 \\
&= 4(2abcd - a^2d^2 - b^2c^2)T_1T_3 + (b^2d^2 - 2abcd + a^2d^2)T_2^2 \\
&= (ad - bc)^2 T_2^2 - 4(ad - bc)T_1T_3 \\
&= \det(g)^2 (T_2^2 - 4T_1T_3).
\end{align*}
\]

Thus the discriminant is fixed under the group action (up to the prefactor). Indeed, the discriminant is fixed under the group action of the special linear group \((i.e., matrices with determinant 1)\).

**9.1.5 Example.** Let $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. Then $gT_1^2 = a^2T_1$. Thus the line $CT_1$ is fixed under the action of upper triangular matrices. However, it is not fixed under the action of lower triangular matrices:

Let $g = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$. Then $gT_1^2 = T_1^2 + 2cT_1T_2 + 2c^2T_1T_3 + c^2T_2^2 + 2c^3T_3 + c^4 T_3^2$. The linear span of these $gT_1^2$ is of dimension at least 5: Their coefficient vectors are $(1, 2c, 2c^2, c^2, c^3, c^4)$, so putting $c = -2, -1, 0, 1, 2$ yields the rank 5 matrix

\[
\begin{pmatrix}
1 & -4 & 8 & 4 & -16 & 16 \\
1 & -2 & 2 & 1 & -2 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 2 & 1 & 2 & 1 \\
1 & 4 & 8 & 4 & 16 & 16
\end{pmatrix}.
\]

**9.1.6 Remark.** It is a small calculation to verify that for $A = \mathbb{C}[X,Y]_2$ we have $\mathbb{C}[A]_2 = \mathbb{C}(T_2^2 - 4T_1T_3) \oplus \langle GL_2 T_1^1 \rangle$.

### 9.2 General representation theory

Remark 9.1.6 gives an interesting example of a decomposition of $\mathbb{C}[A]_\beta$. It is called the decomposition into irreducible representations. To understand this decomposition and how it can be used to restrict the search space for obstructions we now study some basic representation theory. This section and Chapter 10 are based on lecture notes by Peter Bürgisser on “Kombinatorik der Darstellungstheorie symmetrischer Gruppen” (combinatorics of the representation theory of symmetric groups) from 2006 at Paderborn University.

Let $G$ be a group and $V$ be a finite dimensional complex vector space. Recall the definition of a linear monoid action from Chapter 3. Now we restrict our attention to monoids that are groups: A group homomorphism $\phi : G \to \text{GL}(V)$ is called a linear group action or a representation of $G$. If the action is understood, then we just say that $V$ is a representation.
9.2.1 Example. Let $G = \mathfrak{S}_n$. Let $V = \mathbb{C}^n$ and $\varrho(\pi)(e_i) = e_{\pi(i)}$. Then $(\pi \sigma)(e_i) = e_{\pi \sigma i} = \pi(\sigma e_i)$ for all $\pi, \sigma \in \mathfrak{S}_n$. This is called the defining representation of $\mathfrak{S}_n$. The matrices $\varrho(\pi)$ are called permutation matrices. They consist of a single 1 in each row and each column, and the rest is filled with zeros.

9.2.2 Example. Let $C_n := \mathbb{Z}/n\mathbb{Z}$ denote the cyclic group of order $n$. We can think of $C_n \subseteq \mathbb{C}$ as the group of $n$-th roots of unity. The identity element in $C_n$ is denoted by $1_{C_n}$. Let $C_n$ be generated by the element $g$, i.e., $C_n = \langle g \rangle$. Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\} = \mathbb{GL}_1$.

Let $g : C_n \to \mathbb{C}^*$ be a 1-dimensional representation of $C_n$. Since $g^n = 1_{C_n}$ we have $1 = \varrho(1_{C_n}) = \varrho(g^n) = \varrho(g)^n$, thus $\varrho(g) = \zeta^k$ for some $k \in \mathbb{Z}$ and $\zeta := e^{2\pi i/n}$. Indeed, each $k$ gives a representation.

9.2.3 Example. As we saw in Section 9.1, for $\mathbb{A} = \mathbb{C}[X_1, \ldots, X_N]_d$ the vector space $\mathbb{C}[\mathbb{A}]_d$ is a $\mathbb{GL}_N$ representation.

9.2.4 Remark. Representations are sometimes called modules. The term module in algebra is more general though and can be defined for any ring. Representations of $G$ are precisely the finite dimensional modules where the ring is the group algebra of $G$. The group algebra $\mathbb{C}[G]$ is defined as the vector space of formal linear combinations of finitely many group elements. Two elements of $\mathbb{C}[G]$ are multiplied via the obvious convolution: $(\sum_i \alpha_i g_i) \cdot (\sum_j \beta_j g_j) = \sum_{i,j} \alpha_i \beta_j (g_i g_j)$, where $\alpha_i, \beta_j \in \mathbb{C}$ and $g_i, g_j \in G$.

9.2.5 Definition. A linear subspace $W \subseteq V$ of a representation $V$ is called a subrepresentation if it is closed under the action of $G$, i.e.,

\[ \forall g \in G \forall w \in W : gw \in W. \]

The zero vector space and $V$ itself are always subrepresentations.

9.2.6 Example. Let $\mathbb{C}^n$ denote the defining representation of $\mathfrak{S}_n$. Then $w := e_1 + e_2 + \cdots + e_n$ is fixed under the action of $\mathfrak{S}_n$ and hence the line $\mathbb{C}w$ is a subrepresentation.

It is easy to verify that in general, for every element $v \in V$, the linear span $\langle Gv \rangle$ of the orbit $Gv$ is a subrepresentation.

9.2.7 Example. According to Example 9.1.4, for the discriminant, $(\mathbb{GL}_2(b^2 - 4ac))$ is a 1-dimensional subrepresentation of $\mathbb{C}[\mathbb{A}]_2$, where $\mathbb{A} = \mathbb{C}[X,Y]_2$.

9.2.8 Example. Let $Z \subseteq \mathbb{A}$ be a set that is closed under the action of $\mathbb{GL}_N$. Then the vanishing ideal $I(Z)_\delta$ is a subrepresentation of $\mathbb{C}[\mathbb{A}]_\delta$. This can be seen as follows: Let $g \in \mathbb{GL}_N$ and $z \in Z$. If $f$ vanishes on $Z$, then $(gf)(z) = f(g^T z) = 0$ for all $g \in \mathbb{GL}_N$, because $g^T z \in Z$.

9.2.9 Definition. If a representation $V$ only has the two trivial subrepresentations, then $V$ is called irreducible.

From the definition it is clear that every dimensional representation is irreducible.

9.2.10 Lemma. Let $C_n := \mathbb{Z}/n\mathbb{Z}$. Every irreducible $C_n$-representation is 1-dimensional.

Proof. Let $C_n = \langle g \rangle$. Let $V$ be a representation of $C_n$. Consider $\varrho(g) \in \mathbb{GL}(V)$ and let $v \in V \setminus \{0\}$ be an eigenvector of $\varrho(g)$ to some eigenvalue $\beta \in \mathbb{C}$. Then the line $\mathbb{C}v$ is a subrepresentation of $V$, because:

\[ \forall \alpha \in \mathbb{C} : \varrho(\alpha v) = \varrho(g)(\alpha v) = \beta \alpha v \in \mathbb{C}v \]

and thus $\varrho^k(\alpha v) = \varrho(g \cdots (g(\alpha v)) \cdots) = \beta^k \alpha v \in \mathbb{C}v$. Thus if $V$ is irreducible, $V$ is 1-dimensional. \qed
We will see in Corollary 10.2.2 that for separating points $h$ from orbit closures $\overline{Gc}$ it suffices to consider polynomials that lie in irreducible representations. This significantly strengthens Corollary 4.3.5.

### Representations

A group homomorphism $G \to \text{GL}(V)$ is a representation. If the action is understood, we simply call the representation $V$.

The vanishing ideal $I(Z)_δ$ of a set $Z \subset A$ that is closed under the action of $G$ is a subrepresentation of $\mathbb{C}[A]_δ$. 
Chapter 10

Representation theory of finite groups and Maschke’s theorem

Recall from Corollary 4.3.5 that in order to prove that a certain homogeneous polynomial $h$ of degree $\delta$ is not contained in some orbit closure $\overline{GL_N c}$, we want to find a homogenous polynomial $f \in \mathbb{C}[A]$ with $f(GL_N c) = \{0\}$ and $f(h) \neq 0$. Since $GL_N$ is linearly reductive (see below), we can find such an $f$ in an irreducible subrepresentation of $I(\overline{Gc})_H$. While the linear reductivity of $GL_N$ is beyond the scope of these lecture notes, we will prove it for finite groups.

10.1 Maschke’s theorem

10.1.1 Definition. Let $U$ and $W$ be linear subspaces of $V$. We say that $V$ is the direct sum of $U$ and $W$ if for every $v \in V$ there is are unique $u \in U$ and $w \in W$ such that $v = u + w$. We write $V = U \oplus W$.

If $V$ is a $G$-representation and $U$ and $W$ are subrepresentations, then we say that $U$ and $W$ are representation complements.

10.1.2 Example. Let $V^n$ denote the defining representation of $\mathfrak{S}_n$. Then $V^n = \langle e_1 + \cdots + e_n \rangle \oplus W$, where $W = \{w \in V^n \mid w_1 + \cdots + w_n = 0\}$. We have $\dim U = 1$, $\dim W = n - 1$, $U \cap W = 0$, thus $U \oplus W = V^n$.

Does every subrepresentation have a complement? In this chapter we will see that the answer is yes, provided that $G$ is finite.

10.1.3 Definition. An inner product on a finite dimensional complex vector space $V$ is a map $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{C}$

with

- $\langle \alpha_1 v_1 + \alpha_2 v_2, w \rangle = \alpha_1 \langle v_1, w \rangle + \alpha_2 \langle v_2, w \rangle$ for all $\alpha_i \in \mathbb{C}$, $v_i, w \in V$,
- $\langle v, w \rangle = \overline{\langle w, v \rangle}$, where the bar denotes complex conjugation,
- $\langle 0, 0 \rangle = 0$ and $\langle v, v \rangle \in \mathbb{R}_{>0}$ if $v \neq 0$.

For a linear subspace $U \subseteq V$ the orthogonal complement $U^\perp$ is defined as

$U^\perp := \{v \in V \mid \forall u \in U : \langle v, u \rangle = 0\}$. 

54
10.1.4 Lemma. If $U \subseteq V$ is a linear subspace, then $U^\perp \subseteq V$ is a linear subspace and $V = U \oplus U^\perp$.

Proof. If $v_1, v_2 \in U^\perp$ and $\alpha_1, \alpha_2 \in \mathbb{C}$, then let $u \in U$ and calculate
\[
\langle \alpha_1 v_1 + \alpha_2 v_2, u \rangle = \alpha_1 \langle v_1, u \rangle + \alpha_2 \langle v_2, u \rangle = 0 + 0 = 0,
\]
thus $U^\perp \subseteq V$ is a linear subspace. If $u \in U$ and $u \in U^\perp$, then $\langle u, u \rangle = 0$ and hence $u = 0$, thus $U \cap U^\perp = 0$. Since $\dim U^\perp = n - \dim U$ ($U^\perp$ is the vanishing set of $\dim U$ linearly independent linear constraints: $\langle v, u_i \rangle = 0$ for all basis vectors $u_i \in U$), we have $V = U \oplus U^\perp$. \qed

For $\mathbb{C}^n$, we can define the inner product $\langle v, w \rangle := \sum_{k=1}^n v_k \overline{w_k}$, thus every finite dimensional complex vector space has an inner product.

10.1.5 Definition. $\langle ., . \rangle$ is called $G$-invariant, if for all $g \in G$, $v, w \in V$:
\[\langle gv, gw \rangle = \langle v, w \rangle.\]

10.1.6 Lemma. Let $\langle ., . \rangle$ be $G$-invariant and let $U \subseteq V$ be a subrepresentation. Then $U^\perp \subseteq V$ is also a subrepresentation.

Proof. Let $v \in U^\perp$, $g \in G$. We have to show that $gv \in U^\perp$. Let $u \in U$ be arbitrary. Then
\[\langle gv, u \rangle = \langle gv, gg^{-1}u \rangle = \langle v, g^{-1}u \rangle = 0.\]
where $(\ast)$ holds because $\langle ., . \rangle$ is $G$-invariant. \qed

10.1.7 Lemma. Let $G$ be finite and let $V$ be a $G$-representation. Then $V$ has a $G$-invariant inner product.

Proof. Let $\langle ., . \rangle'$ be an inner product on $V$. For $v, w \in V$, we define
\[\langle v, w \rangle := \frac{1}{|G|} \sum_{g \in G} \langle gv, gw \rangle'.\]
It is straightforward to verify that $\langle ., . \rangle$ is an inner product on $V$. We show that $\langle ., . \rangle$ is $G$-invariant:
Let $g' \in G$.
\[\langle g'v, g'w \rangle = \frac{1}{|G|} \sum_{g \in G} \langle gg'v, gg'w \rangle' = \frac{1}{|G|} \sum_{x \in G} \langle xv, xw \rangle' = \langle v, w \rangle.\]
\qed

Of course, dividing by $|G|$ in the proof of Lemma 10.1.7 is optional, but it makes the construction idempotent: if $\langle ., . \rangle'$ is already $G$-invariant, then $\langle ., . \rangle' = \langle ., . \rangle$.

10.1.8 Theorem (Maschke’s theorem). Let $G$ be finite and $V$ be a $G$-representation. Then $V$ decomposes into a direct sum $V = U_1 \oplus U_2 \oplus \cdots \oplus U_t$ of irreducible $G$-representations $U_i$.

Proof. We proceed by induction on $\dim V =: d$. If $d = 0$, then $t = 0$. For the induction step we make a case distinction. If $V$ is irreducible, then we are done. If $V$ is not irreducible, then let $U \in V$ be a nontrivial subrepresentation, i.e., $U \neq 0$ and $U \neq V$. Let $\langle ., . \rangle$ be a $G$-invariant inner product on $V$, which exists by the previous lemma, and let $U^\perp$ be the orthogonal complement of $U$ with respect to this inner product.

Using Lemma 10.1.4 and Lemma 10.1.6 we see that $V = U \oplus U^\perp$ with the $G$-representations $U$ and $U^\perp$. Using the induction hypothesis on $U$ and $U^\perp$ we see that both decompose into a direct sum of irreducibles. Summing up this sum finishes the proof. \qed
10.2 Search space restrictions

10.2.1 Remark. Groups for which every representation decomposes into a direct sum of irreducibles are called linearly reductive or just reductive (which is the same over fields of characteristic 0). We just showed that finite groups are reductive. For us it will be important to know that \( \text{GL}_N \) is reductive. The proof uses the same idea as Maschke’s theorem, but the invariant scalar product is created by using the compact subgroup \( SU_N \subseteq \text{GL}_N \) and normalizing using the so-called Haar measure. We omit the details here, because they require some measure theory.

The next corollary strengthens Corollary 4.3.5 by putting another significant restriction on the search space for our obstructions that we search to find complexity lower bounds.

10.2.2 Corollary. Let \( Z \subseteq h \) be a Zariski-closed cone that is closed under the action of \( \text{GL}_N \). We have seen that \( I(Z)_\delta \) is a \( \text{GL}_N \)-representation and hence \( I(Z)_\delta \) decomposes into a sum of irreducibles. Let \( h \notin Z \). Then there exists an irreducible \( \text{GL}_N \)-representation \( U \subseteq I(Z)_\delta \) that contains an \( f \in U \) such that \( f(h) \neq 0 \).

Proof. Let \( 0 \neq I(Z)_\delta = U_1 \oplus \cdots \oplus U_k \). Pick \( 0 \neq f \in I(Z)_\delta \) with \( f(h) \neq 0 \). Write \( f = f_1 + \cdots + f_k \) with \( f_i \in U_i \), so in particular \( f_i \in I(Z)_\delta \). Since \( f(h) \neq 0 \) there exists \( i \) with \( f_i(h) \neq 0 \). \( \square \)

Search space restrictions

\( \text{GL}_N \) is a reductive group, that is, every \( \text{GL}_N \)-representation decomposes into a direct sum of irreducible \( \text{GL}_N \)-representations.

If \( h \notin \text{GL}_N \), then for some degree \( \delta \) a separating polynomial \( f \) can be found in an irreducible representation of \( I(\text{GL}_N \cap h)_\delta \).
Chapter 11

The irreducible representations of the general linear group: First properties

Since by Corollary 10.2.2 we can find obstructions in irreducible $\text{GL}_n$-representations, we want to understand the structure of irreducible $\text{GL}_n$-representations better. In this chapter we will prove the existence of so-called highest weight vectors in irreducible representations of $\text{GL}_n$. These will be sufficient to separate points from orbit closures.

We follow [Kra85, III.1.3–III.1.4].

11.1 Equivariant maps and isomorphisms

Given two representations $(V, \varphi)$ and $(V', \varphi')$ of a group $G$. A linear map $\varphi : V \to V'$ is called \textit{equivariant} or a $G$-morphism if

$$\forall g \in G, v \in V : g\varphi(v) = \varphi(gv),$$

or in other words, $\varphi'(g)\varphi(v) = \varphi(g(g)v)$. If $\varphi$ is an equivariant vector space isomorphism, then we say that $V$ and $V'$ are \textit{isomorphic representations}.

11.1.1 Definition. Let $G \subseteq \text{GL}_n$ be a subgroup (we will take $G$ to be $\text{GL}_n$ or the group of diagonal matrices). A representation $\varphi : G \to \text{GL}(V)$ is called polynomial if the $\dim(V)^2$ coordinate functions are multivariate polynomials in the $n^2$ coordinate functions of $\text{GL}_n$.

11.1.2 Example. $\mathbb{C}[\Lambda]_{\delta}$ is a polynomial representation, see Proposition 3.6.1. Subrepresentations of polynomial representations are polynomial.

Our goal is to classify the classes of isomorphic polynomial irreducible representations of $\text{GL}_n$.

11.2 The algebraic torus and the weight decomposition

Before we study the irreducible representations of $\text{GL}_n$, in this section we study the subgroup of invertible diagonal matrices in $\text{GL}_n$.

11.2.1 Definition. $T_n := (\mathbb{C}^\times)^n \subseteq \text{GL}_n$ denotes the group of invertible diagonal matrices, also called the algebraic torus.

We prove that $T_n$ is linearly reductive and we fully describe its irreducible representations.
Recall that a matrix $g$ is called diagonalizable if there exists an invertible matrix $P$ such that $P^{-1}gP$ is a diagonal matrix. Matrices with pairwise distinct eigenvalues are diagonalizable, in particular the set of diagonalizable matrices lies dense in the set of all matrices, i.e., every matrix can be approximated arbitrarily closely by diagonalizable matrices via slight perturbations of the entries.

We will use the following lemma for subgroups $H$ which consist of representation matrices $g(t) \in \text{GL}(V)$, where $t \in T_n$.

11.2.2 Lemma (Simultaneous diagonalizability). Let $H \leq \text{GL}(V)$ be an abelian subgroup and each $g \in H$ diagonalizable. Then $H$ is simultaneously diagonalizable, i.e., there exists $P \in \text{GL}(V)$ such that for all $g \in H$ we have that $P^{-1}gP$ is a diagonal matrix.

Proof. The proof is by induction on the size of the matrices. The base case is when all matrices of $H$ have only one eigenvalue. This in particular includes the case, when the matrices have size $1 \times 1$.

If $g = S^{-1}DS$ is diagonalizable (with a diagonal matrix $D$) and has only one eigenvalue $\lambda$, then $g = \text{diag}(\lambda, \ldots, \lambda)$, because $S^{-1}\text{diag}(\lambda, \ldots, \lambda)S = \lambda S^{-1}S = \text{diag}(\lambda, \ldots, \lambda)$. If all $g \in H$ have only one eigenvalue, then there is nothing to show, because all $g$ are diagonal.

Let $g \in H$ with at least 2 eigenvalues. Then find $S^{-1}gS = \text{diag}(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots)$. Note that since $bg = gb$ for all $b \in H$ we have $S^{-1}bSS^{-1}gS = S^{-1}gS\text{diag}(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots)$ and hence all matrices in $S^{-1}HS$ are block diagonal, where the block sizes depend only on the multiplicities of the eigenvalues of $g$.

Then by induction hypothesis the single blocks can be simultaneously diagonalised by matrices $S_1, S_2, \ldots$ Then $\text{diag}(S_1, S_2, \ldots)$ simultaneously diagonalizes $S^{-1}HS$ and thus $P := S \cdot \text{diag}(S_1, S_2, \ldots)$ simultaneously diagonalizes $H$. \hfill $\square$

Let $(V, \varphi)$ be a polynomial representation of $T_n$. Since the elements of $T_n$ commute, all elements $\varphi(t), t \in T_n$, commute. Let $H := \{\varphi(t) \mid t \in T_n\}$. To apply Lemma 11.2.2 we need that each $\varphi(t)$ is diagonalizable. This is achieved in the following lemma.

11.2.3 Lemma. Let $(V, \varphi)$ be a polynomial representation of $T_n$. Then $\varphi(t)$ is diagonalizable for every $t \in T_n$.

Proof. We start with a fact on multivariate interpolation. A multivariate polynomial $f$ of degree $d$ in $n$ variables is uniquely defined by its $(d+1)^n$ evaluations $f(x_1, \ldots, x_n)$ at points $(x_1, \ldots, x_n) \in \mathbb{C}^n$, where we put $d+1$ different values for each of the $x_i$, as can be seen by multivariate interpolation.

Let $\tilde{T}_n$ denote the subgroup of elements $\text{diag}(t_1, \ldots, t_n)$ for which each $t_i$ has finite order (i.e., $t_i$ is a root of unity) by definition $\tilde{T}_n = (\tilde{T}_1)^n$. For every $k$, the primitive $k$-th roots of unity are in $\tilde{T}_1$, in particular $\tilde{T}_1$ has infinitely many elements. By multivariate interpolation we conclude that if $f$ vanishes on $(\tilde{T}_1)^n$, then $f = 0$. We say that $(\tilde{T}_1)^n = \tilde{T}_n$ lies Zariski-dense in $\mathbb{C}^n$.

All elements in $\tilde{T}_n$ commute. Thus all elements in $\varphi(\tilde{T}_n)$ commute. Given $s \in \tilde{T}_n$, let $(s)$ be the cyclic group generated by $s$. Since $(s)$ is a finite cyclic group, it is linearly reductive and its irreducible representations are 1-dimensional (Theorem 10.1.8 and Lemma 9.2.10). Thus we can decompose $V$ into $(s)$-irreducibles, each spanned by a single vector $v_i$. Now $P^{-1}\varphi(s)P$ is diagonal, where the columns of $P$ are given by the $v_i$: For standard basis vectors $e_i$ we have $(P^{-1}\varphi(s)P)e_i = P^{-1}\varphi(s)v_i = P^{-1}av_i = ae_i$ for some $a \in \mathbb{C}$. Therefore $\varphi(s)$ is diagonalizable and using Lemma 11.2.2 we see that $\varphi(\tilde{T}_n)$ is simultaneously diagonalizable: There exists $P$ such that $P^{-1}\varphi(\tilde{T}_n)P$ are all diagonal.

Define $f_{i,j} : \mathbb{C}^n \to \mathbb{C}, (t_1, \ldots, t_n) \mapsto (P^{-1}\varphi(\text{diag}(t_1, \ldots, t_n))P)_{i,j}$. We just saw that $f_{i,j}(s) = 0$ for all $s \in \tilde{T}_n, i \neq j$. Since $\tilde{T}_n$ lies Zariski-dense in $\mathbb{C}^n$ it follows that $f_{i,j}(t) = 0$ for all $t \in \mathbb{C}^n, i \neq j$. Thus $P^{-1}\varphi(t)P$ is diagonal for all $t \in T_n$. \hfill $\square$
11.2.4 Lemma. Given a multivariate polynomial \( \kappa \) in \( n \) variables \( t = (t_1, \ldots, t_n) \) with \( \kappa(t^2) = (\kappa(t))^2 \), where \( t^2 := (t_1^2, \ldots, t_n^2) \). Then \( \kappa \) is a monomial or zero.

Proof. For natural numbers \( k_1, \ldots, k_n \) we have that \( \kappa(\alpha^{k_1}, \alpha^{k_2}, \ldots, \alpha^{k_n}) \) is a univariate nonzero polynomial \( \zeta(\alpha) \). Moreover, \( \zeta(\alpha)^2 = \zeta(\alpha^2) \). The idea is that if \( k_1 \gg k_2 \gg \cdots \gg k_n \), then there is a 1:1 correspondence between the nonzero homogeneous parts \( \zeta_i \) of \( \zeta \)—which are just single monomials, since \( \zeta \) is univariate—and the monomials in \( \kappa \) with nonzero coefficient. (This kind of substitution is also called Kronecker substitution and has been used in polynomial identity testing.) Because of this correspondence it suffices to show that \( \zeta \) is homogeneous, since this implies that \( \kappa \) is a single monomial or zero.

For the sake of contradiction assume that \( \zeta \) is not homogeneous, so assume that \( \zeta = \zeta_i + \zeta_j + \zeta' \) with \( \zeta_i \neq 0 \) being homogeneous of degree \( i \) and \( \zeta_j \neq 0 \) being homogeneous of degree \( j, i > j \), and \( \zeta' \) being of degree less than \( j \). Then \( \zeta(\alpha^2) = \alpha^{2i}\zeta_i(1) + \alpha^{2j}\zeta_j(1) + O(\alpha^{2j-2}) \) and \( \zeta(\alpha^2) = \alpha^{2i}\zeta_i(1)^2 + \alpha^{2j}\zeta_j(1)^2 + 2\alpha^{i+j}\zeta_i(1)\zeta_j(1) + O(\alpha^{i+j-1}) \). Comparing the coefficient of degree \( i + j \) we see that \( 2\alpha^{i+j}\zeta_i(1)\zeta_j(1) \neq 0 \), in contradiction to \( \zeta(\alpha^2) = \zeta(\alpha^2) \).

11.2.5 Theorem. For \( t = \text{diag}(t_1, \ldots, t_n) \in T_n \) and \( \lambda \in \mathbb{N}^n \) we write \( t^\lambda := \prod_{i=1}^n t_i^{\lambda_i} \in \mathbb{C} \).

For every polynomial representation \( \varrho : T_n \rightarrow \text{GL}(V) \) we have that

\[
\mathbb{C} \ni \varrho(\lambda) = \bigoplus_{\lambda \in \mathbb{N}^n} V_\lambda,
\]

where

\[
V_\lambda := \{ v \in V \mid \varrho(t)v = t^\lambda v \text{ for all } t \in T_n \}.
\]

Proof. Using Lemma 11.2.3, \( \{ \varrho(\lambda) \mid \lambda \in \mathbb{N}^n \} \) is simultaneously diagonalizable, so there is \( P \in \text{GL}(V) \) such that \( P^{-1} \varrho(\lambda)P \) is diagonal for every \( t \in T_n \). Therefore the \( i \)-th diagonal entry of \( P^{-1} \varrho(\lambda)P \) is given by a function \( \kappa(\lambda) \). Since \( \varrho \) is a polynomial representation, each \( \kappa \) is a multivariate polynomial in \( n \) variables. Since \( P^{-1} \varrho(\lambda)P \leq \text{GL}(V) \) is a subgroup of diagonal matrices, \( \kappa(t^\lambda) = \kappa(t^\lambda') \), where the product \( t^\lambda t' \) is defined componentwise. Using Lemma 11.2.4 it follows that \( \kappa(t^\lambda) = t^\lambda \) for some \( \lambda \).

The decomposition in Theorem 11.2.5 is called the weight decomposition. \( V_\lambda \) is called the weight space of weight \( \lambda \) and a vector in \( V_\lambda \) is called a weight vector of weight \( \lambda \).

11.2.6 Corollary. The polynomial irreducible representations of \( T_n \) are 1-dimensional and indexed by lists in \( \mathbb{N}^n \).

Proof. Given a polynomial irreducible representation \( V \) of \( T_n \), by Theorem 11.2.5, \( V \) decomposes into a direct sum of weight spaces, each of which decomposes (arbitrarily) into a direct sum of 1-dimensional irreducible \( T_n \)-representations. Since \( V \) is irreducible, there can be only one summand. It follows that each polynomial irreducible representation is 1-dimensional and a weight space to some weight \( \lambda \in \mathbb{N}^n \).

Moreover, let \( \lambda \neq \mu \) and \( V \) and \( W \) be 1-dimensional with weight \( \lambda \) and \( \mu \), respectively. For the sake of readability we use lower dots for scalar multiplication: If \( \varphi : V \rightarrow W \) is \( T_n \)-equivariant, then \( \varphi(v) = \varphi(t^\lambda v) = t^\lambda \varphi(t^\lambda v) \neq t^\mu \varphi(t^\lambda v) = t^\mu \varphi(v) \) and thus \( V \) and \( W \) are not isomorphic. The other direction works analogously: If \( \lambda = \mu \), then \( V \) and \( W \) are isomorphic.

**GL\(_n\) versus \( T_n \)**

Every polynomial \( T_n \)-representation decomposes into a direct sum of weight spaces, indexed by \( \lambda \in \mathbb{N}^n \). This is called the weight decomposition.

Since a polynomial irreducible \( GL_n \)-representation is also a polynomial \( T_n \)-representation, it also has a weight decomposition.
11.3 Highest weight vectors

As seen in the last section, every polynomial irreducible $GL_n$-representation has a weight decomposition. This is a first structural result about irreducible $GL_n$-representations. In this section we fully classify the irreducible $GL_n$-representations.

Embed $\gamma : \mathfrak{S}_n \to GL_n$ via permutation matrices, i.e., the permutation $\pi$ is mapped to the matrix that has entries 1 at positions $(i, \pi(i))$ and zeros everywhere else. One can readily verify that $\gamma$ is a group homomorphism. $\mathfrak{S}_n$ acts on $\mathbb{N}^n$ in the natural way by permuting the positions, so $\pi(\lambda) := (\lambda_{\pi^{-1}(1)}, \ldots, \lambda_{\pi^{-1}(n)})$.

11.3.1 Lemma. Given a $GL_n$-representation $V$ and let $V_\lambda$ denote its $\lambda$ weight space. Let $\mathfrak{S}_n$ act on $V$ via $\gamma : \mathfrak{S}_n \to GL_n$. Then $\pi V_\lambda = V_{\pi(\lambda)}$.

Proof. Let $v \in V_\lambda$, $t = (t_1, \ldots, t_n) \in T_n$ and $\pi \in \mathfrak{S}_n$.

For the sake of readability let the lower dot denote the multiplication with a scalar from the left. We calculate: $\gamma(\pi^{-1})\text{diag}(t_1, \ldots, t_n)\gamma(\pi) = \text{diag}(t_{\pi(1)}, \ldots, t_{\pi(n)}) =: t_\pi$. Since

$$t_\pi v = \text{diag}(t_{\pi(1)}, \ldots, t_{\pi(n)})v = t_{\pi(1)}^{\lambda_1} \cdots t_{\pi(n)}^{\lambda_n}v = t_1^{\lambda_{\pi^{-1}(1)}} \cdots t_n^{\lambda_{\pi^{-1}(n)}} = \pi_v(\lambda)v,$$

we have

$$t(\pi v) = t_{\pi} v = t\pi_v = \pi(t^{-1}t\pi v) = \pi(t_{\pi}v) = \pi(t^{\pi(\lambda)}v) = t^{\pi(\lambda)}(\pi v),$$

and therefore $\pi v \in V_{\pi(\lambda)}$. We conclude $\pi V_\lambda \subseteq V_{\pi(\lambda)}$ and by symmetry $\pi V_\lambda = V_{\pi(\lambda)}$. \qed

11.3.2 Definition. A finite list of natural numbers $\lambda \in \mathbb{N}^n$ is called a partition if it is nonincreasing, i.e., $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. We define $|\lambda| := \sum_{i=1}^n \lambda_i$. More generally, we define $|\lambda| := \sum_{i=1}^n \lambda_i$ for all $\lambda \in \mathbb{Z}^n$. We say that $\lambda$ is a partition of $N$ if $|\lambda| = N$.

On $\mathbb{Z}^n$ we define the following partial order, the so-called dominance order. Two lists $\lambda, \mu \in \mathbb{Z}^n$ satisfy $\lambda \geq \mu$ iff

- for all $1 \leq i \leq n$: $\sum_{j=1}^i \lambda_j \geq \sum_{j=1}^i \mu_j$.

In this situation we say that $\lambda$ dominates $\mu$. Usually when comparing $\lambda$ and $\mu$ we have $|\lambda| = |\mu|$. We write $\lambda \succ \mu$ to denote that both $\lambda \geq \mu$ and $\lambda \neq \mu$ hold.

11.3.3 Example. We have $(6,3,3) \geq (6,2,2,2)$, because $6 \geq 6, 6+3 \geq 6+2, 6+3+3 \geq 6+2+2, \text{ and } 6+3+3+0 \geq 6+2+2+2$.

Dominance is a partial order: $(6,3,1,1) \not\geq (5,3,3)$ and $(6,3,1,1) \not\geq (5,3,3)$.

Let $U_n \leq GL_n$ denote the subgroup of upper triangular matrices with 1s on the diagonal. Analogously, let $U_n^- \leq GL_n$ denote the subgroup of lower triangular matrices with 1s on the diagonal.

11.3.4 Lemma. Let $v$ be a weight vector of weight $\lambda$ and let $g \in U_n$. Then $gv = v + w$, where $w \in \bigoplus_{\mu \succ \lambda} V_\mu$.

If $g \in U_n^-$ instead, then $gv = v + w$, where $w \in \bigoplus_{\mu \succeq \lambda} V_\mu$.

Proof. We only prove the first part. The second part is completely analogous. Again, for the sake of readability we sometimes use the lower dot to denote the multiplication with a scalar from the left.

Define $x_{ij}(\alpha), i \neq j$, to be the identity matrix with a single $\alpha \in \mathbb{C}$ in row $i$, column $j$. We prove the result for $g = x_{ij}(\alpha)$ with $i < j$. This is without loss of generality, because $U_n$ is generated as a group by these $x_{ij}(\alpha)$.

For $t = \text{diag}(t_1, \ldots, t_n) \in T_n$ we have $tx_{ij}(\alpha)t^{-1} = x_{ij}(t_i t^{-1}_j \cdot \alpha)$. 
Let \( \{w_1, \ldots, w_n\} \) be a basis of \( V \). Since \( g \) is a polynomial representation, each coordinate function of \( x_{ij}(\alpha)v \) is a univariate polynomial in \( \alpha \):

\[
x_{ij}(\alpha)v = \sum_{s=1}^{n} \left( \sum_{h \geq 0} c_{h,s} \alpha^h \right) w_h = \sum_{h \geq 0} \alpha^h.v_h
\]

with \( c_{h,s} \in \mathbb{C} \) and \( v_h := \sum_{s=1}^{n} c_{h,s} w_s \).

Since \( x_{ij}(0) = \text{Id}_n \), it follows \( x_{ij}(0)v = v \), thus we get that the constant term \( v_0 = v \). We have

\[
t x_{ij}(\alpha)v = t x_{ij}(\alpha)t^{-1} v = (x_{ij}(t_t^{-1} \alpha))tv = (x_{ij}(t_t^{-1} \alpha))(t^\lambda .v)
\]

\[
= t^\lambda(x_{ij}(t_t^{-1} \alpha))v = t^\lambda \sum_{h \geq 0} t_h^{-h} \alpha^h.v_h = \sum_{h \geq 0} \alpha^h.(t^{\lambda + h \zeta_{ij}}.v_h),
\]

where \( \zeta_{ij} := (0,0,\ldots,0,1,0,\ldots,0,-1,0,\ldots,0) \) with the 1 at position \( i \) and the -1 at position \( j \).

On the other hand

\[
x_{ij}(\alpha)v = t(\sum_{h \geq 0} \alpha^h.v_h) = \sum_{h \geq 0} \alpha^h.tv_h
\]

Comparing coefficients we see that

\[
tv_h = t^{\lambda + h \zeta_{ij}}.v_h,
\]

thus each \( v_h \) is an element of \( V_{\lambda + h \zeta_{ij}} \).

The proof is finished by observing that \( \lambda \triangleleft (\lambda + h \zeta_{ij}) \) for \( i < j \), \( 0 < h \). \( \square \)

Let \( B_n \leq \text{GL}_n \) denote the subgroup of upper triangular matrices. A 1-dimensional linear subspace \( \mathbb{C}v \) of a \( \text{GL}_n \)-representation \( V \) is called a \( B_n \)-stable line, if it is closed under the action of \( B_n \). Since \( T_n \leq B_n \) is a subgroup, in a polynomial \( \text{GL}_n \)-representation every \( B_n \)-stable line is also a 1-dimensional \( T_n \)-representation and hence every \( B_n \)-stable has a weight \( \lambda \in \mathbb{N}^n \).

11.3.5 Corollary. Let \( V \) be a \( \text{GL}_n \)-representation and \( v \in V_\lambda \) for some \( \lambda \in \mathbb{N}^n \) such that \( \mathbb{C}v \) is a \( B_n \)-stable line. Then \( \langle \text{GL}_n.v \rangle \subseteq \mathbb{C}v \oplus \sum_{\mu \triangleleft \lambda} V_\mu \).

Proof. The set \( U^{-1}T_nU_n \subseteq \text{GL}_n \) is dense, because LU factorization of matrices almost always works without pivoting, in other words \( \text{GL}_n = \overline{U_n T_n U_n} \), where the closure is taken in \( \text{GL}_n \). Thus \( \text{GL}_n.v = \overline{\text{U}_n T_n U_n} v \subseteq \overline{\mathbb{C} \cdot U_n v} \subseteq \mathbb{C} \cdot v \). Lemma 11.3.4 yields \( \mathbb{C} \cdot U_n v \subseteq \mathbb{C}v + \sum_{\mu \triangleleft \lambda} V_\mu \). The right hand side is closed, thus \( \mathbb{C} \cdot U_n v \subseteq \mathbb{C}v + \sum_{\mu \triangleleft \lambda} V_\mu \). Therefore \( \text{GL}_n.v \subseteq \mathbb{C}v + \sum_{\mu \triangleleft \lambda} V_\mu \). Since the right hand side is a vector space, it follows \( \langle \text{GL}_n.v \rangle \subseteq \mathbb{C}v + \sum_{\mu \triangleleft \lambda} V_\mu \). \( \square \)

The following theorem completely classifies the irreducible \( \text{GL}_n \)-representations by the weight of their unique \( B_n \)-stable line.

11.3.6 Theorem. a') Let \( V \) be a polynomial \( \text{GL}_n \)-representation with a \( B_n \)-stable line \( \mathbb{C}v \subseteq V \). Then \( \langle \text{GL}_n.v \rangle \) is irreducible.

a) For each irreducible polynomial \( \text{GL}_n \)-representation \( V \) there exists exactly one \( B_n \)-stable line \( \mathbb{C}v \subseteq V \). Let \( \lambda \) be the weight of \( v \), called the highest weight of \( V \). Then the \( \lambda \)-weight space \( V_\lambda = \mathbb{C}v \) is 1-dimensional. Furthermore we have that \( \lambda \) is a partition and for all weights \( \mu \) that appear in \( V \) (i.e., \( V_\mu \neq 0 \)) we have \( \mu \triangleleft \lambda \).

b) Two irreducible polynomial representations \( V \) and \( V' \) are isomorphic, iff their highest weights \( \lambda \) and \( \lambda' \) are equal.
c) Let $\lambda \in \mathbb{N}^n$ be a partition. Then there exists an irreducible polynomial representation of $\text{GL}_n$ with highest weight $\lambda$.

11.3.7 Remark. In the situation a) we call $v$ a highest weight vector (HWV).

Proof. (c) For every partition $\lambda$ we can explicitly construct an irreducible representation. The construction is slightly technical and we postpone it until Section 17.

(a') Let $Cv$ be a $B_n$-stable line of weight $\lambda$ and let $W := \langle \text{GL}_n, v \rangle$. Decompose $W = \bigoplus W_i$ into irreducible $\text{GL}_n$-representations $W_i$. Decompose the $W_i$ further into their weight spaces spanned by weight vectors $v_j$, so that the $v_j$ form a basis of $W$. Since $v \in W$ has weight $\lambda$, one of the $v_j$ must have weight $\lambda$. Let $v_j \in W_i =: W'$. By Cor. 11.3.5 the weight space $W'_{\lambda}$ is 1-dimensional and thus the weight space $W'_{\lambda}$ is also 1-dimensional, in fact $W'_{\lambda} = W_{\lambda}$. Thus $v \in W_{\lambda}$. Therefore:

- The poset (with respect to the dominance order) of weights that occur in $V$ has a maximum: $\lambda$
- In $V$ there is a unique line of weight $\lambda$

We now see that the $B_n$-stable line in $V$ is unique. A second $B_n$-stable $Cw$ line would have a weight $\mu \triangleright \lambda$, but then $\langle \text{GL}_n, w \rangle \subseteq Cw + \sum_{\mu \triangleright \lambda} V_{\mu}$ would not contain $Cv$, which is a contradiction to $V$ being irreducible. Thus the $B_n$-stable line in $V$ is unique.

If $\lambda$ is not a partition, then $\pi(\lambda)$ is a partition for some $\pi \in \mathfrak{S}_n$. By Lemma 11.3.1, $\pi V_{\lambda} = V_{\pi(\lambda)}$. But $\pi(\lambda) \triangleright \lambda$, a contradiction to $\lambda$ dominating all weights in $V$.

It remains to show that there exists a $B_n$-stable line. Take all weights $\mu$ for which $V_{\mu} \neq 0$ and take a maximal element $\lambda$ with respect to the dominance order. Take $0 \neq v \in V_{\lambda}$. Use Lemma 11.3.4 to see that $U_n v = v$. Thus $V$ contains at least the $U_n$-stable line $Cv$. Since $B_n = T_n U_n$ and since $v \in V_{\lambda}$, it follows that $Cv$ is a $B_n$-stable line.

(b) Isomorphic representations clearly have equal highest weights: If $\varphi : V \rightarrow V'$ is an isomorphism, then $v$ is $B_n$-stable iff $\varphi(v)$ is $B_n$-stable. Moreover, $t \varphi(v) = \varphi(tv) = \varphi(t^\lambda v) = t^{\lambda'} \varphi(v)$.

To see the other direction, let $\lambda = \lambda'$, where $Cv \subseteq V$ and $Cw \subseteq V'$ are the $B_n$-stable lines in $V$ and $V'$, respectively. Consider the $\text{GL}_n$-representation $V \oplus V'$, $g(u, u') := (gu, gu')$ for all $u \in V$, $u' \in V'$, $g \in \text{GL}_n$. Then $w := (v, v') \in V \oplus V'$. Let $W := \langle \text{GL}_n, w \rangle \subseteq V \oplus V'$. Since $\lambda = \lambda'$ it follows that $w$ has weight $\lambda$ and $Cw \subseteq W$ is a $B_n$-stable line. By part (a') we have that $W$ is irreducible.

We write $V = V \oplus \{0\}$ and $V' = \{0\} \oplus V'$. Then $W \cap V' \subseteq W$ is a subrepresentation, but $W$ is irreducible, so $W \cap V' = W$ (i.e., $W \subseteq V'$) or $W \cap V' = \{0\}$. Since $w \in W$, but $w \notin V'$, we have $W \cap V' = \{0\}$. But $W \cap V'$ is the kernel of the linear projection map $pr : W \rightarrow V$, $(w, v') \mapsto w$. Thus $pr$ is injective. Moreover, $pr$ is equivariant. Thus $pr : W \rightarrow pr(W)$, $w \mapsto pr(w)$ is a $\text{GL}_n$-isomorphism. Its image is thus a subrepresentation of $V$, isomorphic to $W$. But $V$ is irreducible and $W \neq \{0\}$, thus $V$ and $W$ are isomorphic $\text{GL}_n$-representations ($V \cong W$). Analogously we show that $W \cong V'$ and thus $V \cong V'$.

The previous theorem gives a complete characterization of irreducible polynomial $\text{GL}_n$-representations $V$: Using part (a) we see that $V$ has a unique $B_n$-stable line $Cv$ of some weight $\lambda$. Using part (a') we see that $V = \langle \text{GL}_n, v \rangle$. Using parts (b) and (c) we see that there is a 1:1 correspondence between partitions into at most $n$ parts and isomorphism types of irreducible polynomial $\text{GL}_n$-representations.

11.3.8 Example. Let $\mathbb{A} = \mathbb{C}[X, Y]_2$, $V = \mathbb{C}[\mathbb{A}]_2$. The calculation from Example 9.1.4 shows that the discriminant $f := T_2^2 - 4T_1T_3$ satisfies $gf = \det(g)^2 f$ and thus $\text{diag}(a_1, a_2)f = a_1^2 a_2^2 f$, so $f$ is a weight vector of weight $(2, 2)$. Moreover, $gf = f$ if $\det(g) = 1$, in particular $f$ is fixed under $U_n$. Thus $f$ is a $B_n$-stable line. Hence $f$ is a highest weight vector of weight $(2, 2)$.
Example 9.1.5 shows analogously that the polynomial $T^2_1$ is a highest weight vector of weight $(4,0)$.

Each orbit span of a HWV is an irreducible subrepresentation. Here the orbit span of the discriminant is 1-dimensional, while the orbit span of $T^2_1$ is 5-dimensional. Since $V$ is 6-dimensional this concludes the decomposition into irreducibles, as already pointed out in Remark 9.1.6: the 6-dimensional $\text{GL}_2$-representation $V$ decomposes into a direct sum of two irreducibles: One of type $(4,0)$ and one of type $(2,2)$.

**11.4 Highest weight vector obstructions**

In this section we will see how Corollary 10.2.2 can be strengthened even further with an additional significant search space restriction for obstructions: we only need to consider HWVs, see Corollary 11.4.2.

**11.4.1 Proposition.** Let $\mathbb{A} = \mathbb{C}[X_1, \ldots, X_N]_d$. Then $\mathbb{C}[\mathbb{A}]_\delta$ decomposes into irreducibles as

$$\mathbb{C}[\mathbb{A}]_\delta = \bigoplus_i V_i,$$

where the type of each $V_i$ is a partition of $d\delta$.

**Proof.** We consider the action of $t := \text{diag}(\alpha, \ldots, \alpha) \in T_n$ on $\mathbb{C}[\mathbb{A}]_\delta$. Indeed, $tf = \alpha^{d\delta} f$ for every $f \in \mathbb{C}[\mathbb{A}]_\delta$. Thus each weight vector $f \in \mathbb{C}[\mathbb{A}]_\delta$ of weight $\lambda$ (i.e., $tf = t^\lambda f$) must satisfy

$$\alpha^{d\delta} f = \alpha^\lambda f,$$

thus $\lambda_1 + \cdots + \lambda_n = d\delta$. In particular this is true for highest weight vectors. The statement follows with the classification in Theorem 11.3.6.

**11.4.2 Corollary.** Let $\mathbb{A} = \mathbb{C}[X_1, \ldots, X_N]_d$. Let $c \in \mathbb{A}$, $G = \text{GL}_N$. If $h \not\in \overline{Gc}$, then there exists $\delta \in \mathbb{N}$, $\lambda \in \mathbb{N}^N$ a partition of $d\delta$, and a highest weight vector $f \in \mathbb{C}[\mathbb{A}]_\delta$ of weight $\lambda$, such that $I(\overline{Gc}) = \{0\}$ and for almost all group elements $g \in G$ we have $f(gh) \neq 0$. (“Almost all” means that the set of $g$ for which $f(gh) = 0$ is a Zariski-closed proper subset of $G$.)

**Proof.** From Corollary 10.2.2 we already know that an $f$ exists that is contained in the homogeneous degree $\delta$ part $I(\overline{Gc})_{\delta}$ of the vanishing ideal, but that also satisfies $f(h) \neq 0$. Moreover, $I(\overline{Gc})_\delta$ is a subrepresentation of $\mathbb{C}[\mathbb{A}]_\delta$, so we can decompose it into irreducibles

$$I(\overline{Gc})_\delta = \bigoplus_{j \in \Omega} V_j$$

for a finite index set $\Omega$, where by Prop. 11.4.1 the type of each $V_j$ is a partition of $d\delta$. Now we can write $f = \sum_{j \in \Omega} f_j$, where $f_j \in V_j$. By Theorem 11.3.6(a),(a’) it follows that we can write the finite sum $f_j = \sum_i g_{j,i} f_{j,i}$, where $g_{j,i} \in G$ and $f_{j,i}$ is an HWV (scalars in the linear combination can be merged with the HWVs, so they do not appear in the sum).

Since $f(h) \neq 0$, we have that $(g_{j,i}, f_{j,i})(h) \neq 0$ for some $j, i$. This means $f_{j,i}(g_{j,i}^T h) \neq 0$, which proves the first part of the proposition, choosing $g = g_{j,i}^T$. For the second part we have to analyze the subset of group elements $\tilde{g} \in G$ that satisfy $f_{j,i}(\tilde{g} h) \neq 0$. But $f_{j,i}(\tilde{g} h)$ is a polynomial in the entries of $\tilde{g}$. This finishes the proof.

The following calculation gives a feel that looking at HWVs should be useful.

For $\mathbb{A} = \mathbb{C}[X_1, \ldots, X_M]_d$ we have $\dim \mathbb{C}[\mathbb{A}]_\delta = \left(\delta + \left(\frac{d-d-1}{d}\right)\right)$. Thus if $M = d = \delta = 6$ we have $\dim \mathbb{C}[\mathbb{A}]_\delta = 13949678575756$. But one can compute that the dimension of the vector space of highest weight vectors is only 31781 and the highest dimension of the highest weight subspace in a $V_\lambda$ is 105.
More crucially, the dimensions of the highest weight vector spaces do not change when increasing $M$, but $\dim \mathbb{C}[A]_\delta$ increases significantly! For $A = \mathbb{C}[X_1, \ldots, X_M]_2$ we have $\dim \mathbb{C}[A]_2 = \binom{1+M}{2} = \frac{M^4}{4} + \frac{M^3}{2} + \frac{3M^2}{4} + \frac{M}{4}$, but we will see that the dimension of the space of HWVs is just 2, independent of $M$, provided $M \geq 2$.

**Irreducible representations of $\text{GL}_n$ and HWVs**

The irreducible polynomial representations of $\text{GL}_n$ are indexed by partitions $\lambda \in \mathbb{N}^n$. Each irreducible polynomial representation $V$ has a unique highest weight vector (up to scale): A weight vector that is $B_n$-stable. Its weight determines the isomorphism type of $V$.

For proving lower bounds $h \notin \overline{Cc}$ we can restrict our search for obstructions $f$ to highest weight vectors. This greatly reduces the dimension of the search space.
Chapter 12

Schur’s lemma, multiplicities, and isotypic decompositions

So far we restricted the search space for obstructions further and further. In this chapter we want to present a sufficient criterion for obstructions that is not known to be a necessary criterion: Comparing representation-theoretic multiplicities, see Section 12.4. This strategy for proving \( h \notin \mathcal{G} \) is mathematically beautiful, but still bears many open research questions.

Again we follow Bürgisser’s lecture notes.

12.1 Schur’s lemma

For a group \( G \) and two \( G \)-representations \( V \) and \( W \) we define

\[
\text{Hom}_G(V, W) := \{ \varphi \mid \varphi : V \to W \text{ a } G\text{-morphism} \}.
\]

\( \text{Hom}_G(V, W) \) is a vector space and a linear subspace of \( \text{Hom}(V, W) \). Moreover, define \( \text{End}_G(V) := \text{Hom}_G(V, V) \).

12.1.1 Lemma. Let \( \varphi \in \text{Hom}_G(V, W) \). Then

1. \( \ker \varphi := \{ v \in V \mid \varphi(v) = 0 \} \) is a subrepresentation of \( V \)
2. \( \text{im} \varphi := \{ \varphi(v) \mid v \in V \} \) is a subrepresentation of \( W \)

Proof. It is clear that kernel and image are linear subspaces. We have to verify that both are closed under the group action.

If \( v \in \ker \varphi \) and \( g \in G \), then \( \varphi(gv) = g \varphi(v) = g0 = 0 \) and thus \( gv \in \ker \varphi \).

If \( w \in \text{im} \varphi \) and \( g \in G \), then choose \( v \in V \) such that \( \varphi(v) = w \). Then \( gw = g \varphi(v) = \varphi(gv) \in \text{im} \varphi \).

We write \( V \cong W \) to denote that \( V \) and \( W \) are isomorphic representations, and \( V \not\cong W \) otherwise.

12.1.2 Lemma (Schur’s lemma). Let \( V \) and \( W \) be irreducible \( G \)-representations. Then

1. \( V \not\cong W \Rightarrow \text{Hom}_G(V, W) = 0 \)
2. \( V \cong W \Rightarrow \dim \text{Hom}_G(V, W) = 1 \)

Proof. 1.: We show the contraposition and thus assume the existence of a \( G \)-morphism \( \varphi : V \to W \), \( \varphi \neq 0 \).
CHAPTER 12. SCHUR’S LEMMA, MULTIPL., AND ISOTYPIC DECOMP.

- ker $\varphi \subseteq V$ is a subrepresentation. Since $V$ is irreducible, it follows that ker $\varphi = 0$, thus $\varphi$ is injective.

- $0 \neq \text{im} \varphi \subseteq W$ is a subrepresentation. Since $W$ is irreducible, $\text{im} \varphi = W$, thus $\varphi$ is surjective.

Putting both bullet points together we see that $\varphi$ is bijective. Thus $V \cong W$.

2. We first treat the case $V = W$. Let $\varphi \in \text{End}_G(V)$ be arbitrary. Let $v$ be an eigenvector of $\varphi$ to the eigenvalue $\xi$. Then $\varphi - \text{id} \in \text{End}_G(V)$ and $v \in \ker(\varphi - \text{id})$.

\[ 0 \neq v \in \ker(\varphi - \text{id}) \quad \text{V-irred} \Rightarrow \ker(\varphi - \text{id}) = V, \]

thus $\varphi - \text{id} = 0$, therefore $\varphi = \text{id}$.

For the more general case $V \cong W$ let $\psi \in \text{Hom}_G(V,W)$ be a $G$-isomorphism. Let $\varphi \in \text{Hom}_G(V,W)$ be arbitrary. Then $\psi^{-1} \circ \varphi \in \text{End}_G(V) = \text{Cid}_V$. Thus there exists $\alpha \in \mathbb{C}$ with $\psi^{-1} \circ \varphi = \alpha \text{id}_V$. Therefore $\varphi = \alpha \psi$. We conclude that $\text{Hom}_G(V,W) = \mathbb{C} \psi$.

12.2 Multiplicities

In this section we present the definition of representation-theoretic multiplicities. We will use this to define special types of obstructions, see Section 12.4.

12.2.1 Corollary. Let $V$ be a $G$-representation. Let $V = U_1 \oplus \cdots \oplus U_t$ be a decomposition into irreducibles. Let $W$ be an irreducible $G$-representation. Then $|\{i \mid U_i \cong W\}| = \dim \text{Hom}_G(W,V) = \dim \text{Hom}_G(V,W)$.

Proof. Let $\text{pr}_i : V \to U_i$ denote the $i$-th canonical projection. The following is an isomorphism of vector spaces:

\[
\bigoplus_{i=1}^t \text{Hom}_G(W,U_i) \to \text{Hom}_G(W,V) \quad \text{vectors to functions} \\
(\varphi_1, \ldots, \varphi_t) \mapsto (w \mapsto \varphi_1(w) + \cdots + \varphi_t(w)) \\
(\text{pr}_1 \circ \psi, \ldots, \text{pr}_t \circ \psi) \leftrightarrow \psi
\]

Schur’s lemma implies $\dim \bigoplus_{i=1}^t \text{Hom}_G(W,U_i) = \sum_{i=1}^t \dim \text{Hom}_G(W,U_i) = |\{i \mid U_i \cong W\}|$, which finishes the proof of the first equality. For the second part we proceed analogously with an isomorphism of vector spaces.

\[
\bigoplus_{i=1}^t \text{Hom}_G(U_i,W) \to \text{Hom}_G(V,W) \\
(\varphi_1, \ldots, \varphi_t) \mapsto \varphi_1 \circ \text{pr}_1 + \cdots + \varphi_t \circ \text{pr}_t \\
(\psi|_{U_1}, \ldots, \psi|_{U_t}) \leftrightarrow \psi
\]

Schur’s lemma implies $\dim \bigoplus_{i=1}^t \text{Hom}_G(U_i,W) = \sum_{i=1}^t \dim \text{Hom}_G(U_i,W) = |\{i \mid U_i \cong W\}|$, which finishes the proof of the second equality.

From this corollary we see that $|\{i \mid U_i \cong W\}|$ is independent of the decomposition. This justifies the name “multiplicity” in the following definition.

12.2.2 Definition. For a $G$-representation $V$ and an irreducible $G$-representation $W$ the multiplicity $\text{mult}_W(V)$ of $W$ in $V$ is defined as

\[ \text{mult}_W(V) := \dim \text{Hom}_G(W,V). \]
12.2.3 Corollary. If $U \subseteq V$ is a subrepresentation, then $\text{mult}_W(U) \leq \text{mult}_W(V)$.

Proof. If $U \subseteq V$, then $\text{Hom}_G(W,U)$ is a linear subspace of $\text{Hom}_G(W,V)$.

12.2.4 Corollary. If $U \rightarrow V$ is a $G$-equivariant surjection of representations, then $\text{mult}_W(U) \geq \text{mult}_W(V)$.

Proof. Let $\varphi : U \rightarrow V$ be a $G$-equivariant surjection. Define the linear map $\kappa : \text{Hom}_G(V,W) \rightarrow \text{Hom}_G(U,W)$ by $\kappa(\psi) = \psi \circ \varphi$. It remains to show that $\kappa$ is injective. For this we assume that $\kappa(\psi) = 0$, i.e., $\psi \circ \varphi = 0$. Since $\varphi$ is surjective, it follows $\psi = 0$.

In the case where $G = \text{GL}_N$ we have the following very useful way of determining multiplicities:

12.2.5 Proposition. If $V$ is a $\text{GL}_N$-representation, then $\text{mult}_\lambda(V) = \dim \text{HWV}_\lambda(V)$, where $\text{HWV}_\lambda(V)$ is the linear subspace of highest weight vectors of weight $\lambda$ in $V$.

Proof. Fix an irreducible $\text{GL}_N$-representation $W_\lambda$ and fix a nonzero vector $h$ from the 1-dimensional linear subspace of HWVs in $W_\lambda$. By Theorem 11.3.6(a'), if $0 \neq v \in \text{HWV}_\lambda(V)$, then $(\text{GL}_Nv)$ is irreducible. By Lemma 12.1.2 it follows that $\dim \text{Hom}_G(W_\lambda, (\text{GL}_Nv)) = 1$. Since every equivariant map maps HWVs of weight $\lambda$ to HWVs of weight $\lambda$ or to 0, every element in $\text{Hom}_G(W_\lambda, (\text{GL}_Nv))$ has $\varphi(h) = \alpha v$ for some $\alpha \in \mathbb{C}$. Moreover, for each $\alpha$ there exists such a $G$-isomorphism.

Now we have the following isomorphism of vector spaces $\text{HWV}_\lambda(V) \rightarrow \text{Hom}_G(W_\lambda, V)$:

$$v \mapsto \left( \varphi \in \text{Hom}_G(W_\lambda, (\text{GL}_Nv)), \varphi(h) = v \right)$$

with inverse map $\varphi \mapsto \varphi(h)$.

12.3 Isotypic components

In this section we assume that our group $G$ is linearly reductive. The decomposition into irreducible representations might not be unique as soon as the multiplicity of some isomorphism type $\lambda$ exceeds 1. In this section we group together isomorphic copies of the same irreducible representation to obtain the unique isotypic decomposition.

A representation $V$ is called isotypic if $V$ is a (not necessarily direct or finite) sum of irreducible representations that are all isomorphic.

12.3.1 Definition. Let $G$ be a group and let $V$ be a $G$-representation (in particular finite dimensional). Let $W$ be an irreducible $G$-representation and define $\lambda$ to be its isomorphism type. The isotypic component $V_\lambda$ of type $\lambda$ is defined as the (possibly infinite) sum $\sum_i V_i$ of all irreducible subrepresentations of type $\lambda$.

For example the weight spaces in section 11 are isotypic components, where the group $G$ is the algebraic torus.

12.3.2 Lemma. An isotypic representation of type $\lambda$ decomposes into a direct sum of irreducibles of type $\lambda$.

Proof. Let $V$ be isotypic and write $V = E_1 + \cdots + E_t$ with $E_i$ irreducible of type $\lambda$ and $t$ minimal. Clearly $t$ is finite because $\dim V$ is finite and $\dim E_i \geq 1$. For the sake of contradiction assume that the sum is not direct: There exists $x_i \in E_i$ such that $x_1 + \cdots + x_t = 0$ and w.l.o.g. $x_t \neq 0$. Thus $x_t = -(x_1 + \cdots + x_{t-1})$ and hence $(E_1 + \cdots + E_{t-1}) \cap E_t \neq 0$. Since $E_t$ is irreducible: $E_t \subseteq E_1 + \cdots + E_{t-1}$, which is a contradiction to $t$ being minimal.

12.3.3 Proposition. Every representation $V$ decomposes into a direct sum of isotypic representations $V = \bigoplus \lambda V_\lambda$, where $\lambda$ runs over all types of irreducible representations.
CHAPTER 12. SCHUR’S LEMMA, MULTIPL., AND ISOTYPIC DECOMP.

68

12.4 Using multiplicities or occurrences as obstructions

Let $\mathbb{A} = \mathbb{C}^n$. For a Zariski-closed set $Z \subseteq \mathbb{A}$ we define the coordinate ring

$$C[Z] := \mathbb{C}[\mathbb{A}] / I(Z)$$

If $Z$ is a cone, then $C[Z]$ is graded via $C[Z]_\delta = \mathbb{C}[\mathbb{A}]_\delta / I(Z)_\delta$.

**12.4.1 Lemma.** For a $G$-representation $V$ and a subrepresentation $U$ the quotient $V/U$ is also a $G$-representation.

Let $G$ by a group, $V$ be a $G$-representation, and let there exist a $G$-invariant inner product on $V$. If $U \subseteq V$ is a subrepresentation, then the quotient $V/U$ is also a $G$-representation. More precisely, $V \cong U \oplus V/U$.

**Proof.** Since $U$ is a $G$-representation, for $g \in G$ we have $gU = U$ as a set. Thus if $v \in V$, then $v + U \in V/U$ and $g(v + U) = g v + gU = g v + U \in V/U$.

If we have a $G$-invariant inner product, then $V = U \oplus U^\perp$. We now show that $U^\perp \cong V/U$.

Let $p : V \rightarrow U^\perp$ be the projection that sends $U$ to 0. The equivariant isomorphism $V/U \rightarrow U^\perp$ is given by $v + U \mapsto p(v + U) = p(v)$ with inverse map $w \mapsto w + U$.

From the lemma we conclude that if $C[\mathbb{A}]_\delta$ is a $G$-representation and $I(Z)_\delta$ is a $G$-representation, then $C[Z]_\delta$ is a $G$-representation.

An approach towards proving complexity lower bounds goes as follows. Let $Z' \subseteq Z$ be a Zariski-closed cone that is closed under the action of $GL_M$. Think of $GL_{n^2+1} T^{n-m} \per_m \subseteq GL_{n^2+1} \det_n$ or of $GL_{n^2} X_{1,1}^{n-m} \per_m \subseteq GL_{n^2} \det_m$ for some fixed values of $n$ and $m$. Then $I(Z)_\delta \subseteq I(Z')_\delta$ and thus we obtain a canonical $GL_{n^2}$-equivariant surjection $C[Z]_\delta \twoheadrightarrow C[Z']_\delta$. By Schur’s lemma (Cor. 12.2.4) this implies $\text{mult}_\lambda(C[Z]) \geq \text{mult}_\lambda(C[Z'])$.

Thus if we want to prove $Z' \not\subseteq Z$, it is sufficient to show the existence of some $\lambda$ that satisfies $\text{mult}_\lambda(C[Z]) < \text{mult}_\lambda(C[Z'])$. Such $\lambda$ are called representation theoretic multiplicity obstructions. If $\text{mult}_\lambda(C[Z]) = 0 < \text{mult}_\lambda(C[Z'])$, then these $\lambda$ are called occurrence obstructions.

Mulmuley and Sohoni conjectured that one could separate $\text{VNP} \not\subseteq \text{VP}_{\text{nos}}$ by using occurrence obstructions, but this was recently rejected:

**12.4.2 Conjecture.** For every polynomial $p$ there exist infinitely many $m$ and $n \geq p(m)$ with:

If $Z' := \overline{GL_n^2 X_{1,1}^{n-m} \per_m}$ and $Z := \overline{GL_n^2 \det_n}$, then there exists $\lambda$ with $\text{mult}_\lambda(C[Z']) > 0 = \text{mult}_\lambda(C[Z])$.

**12.4.3 Theorem ([BIP16]).** Let $n \geq m^{25}$ and let $Z' := \overline{GL_n^2 X_{1,1}^{n-m} \per_m}$ and $Z := \overline{GL_n^2 \det_n}$. If $\text{mult}_\lambda(C[Z']) > 0$, then $\text{mult}_\lambda(C[Z]) > 0$. 

Proof. Let $V = M_1 \oplus \cdots \oplus M_k$ be a decomposition into irreducibles. Then $\text{mult}_\lambda(V)$ equals the number of times for which $M_i$ is of type $\lambda$. Define the direct sum

$$M := \bigoplus_{i \text{ with } M_i \text{ of type } \lambda} M_i,$$

so $\dim M = \text{mult}_\lambda(V) \cdot \dim \lambda$, where $\dim \lambda$ denotes the dimension of the irreducible representation of type $\lambda$. It remains to show that $M = V_\lambda$, because then we see that the direct sum of isotypic components results from adding up isomorphic copies of irreducible representations. Clearly $M \subseteq V_\lambda$.

Since $V_\lambda$ decomposes into a direct sum of irreducibles of type $\lambda$ (Lemma 12.3.2), the number of summands in this decomposition is $\text{mult}_\lambda(V_\lambda)$ by Cor. 12.2.1. Therefore $\dim V_\lambda = \text{mult}_\lambda(V_\lambda) \cdot \dim \lambda$.

Since $V_\lambda \subseteq V$ it follows $\text{mult}_\lambda(V_\lambda) \leq \text{mult}_\lambda(V)$ and thus $\dim V_\lambda \leq \text{mult}_\lambda(V) \cdot \dim \lambda = \dim M$. 

\[\square\]
It is an open problem if multiplicities can be used to separate orbit closures. More specifically, it is open if $\text{VNP} \nsubseteq \text{VP}_{\text{ws}}$ can be proved using representation theoretic multiplicity obstructions.

12.4(i) Plethysm coefficients

When we comparing multiplicities, then a first comparison is always with the plethysm coefficient that we explain in this section (see Lemma 12.4.4).

Let $A = \mathbb{C}[X_1, \ldots, X_M]_d$. Fix $\delta \in \mathbb{N}$. Let $\lambda$ be a partition of $d\delta$. Define the plethysm coefficient as

$$a_{\lambda}(\delta, d) := \text{mult}_{\lambda}(\mathbb{C}[A]_\delta).$$

We will see later (Proposition 19.3.9) that $a_{\lambda}(\delta, d)$ basically does not depend on $M$: Define $\ell(\lambda) := \max\{i \mid \lambda_i > 0\}$. If $\ell(\lambda) > M$, then $a_{\lambda}(\delta, d) = 0$. On the other hand $a_{\lambda}(\delta, d)$ has the same value for all $M \geq \ell(\lambda)$. Therefore we define $a_{\lambda}(\delta, d)$ to be the value for large $M$.

Finding a combinatorial description for $a_{\lambda}(\delta, d)$ is a major open problem in algebraic combinatorics. It is problem 9 on Stanley’s problem list from 2000 [Sta00]. In terms of theoretical computer science, this question can be phrased as: Is the function $(\lambda, \delta, d) \mapsto a_{\lambda}(\delta, d)$ in the complexity class $\text{#P}$? Here we are allowed to encode the partition $\lambda$ in unary.

The Schur software and the LiE software can compute plethysm coefficients.

12.4.4 Lemma. There exists $\lambda$ with $\text{mult}_\lambda(\mathbb{C}[Gv]_\delta) < a_{\lambda}(\delta, d)$ iff the type $\lambda$ occurs in the vanishing ideal $I(Gv)_\delta$.

Proof. $\mathbb{C}[A]_\delta = I(Gv)_\delta \oplus \mathbb{C}[Gv]_\delta$ and thus $a_{\lambda}(\delta, d) = \text{mult}_\lambda(\mathbb{C}[Gv]_\delta) + \text{mult}_\lambda(I(Gv)_\delta)$. \qed

### Multiplicities

Representation-theoretic multiplicities count how often an irreducible representation occurs in a decomposition into irreducibles.

The vanishing ideal and the coordinate ring are dual notions. Their multiplicities add up to the plethysm coefficient.

An attack route towards finding obstructions goes via comparing multiplicities in coordinate rings of orbit closures. Occurrence obstructions are known not to separate $\text{VP}_{\text{ws}}$ from VNP.
Chapter 13

Tensors for computer scientists

In this chapter we discuss tensors. This will serve mainly two purposes: To discuss the computational complexity of bilinear maps using geometric complexity theory, and to explicitly construct the irreducible representations of $\text{GL}_n$ and their highest weight vectors.

13.1 Bilinear forms

Let $U$ and $V$ be vector spaces over $F$. All vector spaces are assumed to be finite dimensional. Let $f : U \times V \to F$ be a bilinear form, that is, a form which is linear in both components. We denote the set of all bilinear forms by $\text{Bil}(U, V; F)$. A linear form $\ell : U \to F$ is uniquely determined when we know its values at any basis $u_1, \ldots, u_m$ of $U$. How about $f$?

13.1.1 Lemma. Let $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ be bases of $U$ and $V$, respectively. Then $f$ is uniquely determined by the values $f_{i,j} := f(u_i, v_j)$, $1 \leq i \leq m$, $1 \leq j \leq n$.

Proof. Let $g : U \times V \to F$ be another bilinear form with $g(u_i, v_j) = f_{i,j}$, $1 \leq i \leq m$, $1 \leq j \leq n$. Let $u = \sum_{i=1}^m \alpha_i u_i$ and $v = \sum_{j=1}^n \beta_j v_j$ be arbitrary. We have

$$g(u, v) = g(\sum_{i=1}^m \alpha_i u_i, \sum_{j=1}^n \beta_j v_j)$$

$$= \sum_{i=1}^m \alpha_i g(u_i, \sum_{j=1}^n \beta_j v_j)$$

$$= \sum_{i=1}^m \sum_{j=1}^n \alpha_i \beta_j g(u_i, v_j)$$

$$= \sum_{i=1}^m \sum_{j=1}^n \alpha_i \beta_j f_{i,j}$$

$$= f(u, v).$$

Note that to get the last line, we used bilinearity again. □

By choosing the bases, we identify $U$ with $F^m$ and $V$ with $F^n$. Now, we can write $f$ even more concretely as

$$f(x, y) = \sum_{i=1}^m \sum_{j=1}^n f_{i,j} x_i y_j.$$
(As a golden rule, you should avoid specifying a basis unless it is really necessary. However, it is at first more intuitive to think in terms of bases.) You usually think of \( x_i \) as an indeterminate, and to evaluate \( f \) we substitute the value \( \alpha_i \) for \( x_i \). But you can also think of \( x_i \) being a linear form mapping (by substitution) a vector \( \sum_{i=1}^{m} \alpha_i u_i \) to \( \alpha_i \), that is, \( x_1, \ldots, x_m \) is a dual basis to \( u_1, \ldots, u_m \). The same is true for \( y_1, \ldots, y_n \). The products of linear forms \( x_i y_j \) form a basis of the linear space of bilinear forms \( U \times V \rightarrow \mathbb{F} \). Recall that the set of all linear forms on \( U \) or \( V \) are denoted by \( U^* \) of \( V^* \), respectively.

13.1.2 Definition. The space of all bilinear forms \( U \times V \rightarrow \mathbb{F} \) is called the tensor product of \( U^* \) and \( V^* \) and is denoted by \( U^* \otimes V^* \).

Let \( x = \sum_{i=1}^{m} \alpha_i x_i \in U^* \) and \( y = \sum_{j=1}^{n} \beta_j y_j \in V^* \). We have

\[
xy = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \beta_j x_i y_j.
\]

How do you get the tensor product \( U \otimes V \)? You simply start with bilinear forms \( U^* \times V^* \rightarrow \mathbb{F} \). While this looks complicated at a first glance—bilinear forms mapping pairs of linear forms to scalars—also \( U^* \) is just a vector space and once you choose a basis, everything is isomorphic to some \( \mathbb{F}^m \).

13.1.3 Exercise. Prove that \( \text{Hom}(U, V) \cong U^* \otimes V^* \). While this is absolutely clear to every mathematician, computer scientist tend to forget about this pretty soon. (Or even never learned it this way.)

The previous exercise identifies linear maps \( U \rightarrow V \) with bilinear forms on \( U \times V^* \). Both objects are specified by a two-dimensional array of field elements and we interpret this data in two different ways. So \( \text{Hom}(U, V) \cong \text{Bil}(U, V^*; \mathbb{F}) \) essentially says nothing. However, by re-interpreting objects in the right way, one can often prove astonishing facts very quickly. You should get used to this!

13.2 Universal property

We can define tensor products also in terms of a universal property. A tensor product of two spaces \( U \) and \( V \) is a vector space, denoted by \( U \otimes V \), together with a bilinear map \( \phi: U \times V \rightarrow U \otimes V \) such that for any bilinear map \( b: U \times V \rightarrow W \), there is a unique linear map \( \ell: U \otimes V \rightarrow W \) such that \( b = \ell \circ \phi \). Given a tensor product, we set \( u \otimes v := \phi(u, v) \) for every \( u \in U \) and \( v \in V \).

13.2.1 Theorem. Let \( U \) and \( V \) be (finite-dimensional) vector spaces.

1. \( U \) and \( V \) have a tensor product.

2. Any two tensor products of \( U \) and \( V \) are isomorphic.

3. If \( u_1, \ldots, u_m \) is a basis of \( U \) and \( v_1, \ldots, v_n \) is a basis of \( V \), then \( u_i \otimes v_j, 1 \leq i \leq m, 1 \leq j \leq n \) is a basis of \( U \otimes V \).

Proof. To prove the first item, we construct an explicit tensor product. It will be the construction of the previous section. We set \( U \otimes V = \text{Bil}(U^*, V^*; \mathbb{F}) \) and \( \phi(u, v)(x, y) = x(u) \cdot y(v) \). Then the third item immediately follows from the discussion right after Lemma 13.1.1.
CHAPTER 13. TENSORS FOR COMPUTER SCIENTISTS

Let \( b : U \times V \to W \) be a bilinear map. We choose a basis \( u_1, \ldots, u_m \) of \( U \) and a basis \( v_1, \ldots, v_n \) of \( V \). To finish the proof of the first item, we define the linear map \( \ell : U \otimes V \to W \) by

\[
\ell(u \otimes v) = \ell \left( \sum_{i=1}^{m} \alpha_i u_i \otimes \sum_{j=1}^{n} \beta_j v_j \right)
\]

Note that for \( u = \sum_{i=1}^{m} \alpha_i u_i \) and \( \sum_{j=1}^{n} \beta_j v_j \), we have

\[
\ell(u \otimes v) = \left( \sum_{i=1}^{m} \alpha_i u_i \otimes \sum_{j=1}^{n} \beta_j v_j \right) = \left( \sum_{i=1}^{m} \alpha_i \beta_j \ell(u_i \otimes v_j) \right) = \left( \sum_{i=1}^{m} \alpha_i \beta_j \ell(u_i, v_j) \right) = b(u, v),
\]

so \( \ell \circ \phi \) and \( b \) are equal. \( \ell \) is unique, since it is defined on a basis of \( U \times V \).

It remains to prove the second item. Assume we have two tensor products \( U \otimes V, \phi \) and \( U \otimes V, \phi' \). We apply the definition of tensor product to \( U \otimes V \) and \( \phi \) and let the bilinear map \( b = \phi' \) and the vector space \( W = U \otimes' V \). We get a linear map \( \ell : U \otimes W \to U \otimes' V \) such that \( \ell \circ \phi = \phi' \). In the same way, by interchanging the roles of the two tensor products, we get a linear map \( \ell' : U \otimes' V \to U \otimes V \). The situation is depicted below:

We have \( \ell' \circ \ell \circ \phi = \ell' \circ \phi' = \phi \). We apply the definition of tensor product to \( U \otimes V \) and \( \phi \) and let the bilinear map be \( \phi \) and the vector space \( W = U \times V \). Then the linear map can be the identity and it can be \( \ell' \circ \ell \) by the equation above. By the uniqueness of the linear map, we get that \( \ell' \circ \ell \) is the identity (on \( U \otimes V \)). In the same way, we get that \( \ell \circ \ell' \) is the identity (on \( U \otimes' V \)). Thus \( \ell \) and \( \ell' \) are isomorphisms.

13.2.2 Exercise. Let \( U, V, \) and \( W \) be vector spaces. Prove the following:

1. \( U \otimes V \cong V \otimes U \).
2. \( U \otimes (V \otimes W) \cong (U \otimes V) \otimes W \).
3. \( \dim (U \otimes V) = \dim U \cdot \dim V \).

The second item says that the tensor product is associative (up to isomorphisms), therefore, we simply can write \( U \otimes V \otimes W \). One could also define a threefold tensor product directly by defining it as the vector space of trilinear form \( U^* \times V^* \times W^* \to \mathbb{F} \). In the same way, we can build the tensor product of an arbitrary number of vector spaces.
13.3 Tensor rank

Elements of the form $u \otimes v \in U \otimes V$ are called elementary or decomposable or rank-one tensors or triads. Not all elements are elementary, for instance $u_1 \otimes v_1 + u_2 \otimes v_2$ is not elementary when $u_1$ and $u_2$ as well as $v_1$ and $v_2$ are linearly independent. In general, if we have a tensor product $V_1 \otimes \cdots \otimes V_k$, we call elements of the form $v_1 \otimes \cdots \otimes v_k$ with $v_i \in V_i$ elementary (or decomposable or rank-one tensors).

The rank of a matrix $M$ can be defined as the minimum number $r$ of rank one matrices $S_1, \ldots, S_r$ such that $M = S_1 + \cdots + S_r$. In the same way, we define the rank of a tensor $t \in V_1 \otimes \cdots \otimes V_k$ as the minimum number of rank-one tensors $s_1 \otimes \cdots \otimes s_r \in V_1 \otimes \cdots \otimes V_k$ such that

$$t = s_1 + \cdots + s_r.$$

We denote the rank of a tensor by $R(t)$.

Note that this generalizes the rank of a matrix. Any matrix $M$ can be interpreted as an element of $U^* \otimes V$. A rank-one matrix $S$ can be written as $S = a \cdot b$ where $a$ is a column vector and $b$ is a row vector. Then for any column vector $x$,

$$S \cdot x = (a \cdot b) \cdot x = (b \cdot x) \cdot a,$$

because $b \cdot x$ is a $1 \times 1$ matrix. In this way, we can interpret $b$ as a linear form on $U$.

Note that for matrices, we have further equivalent definitions of rank. In particular, there are efficient algorithms for computing the rank. This is not true for tensors in a threefold (or higher) tensor product. Here the problem is NP-hard [Hás90] and even hard for the existential theory over the underlying ground field $F$ [SS16, Shi16].

13.4 Actions on tensor products

Let $V_1, \ldots, V_k$ and $U_1, \ldots, U_k$ be vector spaces and let $A_i \in \text{Hom}(V_i, U_i)$, $1 \leq i \leq k$. We can extend the $A_i$ to a homomorphism

$$A_1 \otimes \cdots \otimes A_k : V_1 \otimes \cdots \otimes V_k \rightarrow U_1 \otimes \cdots \otimes U_k$$

in the following way: Let $v_1 \otimes \cdots \otimes v_k \in V_1 \otimes \cdots \otimes V_k$. We set

$$A_1 \otimes \cdots \otimes A_k(v_1 \otimes \cdots \otimes v_k) = A_1(v_1) \otimes \cdots \otimes A_k(v_k)$$

and extend $A_1 \otimes \cdots \otimes A_k$ to $V_1 \otimes \cdots \otimes V_k$ by linearity.

13.4.1 Exercise. Prove that $A_1 \otimes \cdots \otimes A_k$ is well-defined, that is, if we decompose a tensor $t$ in two different ways into rank-one tensors, then we get the same result.

13.4.2 Definition. Let $t \in V_1 \otimes \cdots \otimes V_k$ and $s \in U_1 \otimes \cdots \otimes U_k$. We call $s$ a restriction of $t$ and write $s \leq t$ if there are $A_i \in \text{Hom}(V_i, U_i)$, $1 \leq i \leq k$, such that $A_1 \otimes \cdots \otimes A_k(t) = s$.

The proof of the following lemma is obvious.

13.4.3 Lemma. If $s$ is a restriction of $t$, then $R(s) \leq R(t)$.

We can let $\text{End}(V_i) \times \cdots \times \text{End}(V_k)$ act on $V_1 \otimes \cdots \otimes V_k$ by

$$(A_1, \ldots, A_k)t = A_1 \otimes \cdots \otimes A_k(t).$$

If $U_i$ is a subspace of $V_i$, then we can write the fact that $s$ is a restriction of $t$ as an orbit problem, namely, $s \leq t$ if

$$s \in (\text{End}(V_1) \times \cdots \times \text{End}(V_k))t.$$

Note that by Lemma 13.4.3, this means that $R(s) \leq R(t)$. In the next chapter, we will see how we can interpret this in terms of complexity.
The language of tensors

The language of tensors is a natural way of describing multilinear maps. This will help us in the study of the complexity of bilinear maps. Moreover, tensor products are fundamental building blocks in the representation theory of $\text{GL}_n$. 
Chapter 14

Complexity of bilinear maps

The following two chapters give a brief introduction to the tensor rank problem and its relation to fast matrix multiplication. Many results have been taken from [Blä13], nevertheless we decided to restate them explicitly for the reader's convenience. For even more details, the reader is referred to [Blä13] and the references given there.

14.1 Strassen’s algorithm

Given a \( k \times m \)-matrix \( x = (x_{hi}) \) and an \( m \times n \)-matrix \( y = (y_{ij}) \) whose entries are indeterminates over some field \( \mathbb{F} \), we want to compute their product \( xy = (z_{hj}) \). The entries \( z_{hj} \) are given by

\[
\begin{align*}
  z_{hj} &= \sum_{i=1}^{m} x_{hi} y_{ij}, \quad 1 \leq h \leq k, \quad 1 \leq j \leq m. 
\end{align*}
\]  

(14.1.1)

In 1969, Strassen [Str69] found a way to multiply \( 2 \times 2 \)-matrices with only 7 multiplications but 18 additions.

Let \( z_{ij} \), \( 1 \leq i, j \leq 2 \), be given by

\[
\begin{pmatrix}
  z_{11} & z_{12} \\
  z_{21} & z_{22}
\end{pmatrix} = 
\begin{pmatrix}
  x_{11} & x_{12} \\
  x_{21} & x_{22}
\end{pmatrix} 
\begin{pmatrix}
  y_{11} & y_{12} \\
  y_{21} & y_{22}
\end{pmatrix}.
\]

We compute the seven products

\[
\begin{align*}
  p_1 &= (x_{11} + x_{22})(y_{11} + y_{22}), \\
  p_2 &= (x_{21} + x_{22})y_{11}, \\
  p_3 &= x_{11}(y_{12} - y_{22}), \\
  p_4 &= x_{22}(-y_{11} + y_{21}), \\
  p_5 &= (x_{11} + x_{12})y_{22}, \\
  p_6 &= (-x_{11} + x_{21})(y_{11} + y_{12}), \\
  p_7 &= (x_{12} - x_{22})(y_{21} + y_{22}).
\end{align*}
\]

We can express each of the \( z_{ij} \) as a linear combination of these seven products, namely,

\[
\begin{pmatrix}
  z_{11} & z_{12} \\
  z_{21} & z_{22}
\end{pmatrix} = 
\begin{pmatrix}
  p_1 + p_4 - p_5 + p_7 \\
  p_2 + p_4 \\
  p_3 + p_7 \\
  p_2 + p_4 \\
  p_1 + p_3 - p_2 + p_6
\end{pmatrix}.
\]

By applying this construction recursively, we get the well-known algorithm which multiplies matrices in time \( O(n \log_2 7) \). To make the recursion work, it is crucial that the entries are bilinear products.
14.2 Relation to tensor rank

Assume that in general, we have $k$ bilinear forms

$$z_h = \sum_{i=1}^{m} \sum_{j=1}^{n} t_{i,j,h} x_i y_j, \quad h = 1, \ldots, k$$

and we have $r$ bilinear products

$$p_\rho = (u_{\rho,1} x_1 + \cdots + u_{\rho,m} x_m)(v_{\rho,1} y_1 + \cdots + v_{\rho,m} y_m)$$

such that we can write each $z_h$ as a linear combination of them, that is,

$$z_h = w_{1,h} p_1 + \cdots + w_{r,h} p_r.$$

We can view the "array" $t = (t_{h,i,j})$ as a tensor in $F^k \otimes F^m \otimes F^n$. The products $p_1, \ldots, p_r$ correspond to a decomposition of $t$ into rank-one tensors: Namely, let $w_\rho = (w_{\rho,1}, \ldots, w_{\rho,k})$, $1 \leq \rho \leq r$ and define $u_\rho$ and $v_\rho$ accordingly. Then

$$w_\rho \otimes u_\rho \otimes v_\rho = (w_{\rho,h} u_{\rho,i} v_{\rho,j})$$

and by comparing coefficients, we get that

$$t = \sum_{\rho=1}^{r} w_\rho \otimes u_\rho \otimes v_\rho.$$ 

In the same way, if we have a decomposition of $t$ into $r$ rank-one tensors, then we can obtain $r$ bilinear products such that each $z_h$ is contained in their linear span. Therefore, the minimal number of such products is precisely $R(t)$.

14.3 The exponent of matrix multiplication

We denote the tensor of the multiplication of $k \times m$-matrices with $m \times n$-matrices by $\langle k, m, n \rangle$. The corresponding tensor lives in $F^k \otimes F^m \otimes F^n$. We here transpose the matrices in the last component for symmetry reasons. Note that every component is indexed by double-indices. We have

$$z'_{h',j} = \sum_{i=1}^{m} x'_{h',i} y'_{i,j} = \sum_{h'=1}^{k} \sum_{i=1}^{m} \sum_{j=1}^{n} \delta_{h,h'} \delta_{i,i'} \delta_{j,j'} x'_{h',i} y'_{i,j},$$

Thus $\langle k, m, n \rangle = (\delta_{h,h'} \delta_{i,i'} \delta_{j,j'})$. Figure 14.1 contains an explicit description of the tensor of $2 \times 2$-matrix multiplication.

14.3.1 Definition. $\omega = \inf \{ \beta \mid R(\langle n, n, n \rangle) \leq O(n^\beta) \}$ is called the exponent of matrix multiplication.

In the definition of $\omega$ above, we only count bilinear products. For the asymptotic growth, it does not matter whether we count all operations or only bilinear products. Let $\tilde{\omega}$ be the infimum over all $\beta$ such that there is a family of arithmetic circuits of size $O(n^\beta)$ computing the product of two $n \times n$-matrices. Since these circuits compute forms of degree two, we can make these circuits homogeneous such that the only nonscalar multiplications are products of linear forms.

14.3.2 Theorem. $\omega = \tilde{\omega}$. 


We first prove \( \omega \leq \tilde{\omega} \): Consider an arbitrary circuit computing the product of two matrices. Let \( r \) be number of nonscalar multiplications in it. As in the transformation of arbitrary circuits into homogeneous ones, we now compute with each homogeneous component separately. Note that since the output of each circuit is homogeneous of degree two, we only need to keep the components of degree up to two. The only nonscalar multiplications that we need to perform are the multiplications between the degree-one-terms, which is a product of linear forms. Therefore, we can modify the circuit as follows: We first compute several linear forms, then we perform \( r \) multiplications in them and then we compute linear combinations of the \( r \) products.

Does this prove that the rank is bounded by \( r \), too? Not quite. The linear forms can be linear forms in the entries of both matrices. Consider such a product \( u(X,Y)v(X,Y) \). We can write \( u(X,Y) = u'(X) + u''(Y) \). We do the same for \( v \). Then

\[
 u(X,Y)v(X,Y) = u'(X)v'(X) + u''(Y)v'(X) + u''(Y)v''(X).
\]

Since the outputs are all bilinear forms, the contribution of all \( u'(X)v'(X) \) and all \( u''(Y)v''(Y) \) cancel. Therefore, we can replace the product above by two bilinear products \( u'(X)v''(Y) + u''(Y)v'(X) \). Therefore, the rank is bounded by \( 2r \) and \( \omega \leq \tilde{\omega} \).

For the other inequality, note that from the definition of \( \omega \), it follows that

\[
 \forall \epsilon > 0 : \exists \alpha \text{ and } m_0 > 1 : \forall m \geq m_0 : R(\langle m, m, m \rangle) \leq \alpha \cdot m^{\omega + \epsilon}.
\]

Let \( \epsilon > 0 \) be given and choose \( m \) large enough. Let \( r = R(\langle m, m, m \rangle) \).

To multiply \( m^i \times m^j \)-matrices we decompose them into blocks of \( m^{i-1} \times m^{j-1} \)-matrices and apply recursion. To multiply matrices of arbitrary sizes, we can pad with 0 to the next power of \( m \). Let \( A(n) \) be the number of arithmetic operations for the multiplication of \( n \times n \)-matrices with this approach. We obtain

\[
 A(n) \leq rA(n/m) + c(n/m)^2
\]

where \( c \) is the number of additions and scalar multiplications that are performed by the chosen bilinear algorithm for \( \langle m, m, m \rangle \) with \( r \) bilinear multiplications. Solving the recursion using the master theorem [CLRS09], we get \( A(n) = O(n^{\log_m r}) \). (Note that \( r > m^2 \) in general, so \( \log_m r > 2 \) and we are in the first case of the master theorem.)

Since \( r \leq \alpha \cdot m^{\omega + \epsilon} \), we have \( \log_m r \leq \omega + \epsilon + \log_m \alpha \). With \( \epsilon' = \epsilon + \log_m \alpha \),

\[
 L(\langle n, n, n \rangle) = O(n^{\log_m r}) = O(n^{\omega + \epsilon'}).
\]

Thus

\[
 \tilde{\omega} \leq \omega + \epsilon \quad \text{for all } \epsilon > 0,
\]

since \( \log_m \alpha \to 0 \) if \( m \to \infty \). This means \( \tilde{\omega} = \omega \), since \( \tilde{\omega} \) is an infimum. \( \square \)
14.4 Rank and restrictions

In the following, \( \langle r \rangle \) denotes the tensor in \( \mathbb{F}^r \otimes \mathbb{F}^r \otimes \mathbb{F}^r \) that has a 1 in the positions \( (\rho, \rho, \rho) \), \( 1 \leq \rho \leq r \), and 0s elsewhere (a “diagonal”, the three-dimensional analogue of the identity matrix). This tensor corresponds to the \( r \) bilinear forms \( x_\rho y_\rho \), \( 1 \leq \rho \leq r \) (\( r \) independent products) and is called the unit tensor.

14.4.1 Lemma. \( R(t) \leq r \Leftrightarrow t \leq \langle r \rangle \).

Proof. "\( \Leftarrow \)" follows immediately from the observations that \( s \leq s' \) implies \( R(s) \leq R(s') \).

"\( \Rightarrow \)"  \( \langle r \rangle = \sum_{\rho=1}^{r} e_\rho \otimes e_\rho \otimes e_\rho \), where \( e_\rho \) is the \( \rho \)th unit vector. If the rank of \( t \) is \( \leq r \), then we can write \( t \) as the sum of \( r \) triads,

\[
t = \sum_{\rho=1}^{r} u_\rho \otimes v_\rho \otimes w_\rho.
\]

We define three homomorphisms

\[
\begin{align*}
\alpha : & e_\rho \mapsto u_\rho, & 1 \leq \rho \leq r, \\
\beta : & e_\rho \mapsto v_\rho, & 1 \leq \rho \leq r, \\
\gamma : & e_\rho \mapsto w_\rho, & 1 \leq \rho \leq r.
\end{align*}
\]

By construction,

\[
(\alpha \otimes \beta \otimes \gamma)(r) = \sum_{\rho=1}^{r} \alpha(e_\rho) \otimes \beta(e_\rho) \otimes \gamma(e_\rho) = t.
\]

Thus we can rephrase the question whether \( R((n,n,n)) \leq r \) as \( \langle n,n,n \rangle \leq \langle r \rangle \). Note that \( \langle n,n,n \rangle \) and \( \langle r \rangle \) live in general in different spaces, \( (\mathbb{F}^{n\times n})^{\otimes 3} \) and \( (\mathbb{F}^r)^{\otimes 3} \). For \( r \geq n^2 \) we can embed \( \langle n,n,n \rangle \) into \( (\mathbb{F}^r)^{\otimes 3} \) by padding the tensor with zeros. Therefore, the question \( R((n,n,n)) \leq r \) is equivalent whether the padded \( \langle n,n,n \rangle \) is in the \( \text{End}(\mathbb{F}^r)^{\otimes 3} \)-orbit of \( \langle r \rangle \).

14.5 Permutations of matrix multiplication tensors

Let \( t \in \mathbb{F}^k \otimes \mathbb{F}^m \otimes \mathbb{F}^n \) and \( t = \sum_{j=1}^{r} t_j \) with rank-one tensors \( t_j = a_{j1} \otimes a_{j2} \otimes a_{j3} \), \( 1 \leq j \leq r \).

Let \( \pi \in S_3 \), where \( S_3 \) denotes the symmetric group on \( \{1,2,3\} \). For a rank-one tensor \( t_j \), let \( \pi t_j = a_{j\pi^{-1}(1)} \otimes a_{j\pi^{-1}(2)} \otimes a_{j\pi^{-1}(3)} \) and \( \pi t = \sum_{j=1}^{r} \pi t_j \). It is an easy exercise to prove that \( \pi t \) is well-defined. The proof of the following lemma is obvious.

14.5.1 Lemma. \( R(t) = R(\pi t) \).

Let \( t = (t_{h',i',j',h}) = \langle k,m,n \rangle \) and \( \pi = (123) \). Then for \( \pi t =: t' \in \mathbb{F}^{(n\times k)} \otimes \mathbb{F}^{(k\times m)} \otimes \mathbb{F}^{(m\times n)} \), we have

\[
\begin{align*}
\delta_{j,j'} \delta_{h,h'} \delta_{i,i'} & = \delta_{j,j'} \delta_{h,h'} \\
& = \delta_{i,i'} \delta_{j,j'} \delta_{h,h'} \\
& = t_{h',i',j',h'}
\end{align*}
\]

Therefore,

\[
R(\langle k,m,n \rangle) = R(\langle n,k,m \rangle) = R(\langle m,n,k \rangle).
\]
Now, let \( t'' = (t_{h',i',v',j',h'}) \). We have \( R(t) = R(t'') \), since permuting the “inner” indices corresponds to permuting the slices of the tensor.

Next, let \( \pi = (12)(3) \). Let \( \pi t'' = t''' \in \mathbb{F}^{(n \times m)} \otimes \mathbb{F}^{(m \times k)} \otimes \mathbb{F}^{(k \times n)} \). We have,

\[
t'''_{h',i',v',h,j} = \delta_{i,\bar{h}} \delta_{h,v} \delta_{j,\bar{j}}
= t_{h',i',v',j',h}.
\]

Therefore,

\[
R((k,m,n)) = R((n,m,k))
\]

The second transformation corresponds to the well-known fact that \( AB = C \) implies \( B^T A^T = C^T \).

To summarize:

14.5.2 Lemma. \( R((k,m,n)) = R((n,m,k)) = R((m,n,k)) = R((k,n,m)) \).

14.6 Products of matrix multiplication tensors

If we have two tensors \( t \in U \otimes V \otimes W \) and \( t' \in U' \otimes V' \otimes W' \), we can view their product \( t \otimes t' \in (U \otimes V \otimes W) \otimes (U' \otimes V' \otimes W') \) as a tensor in \( (U \otimes U') \otimes (V \otimes V') \otimes (W \otimes W') \) by using the natural isomorphisms.

14.6.1 Lemma. \( R(t \otimes t') \leq R(t) R(t') \).

Proof. Let \( t = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i \) and \( t' = \sum_{j=1}^{r'} u'_j \otimes v'_j \otimes w'_j \). We have

\[
t \otimes t' = \left( \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i \right) \otimes \left( \sum_{j=1}^{r'} u'_j \otimes v'_j \otimes w'_j \right)
= \sum_{i=1}^{r} \sum_{j=1}^{r'} (u_i \otimes v_i \otimes w_i) \otimes (u'_j \otimes v'_j \otimes w'_j)
= \sum_{i=1}^{r} \sum_{j=1}^{r'} (u_i \otimes u'_j) \otimes (v_i \otimes v'_j) \otimes (w_i \otimes w'_j).
\]

Note that for the rank, it can make a difference whether we view \( t \otimes t' \) has a tensor in \( (U \otimes U') \otimes (V \otimes V') \otimes (W \otimes W') \) or \( U \otimes U' \otimes V \otimes V' \otimes W \otimes W' \). In the first case the number of inputs stays the same, we still compute bilinear forms. But the size of each input increases. In the second case, we would have five inputs, but their size stays the same. For complexity applications, we choose the first point of view.

Let \( u_1, \ldots, u_k \) be a basis of \( U \), \( v_1, \ldots, v_m \) of \( V \), and \( w_1, \ldots, w_k \) of \( W \). Let \( t_{h,i,j} \) be the coefficient of \( t \) of \( u_h \otimes v_i \otimes w_j \). In the same way, choose bases for the other three spaces and let \( t'_{h',i',j'} \) be the coefficient of \( t' \) of \( u_{h'} \otimes v'_{i'} \otimes w'_{j'} \). Then the coefficient of \( t \otimes t' \) of \( (u_h \otimes u'_{h'}) \otimes (v_i \otimes v'_{i'}) \otimes (w_j \otimes w'_{j'}) \) is \( t_{h,i,j} t'_{h',i',j'} \).

For the tensor product of matrix multiplications, we have

\[
<k,m,n> \otimes <k',m',n'> = \left( \delta_{\alpha,\bar{\alpha}} \delta_{\mu,\bar{\mu}} \delta_{\nu,\bar{\nu}} \delta_{\rho,\bar{\rho}} \delta_{\sigma,\bar{\sigma}} \right)
= \left( \delta_{\alpha,\bar{\alpha}} \delta_{\mu,\bar{\mu}} \delta_{\nu,\bar{\nu}} \delta_{\rho,\bar{\rho}} \delta_{\sigma,\bar{\sigma}} \right)
= <k,k',m,m'> \otimes <n,n'>
\]

\[
\langle k,m,n \rangle \otimes \langle k',m',n' \rangle = \left( \delta_{\alpha,\bar{\alpha}} \delta_{\mu,\bar{\mu}} \delta_{\nu,\bar{\nu}} \delta_{\rho,\bar{\rho}} \delta_{\sigma,\bar{\sigma}} \right)
= \left( \delta_{\alpha,\bar{\alpha}} \delta_{\mu,\bar{\mu}} \delta_{\nu,\bar{\nu}} \delta_{\rho,\bar{\rho}} \delta_{\sigma,\bar{\sigma}} \right)
= \langle k,k',m,m' \rangle \otimes \langle n,n' \rangle
\]
Thus, the tensor product of two matrix multiplication tensors is a bigger matrix multiplication tensor. This corresponds to the well known identity $(A \otimes B)(A' \otimes B') = (AA' \otimes BB')$ for the Kronecker product of matrices. (Note that we use quadruple indices to address the entries of the Kronecker products and also of the slices of $(k, m, n) \otimes (k', m', n').$ It follows that the inequality in Lemma 14.6.1 can be strict. We have $R(\langle 2, 2, 2 \rangle) = 7$, but there are faster ways to multiply matrices than Strassen’s algorithm.

Using this machinery, we can show that whenever we can multiply matrices of a fixed format efficiently, then we get good bounds for $\omega$.

14.6.2 Theorem. If $R(\langle k, m, n \rangle) \leq r$, then $\omega \leq 3 \cdot \log_{kmn} r$.

Proof. If $R(\langle k, m, n \rangle) \leq r$, then $R(\langle n, k, m \rangle) \leq r$ and $R(\langle m, n, k \rangle) \leq r$ by Lemma 14.5.2. Thus, by Lemma 14.6.1,

$$R(\langle k, m, n \rangle \otimes \langle n, k, m \rangle \otimes \langle m, n, k \rangle) \leq r^3$$

and, with $N = kmn$,

$$R(\langle N^i, N^i, N^i \rangle) \leq r^{3i} = (N^{3 \log_N r})^i = (N^i)^{3 \log_N r}$$

for all $i \geq 1$. Therefore, $\omega \leq 3 \log_N r$. \hfill \square

Thus, to get a fast matrix multiplication algorithm, it suffices to get a good upper bound on the rank of some fixed matrix multiplication tensor.

Tensor rank and the exponent of matrix multiplication

The exponent of matrix multiplication $\omega$ can be expressed equivalently using arithmetic circuits and tensor rank.

Tensor rank can be studied via the restrictions of the unit tensor.
Chapter 15

Border rank

15.1 Approximate computations

Over $\mathbb{R}$ or $\mathbb{C}$, the rank of matrices is semi-continuous. Let

$$C^{n \times n} \ni A_j \to A = \lim_{j \to \infty} A_j$$

If for all $j$, $\text{rk}(A_j) \leq r$, then $\text{rk}(A) \leq r$ as $\text{rk}(A_j) \leq r$ means all $(r+1) \times (r+1)$ minors vanish.

But since minors are continuous functions, all $(r+1) \times (r+1)$ minors of $A$ vanish, too.

The same is not true for 3-dimensional tensors. Consider the multiplication of univariate polynomials of degree one modulo $X^2$:

$$(a_0 + a_1 X)(b_0 + b_1 X) = a_0 b_0 + (a_1 b_0 + a_0 b_1)X + a_1 b_1 X^2$$

The tensor corresponding to the two bilinear forms $a_0 b_0$ and $a_1 b_0 + a_0 b_1$ consists of the two slices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It has rank 3: To show the lower bound, we use the substitution method. We first set $a_0 = 0$, $b_0 = 1$. Then we still compute $a_1$. Thus there is a product that depends on $a_1$, say one factor is $\alpha a_0 + \beta a_1$ with $\beta \neq 0$. When we replace $a_1$ by $-\frac{\alpha}{\beta} a_0$, we kill one product. We still compute $a_0 b_0$ and $-\frac{\alpha}{\beta} a_0 b_0 + a_0 b_1$. Next, set $a_0 = 1$, $b_0 = 0$. Then we still compute $b_1$. We can kill another product by substituting $b_1$ as above. After this, we still compute $a_0 b_0$, which needs one product.

However, we can approximate the tensor above by tensors of rank two. Let

$$t(\epsilon) = (1, \epsilon) \otimes (1, \epsilon) \otimes (0, \frac{1}{2}) + (1, 0) \otimes (1, 0) \otimes (1, -\frac{1}{2})$$

t(\epsilon) obviously has rank two for every $\epsilon > 0$. The slices of $t(\epsilon)$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & \epsilon \end{pmatrix}$$

Thus $t(\epsilon) \to t$ if $\epsilon \to 0$.

Bini, Capovani, Lotti and Romani [BCLR79] used this effect to design better matrix multiplication algorithms. They started with the following partial matrix multiplication tensor that we denote by $\{z_{11}, z_{12}, z_{21}\}$:

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & * \end{pmatrix}$$
where we only want to compute three entries of the result. It can be shown using the substitution method that \( R(\{z_{11}, z_{12}, z_{21}\}) = 6 \), but we can approximate \( \{z_{11}, z_{12}, z_{21}\} \) with only five products. Consider the following five products:

\[
\begin{align*}
p_1 &= (x_{12} + \epsilon x_{22})y_{21}, \\
p_2 &= x_{11}(y_{11} + \epsilon y_{12}), \\
p_3 &= x_{12}(y_{11} + y_{21} + \epsilon y_{22}), \\
p_4 &= (x_{11} + x_{12} + \epsilon x_{21})y_{11}, \\
p_5 &= (x_{12} + \epsilon x_{21})(y_{11} + \epsilon y_{22}).
\end{align*}
\]

We have

\[
\begin{align*}
\epsilon z_{11} &= \epsilon p_1 + \epsilon p_2 + O(\epsilon^2), \\
\epsilon z_{12} &= p_2 - p_4 + p_5 + O(\epsilon^2), \\
\epsilon z_{21} &= p_1 - p_3 + p_5 + O(\epsilon^2).
\end{align*}
\]

Here, \( O(\epsilon^i) \) collects terms of degree \( i \) or higher in \( \epsilon \). Now we take a second copy of the partial matrix multiplication above, with new variables. With these two copies, we can multiply \( 2 \times 2 \)-matrices with \( 2 \times 3 \)-matrices (by identifying some of the variables in the copy). So we can approximate \( (2, 2, 3) \) with 10 multiplications. If approximation would be as good as exact computation, then we would get \( \omega \leq 2.78 \) out of this, an improvement over Strassen’s algorithm.

We will formalize the concept of approximation. Let \( K \) be a field and \( K[[\epsilon]] = K \). The role of the small quantity \( \epsilon \) in the beginning of this chapter is now taken by the indeterminate \( \epsilon \).

15.1.1 Definition. Let \( h \in \mathbb{N}, t \in \mathbb{F}^k \otimes \mathbb{F}^m \otimes \mathbb{F}^n \).

1. \( R_h(t) = \min \{ r \mid \exists u_p \in \mathbb{F}[\epsilon]^h, v_p \in \mathbb{F}[\epsilon]^m, w_p \in \mathbb{F}[\epsilon]^n : \sum_{\rho=1}^r u_p \otimes v_p \otimes w_p = \epsilon^ht + O(\epsilon^{h+1}) \} \).

2. \( \mathbb{R}(t) = \min_h R_h(t). \mathbb{R}(t) \) is called the border rank of \( t \).

15.1.2 Remark. 1. \( R_0(t) = R(t) \).

2. \( R_0(t) \geq R_1(t) \geq ... = \mathbb{R}(t) \).

3. For \( R_h(t) \) it is sufficient to consider powers up to \( \epsilon^h \) in \( u_p, v_p, w_p \).

Above, we have used an algebraic definition of border rank. There is an equivalent geometric definition (see the end of this chapter), but the proof of equivalence is beyond the scope of this lecture.

15.1.3 Theorem (Alder). Let \( U, V, \) and \( W \) be vector spaces over an algebraically closed field. The set of all tensors \( t \in U \otimes V \otimes W \) with \( \mathbb{R}(t) \leq r \) is the closure of the set of all tensors in \( s \in U \otimes V \otimes W \) with \( R(s) \leq r \).

15.2 Properties of border rank

15.2.1 Theorem. Let \( t \in \mathbb{F}^k \otimes \mathbb{F}^m \otimes \mathbb{F}^n, t' \in \mathbb{F}^{k'} \otimes \mathbb{F}^{m'} \otimes \mathbb{F}^{n'} \). We have

1. \( \forall \pi \in \mathfrak{S}_3 : R_h(\pi t) = R_h(t) \).

2. \( R_{h+h'}(t \otimes t') \leq R_h(t) \cdot R_{h'}(t') \).

2. Let \( t = (t_{i,j,l}) \) and \( t' = (t'_{i',j',l'}) \). We have \( t \otimes t' = (t_{i,j,l} \cdot t'_{i',j',l'}) \in \mathbb{F}^{kk'} \otimes \mathbb{F}^{mm'} \otimes \mathbb{F}^{nn'} \). Take two approximate computations for \( t \) and \( t' \) as above. Viewed as exact computations over \( \mathbb{F}[[\epsilon]] \), their tensor product computes over the following:

\[
T = \epsilon^h t + \epsilon^{h+1} s \quad T' = \epsilon^h t' + \epsilon^{h+1} s'
\]

with \( s \in \mathbb{F}[\epsilon]^k \otimes \mathbb{F}[\epsilon]^m \otimes \mathbb{F}[\epsilon]^n \) and \( s' \in \mathbb{F}[\epsilon]^{k'} \otimes \mathbb{F}[\epsilon]^{m'} \otimes \mathbb{F}[\epsilon]^{n'} \). The tensor product of these two computations computes:

\[
T \otimes T' = (\epsilon^h t_{ijl} + \epsilon^{h+1} s_{ijl})(\epsilon^h t'_{i'j'l'} + \epsilon^{h+1} s'_{i'j'l'}) = \epsilon^{h+h'} t \otimes t' + O(\epsilon^{h+h'+1})
\]

But this is an approximate computation for \( t \otimes t' \).

\( \square \)

15.3 From approximate to exact computations

The next lemma shows that we can turn approximate computations for matrix multiplication into exact ones. So for matrix multiplication, border rank is the right measure.

15.3.1 Lemma. There is a constant \( c_h \) such that for all \( t \): \( R(t) \leq c_h R_h(t) \). \( c_h \) depends polynomially on \( h \), in particular \( c_h \leq \left( \frac{h+2}{2} \right) \).

15.3.2 Remark. Over infinite fields, even \( c_h = 1 + 2h \) works.

Proof. Let \( t \) be a tensor with border rank \( r \) and let

\[
\sum_{\rho=1}^r \left( \sum_{\alpha=0}^h \epsilon^\alpha u_{\rho\alpha} \right) \otimes \left( \sum_{\beta=0}^h \epsilon^\beta v_{\rho\beta} \right) \otimes \left( \sum_{\gamma=0}^h \epsilon^\gamma w_{\rho\gamma} \right) = \epsilon^h t + O(\epsilon^{h+1})
\]

The left-hand side of the equation can be rewritten as follows:

\[
\sum_{\rho=1}^r \sum_{\alpha=0}^h \sum_{\beta=0}^h \sum_{\gamma=0}^h \epsilon^{\alpha+\beta+\gamma} u_{\rho\alpha} \otimes v_{\rho\beta} \otimes w_{\rho\gamma}
\]

By comparing the coefficients of \( \epsilon \) powers, we see that \( t \) is the sum of all \( u_{\rho\alpha} \otimes v_{\rho\beta} \otimes w_{\rho\gamma} \) with \( \alpha + \beta + \gamma = h \). Thus to compute \( t \) exactly, it is sufficient to compute \( \left( \frac{h+2}{2} \right) \) products for each product in the approximate computation. \( \square \)

The following theorem is the border rank version of Theorem 14.6.2.

15.3.3 Theorem. If \( R((k,m,n)) \leq r \) then \( \omega \leq 3 \log_{kmn} r \).

Proof. Let \( N = kmn \) and let \( R_h((k,m,n)) \leq r \). By Theorem 15.2.1, we get \( R_{3h}((N,N,N)) \leq r^3 \) and \( R_{3hs}((N^s,N^s,N^s)) \leq r^{3s} \) for all \( s \). By Lemma 15.3.1, this yields \( R((N^s,N^s,N^s)) \leq c_{3hs} r^{3s} \). Therefore,

\[
\omega \leq \log_{N^s}(c_{3hs}) \leq 3s \log_{N^s}(r) + \log_{N^s}(c_{3hs}) = 3 \log_{N}(r) + \frac{1}{s} \log_{N}(\text{poly}(s)) \to 0
\]

for \( s \to \infty \). Since \( \omega \) is an infimum, we get \( \omega \leq 3 \log_{N}(r) \). \( \square \)

15.3.4 Corollary. \( \omega \leq 2.78 \).

Proof. Combine Theorem 15.3.3 with \( R((2,2,3)) \leq 10 \). \( \square \)
15.4 Degeneration

Degenerations relate to border rank like restrictions relate to rank. Again, we only give an algebraic definition of degenerations and we simply state an equivalent topological definition. Furthermore, we choose coordinate right from the beginning, since it simplifies the notations somewhat.

15.4.1 Definition. Let \( t \in \mathbb{F}^k \otimes \mathbb{F}^m \otimes \mathbb{F}^n \), \( t' \in \mathbb{F}^{k'} \otimes \mathbb{F}^{m'} \otimes \mathbb{F}^{n'} \).

1. Let \( t' = \sum_{\rho=1}^r u_\rho \otimes v_\rho \otimes w_\rho \) as well as \( A(\epsilon) \in \mathbb{F}[\epsilon]^{k \times k'} \), \( B(\epsilon) \in \mathbb{F}[\epsilon]^{m \times m'} \), and \( C(\epsilon) \in \mathbb{F}[\epsilon]^{n \times n'} \).

Define \( (A(\epsilon) \otimes B(\epsilon) \otimes C(\epsilon))t' = \sum_{\rho=1}^r A(\epsilon)u_\rho \otimes B(\epsilon)v_\rho \otimes C(\epsilon)w_\rho. \)

(This is well-defined.)

2. \( t \) is a degeneration of \( t' \) if there are \( A(\epsilon) \in \mathbb{F}[\epsilon]^{k \times k'} \), \( B(\epsilon) \in \mathbb{F}[\epsilon]^{m \times m'} \), and \( C(\epsilon) \in \mathbb{F}[\epsilon]^{n \times n'} \), and \( q \in \mathbb{N} \) such that \( \epsilon^q t = (A(\epsilon) \otimes B(\epsilon) \otimes C(\epsilon))t' + O(\epsilon^{q+1}). \)

We will write \( t \preceq_q t' \) or simply \( t \preceq t' \).

As for the rank, it is very easy to prove the following lemma.

15.4.2 Lemma. Let \( s \) and \( t \) be tensors.

1. \( t \preceq s \Rightarrow R(t) \leq R(s) \).
2. \( R(t) \leq r \Leftrightarrow t \preceq \langle r \rangle. \)

The proof of the following theorem is beyond the scope of this lecture.

15.4.3 Theorem (Strassen). Let \( \mathbb{F} \) be algebraically closed. Let \( U \), \( V \), and \( W \) be vector spaces over \( \mathbb{F} \). Let \( t \in U \otimes V \otimes W \).

\[ \{ s \in U \otimes V \otimes W \mid s \preceq t \} = \{ s \in U \otimes V \otimes W \mid s \preceq \langle r \rangle \}. \]

Let \( t \in V \otimes V \otimes V \) and \( \dim V = r \). \( R(t) \leq r \) is equivalent to \( t \preceq \langle r \rangle \). If \( t \) lives in a smaller space \( U \otimes U \otimes U \), we first embed it into \( V \otimes V \otimes V \) by choosing a injective linear map \( U \rightarrow V \). By the above theorem, \( t \preceq \langle r \rangle \) is equivalent to \( t \in \overline{\text{End}(V)}^{\times 3}(\langle r \rangle) \), so again, we have a (monoid) orbit closure problem. Since \( \text{GL}(V) \) lies dense in \( \text{End}(V) \), we can even just look at group orbits:

\[ t \in \overline{\text{GL}(V)}^{\times 3}(\langle r \rangle). \]

Border rank: a geometric complexity measure

In the same way as defining the border determinantal complexity via the determinantal complexity, we define the border rank via the tensor rank.

As for the border determinantal complexity, border rank lower bounds can be defined via an orbit closure question.
Chapter 16
Symmetric and alternating tensors

Let $V$ be a vector space. $S_d$ acts on $V^\otimes d$ by $(\pi, t) = v_1 \otimes \cdots \otimes v_d \mapsto \pi t = v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(d)}$ and linear extension to higher rank tensors. (In the previous chapters, we defined this for $d = 3$.)

16.1 $S^2 V$ and $\Lambda^2 V$

We start with the simplest examples. Let $v_1, \ldots, v_n$ be a basis of $V$. The space $S^2 V$ is defined as

$$S^2 V = \langle v_i \otimes v_j + v_j \otimes v_i \mid 1 \leq i, j \leq n \rangle.$$ 

We call it the space of symmetric 2-tensors of $V$.

16.1.1 Proposition. 1. $S^2 V = \langle v \otimes v \mid v \in V \rangle$.

2. For $t \in V \otimes V$, $t \in S^2 V$ iff $(1, 2)t = t$.

Proof. We start with the first item. We have

$$(v_i + v_j) \otimes (v_i + v_j) - (v_i - v_j) \otimes (v_i - v_j) = 2(v_i \otimes v_j + v_j \otimes v_i).$$

Therefore, the left-hand side is contained in the right-hand side. On the other hand, if $v = \alpha_1 v_1 + \cdots + \alpha_d v_d$, then

$$v \otimes v = \sum_{i<j} \alpha_i \alpha_j (v_i \otimes v_j + v_j \otimes v_i) + \sum_i \alpha_i^2 (v_i \otimes v_i).$$

Thus, the right-hand side is also contained in the left-hand side.

For the second item, notice that every tensor $t \in S^2 V$ fulfills $(1, 2)t = t$, since the basis does. For the other direction, let $t = \sum_{i,j} \alpha_{i,j} v_i \otimes v_j$. If $(1, 2)t = t$, then $\alpha_{i,j} = \alpha_{j,i}$ for all $i, j$. Therefore, $t \in S^2 V$.

The skew-symmetric 2-tensors of $V$ are defined as

$$\Lambda^2 V = \langle v_i \otimes v_j - v_j \otimes v_i \mid 1 \leq i, j \leq n \rangle.$$ 

16.1.2 Exercise. Prove the following:

1. $\Lambda^2 V = \langle v \otimes w - w \otimes v \mid v, w \in V \rangle$.

2. For all $t \in V^\otimes 2$, $(1, 2)t = -t$. 

85
16.2 Symmetric tensors

Let \( \pi_S : V^{\otimes d} \to V^{\otimes d} \) be the map that is defined on rank-one tensors by

\[
\pi_S(v_1 \otimes \cdots \otimes v_d) = \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}.
\]

16.2.1 Definition. The \( d \)th symmetric power of \( V \) is defined as \( S^dV := \pi_S(V^{\otimes d}) \).

Note that this generalises the definition of \( S^2V \) in the previous section. We write \( v_1 v_2 \ldots v_d := \pi_S(v_1 \otimes v_2 \otimes \cdots \otimes v_d) \).

16.2.2 Proposition. For all \( t \in V^{\otimes d} \), \( \pi_S(\pi_S(t)) = \pi_S(t) \), that is, \( \pi_S \) is a projection.

Proof. We have

\[
\pi_S(\pi_S(v_1 \otimes \cdots \otimes v_d)) = \frac{1}{d!} \sum_{\tau \in \mathfrak{S}_d} \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} v_{\tau(\sigma(1))} \otimes \cdots \otimes v_{\tau(\sigma(d))}.
\]

Since \( \tau \) is a bijection on \( \mathfrak{S}_d \), all \( d! \) inner sums are the same.

16.2.3 Proposition. If \( w_1, \ldots, w_n \) is a basis of \( V \), then \( (w_{j_1} \cdots w_{j_d})_{1 \leq j_1 \leq \cdots \leq j_d \leq n} \) is a basis of \( S^dV \).

Proof. If \( \{j_1, \ldots, j_d\} = \{i_1, \ldots, i_d\} \), then

\[
\pi_S(w_{j_1} \cdots \otimes w_{j_d}) = \pi_S(w_{i_1} \cdots \otimes w_{i_d}).
\]

On the other hand, if \( \{j_1, \ldots, j_d\} \neq \{i_1, \ldots, i_d\} \), then the terms appearing in the sums \( \pi_S(w_{j_1} \otimes \cdots \otimes w_{j_d}) \) and \( \pi_S(w_{i_1} \otimes \cdots \otimes w_{i_d}) \) are all distinct. Therefore, any linear dependency between \( w_{j_1}, \ldots, w_{j_d} \), \( 1 \leq j_1 \leq \cdots \leq j_d \leq n \) would translate into a linear dependency between \( w_{j_1} \otimes \cdots \otimes w_{j_d} \), \( 1 \leq j_1, \ldots, j_d \leq n \).

\[
16.2.4 \text{Corollary.} \quad \dim S^dV = \binom{n+d-1}{d}
\]

16.3 Alternating tensors

Let \( \pi_A : V^{\otimes d} \to V^{\otimes d} \) be the map that is defined on rank-one tensors by

\[
\pi_A(v_1 \otimes \cdots \otimes v_d) = \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \text{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}.
\]

16.3.1 Definition. The \( d \)th alternating power of \( V \) is defined as \( \Lambda^dV := \pi_A(V^{\otimes d}) \).

16.3.2 Exercise. \( \pi_A \) is a projection.
CHAPTER 16. SYMMETRIC AND ALTERNATING TENSORS

Again, this generalises the space \( \Lambda^2 V \) of the first section. We write \( v_1 \wedge v_2 \wedge \cdots \wedge v_d := \pi_V(v_1 \otimes v_2 \otimes \cdots \otimes v_d) \).

16.3.3 Proposition. \( v_{\tau(1)} \wedge \cdots \wedge v_{\tau(d)} = \text{sgn}(\tau)v_1 \wedge \cdots \wedge v_d \).

Proof. We have:

\[
v_{\tau(1)} \wedge \cdots \wedge v_{\tau(d)} = \frac{1}{d!} \sum_{\sigma \in S_d} \text{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}
\]

\[
= \frac{1}{d!} \sum_{\sigma \in S_d} \text{sgn}(\sigma \circ \tau^{-1}) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}
\]

\[
= \text{sgn}(\tau) \frac{1}{d!} \sum_{\sigma \in S_d} \text{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}
\]

\[
= \text{sgn}(\tau)v_1 \wedge \cdots \wedge v_d.
\]

The third line follows from the fact that \( \sigma \mapsto \sigma \circ \tau^{-1} \) is a bijection and the last line from the fact that \( \text{sgn}(\tau) = \text{sgn}(\tau^{-1}) \).

16.3.4 Proposition. We have \( v = v_1 \wedge \cdots \wedge v_d = 0 \) if and only if \( v_1, \ldots, v_d \) are linearly dependent.

Proof. If two of the vectors are the same, say \( v_1 = v_2 \), then we can group the summands in \( \pi_V(v) \) into pairs such that the two summands in the pair cancel. (Namely, if we switch the two identical vectors, we get the same tensor product but with opposite sign.)

In the general case, we can w.l.o.g. write \( v_1 = \alpha_2 v_2 + \cdots + \alpha_d v_d \). Now by using linearity, we get a sum of tensors, each of which has two identical vectors.

For the other direction, assume that \( v_1, \ldots, v_d \) are linearly independent. Enlarge the set \( \{v_1, \ldots, v_d\} \) to a basis of \( V \). Then in

\[
v_1 \wedge \cdots \wedge v_d = \frac{1}{d!} \sum_{\sigma \in S_d} \text{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)},
\]

all \( v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)} \) are distinct basis vectors of \( V \otimes d \), hence \( v_1 \wedge \cdots \wedge v_d \) cannot vanish.

16.3.5 Proposition. If \( w_1, \ldots, w_n \) is a basis of \( V \), then \( w_{j_1} \wedge \cdots \wedge w_{j_d}, 1 \leq j_1 < \cdots < j_d \leq n \), is a basis of \( \Lambda^d V \).

Proof. Given any tensor \( v_1 \wedge \cdots \wedge v_d \), we can express each \( v_i \) as a linear combination of the basis vectors. Using linearity, we get a sum of alternating products of the basis vectors. Whenever two of the basis vectors are the same, the product vanishes by the previous proposition. Each product with pairwise distinct basis vectors can be brought into the form of the statement by permuting the vectors.

The vectors in the statement are obviously independent.

16.3.6 Corollary. \( \dim \Lambda^d V = \binom{n}{d} \).

Now let \( d = n = \dim V \). Then \( \Lambda^n V \cong \mathbb{C} \). \( \text{GL}(V) \) acts on \( \Lambda^n V \) by

\[
g(v_1 \wedge \cdots \wedge v_n) = gv_1 \wedge \cdots \wedge gv_n.
\]

Let \( gv_i = \sum_{j=1}^{n} \gamma_{j,i} v_j \). Then

\[
g(v_1 \wedge \cdots \wedge v_n) = \left( \sum_{j=1}^{n} \gamma_{j,1} v_j \right) \wedge \cdots \wedge \left( \sum_{j=1}^{n} \gamma_{j,n} v_j \right)
\]

\[
= \sum_{j_1, \ldots, j_n} \gamma_{j_1,1} \cdots \gamma_{j_n,n} v_{j_1} \wedge \cdots \wedge v_{j_n}.
\]
In the last sum, only summands with pairwise distinct indices $j_1, \ldots, j_n$ are non-zero. Let $\sigma$ be the permutation such that $\sigma(h) = j_h$ for all $h$. By Proposition 16.3.3, we have $v_{j_1} \wedge \cdots \wedge v_{j_n} = \text{sgn}(\sigma)v_1 \wedge \cdots \wedge v_n$. Thus

$$g(v_1 \wedge \cdots \wedge v_n) = \det(g)v_1 \wedge \cdots \wedge v_n.$$ 

This is called the alternating representation.

### Symmetric and alternating tensors

Two types of tensors of high importance are the symmetric and alternating tensors. They are defined by symmetrization and skew-symmetrization of tensors, respectively. These important tensors will serve as building blocks in the representation theory of $\text{GL}_n$. 


Chapter 17

The construction of the irreducible representations of the general linear group

We wish to understand much better the representation theory of coordinate rings of orbit closures. A first step into the right direction is to understand the building blocks: The irreducible representations of $\text{GL}_n$. They are completely understood in terms of combinatorial objects called Young tableaux. We follow the exposition in [Ful97, Ch. 8] very closely.

17.1 Young tableaux

A Young diagram is a left-justified top-aligned array of boxes. To each partition $\lambda$ we assign its Young diagram by interpreting $\lambda_i$ as the number of boxes in row $i$. For example the Young diagram to the partition $(5, 3, 3, 1)$ is

```
  +-----+-----+-----+-----+-----+-----+
  |     |     |     |     |     |     |
  |     |     |     |     |     |     |
  |     |     |     |     |     |     |
  |     |     |     |     |     |     |
  +-----+-----+-----+-----+-----+-----+
```

We often identify partitions with their Young diagrams. The number of boxes in a Young diagram shall be denoted by $|\lambda| := \sum \lambda_i$.

If we fill the boxes of a Young diagram with numbers, we obtain a so-called Young tableau. For example,

```
  6 2 4 4 8
  2 1 6
  4 3 1
  3
```

is a Young tableau. The partition corresponding to its Young diagram is called the shape of the Young diagram.

To simplify the notation, we define $\mu_i$ to be the number of boxes of the $i$-th column of $\lambda$. We call $\mu = (\mu_1, \mu_2, \ldots)$ the transpose of $\lambda$. The Young diagram of $\mu$ is obtained by transposing the Young diagram of $\lambda$.

We will need the notion of an exchange. This depends on a choice of two columns and a choice of $k$ boxes in each column. For a Young tableau $T$ of shape $\lambda$ (with entries in any set) the corresponding exchange is the Young tableau $S$ obtained from $T$ by interchanging the entries
in the two chosen sets of boxes, maintaining the vertical order in these; the entries outside these boxes are unchanged.

For example, if \( \lambda = (4,3,3,2) \) and the chosen boxes are the top two in the third column and the second and forth in the second column, then the exchange takes

\[
T = \begin{array}{cccc}
1 & 5 & 2 & 1 \\
1 & 3 & 4 & \\
2 & 4 & 5 & \\
3 & 5 & & 
\end{array}
\quad \text{to} \quad S = \begin{array}{cccc}
1 & 5 & 3 & 1 \\
1 & 2 & 5 & \\
2 & 4 & 5 & \\
3 & 4 & & 
\end{array}.
\]

Sometimes we fix two columns and fix a subset of boxes in the right chosen column. The set of all corresponding exchanges are defined to have the same exchange type, i.e., an exchange type is a pair of columns together with a set of boxes from the right column.

17.2 Construction as a quotient space

Let \( E = \mathbb{C}^n \) with the standard action of \( \text{GL}_n \). All linear maps in the following constructions are equivariant, which defines the action of \( \text{GL}_n \) on the target space.

We write \( E^{\times \lambda} = E \oplus E \oplus \cdots \oplus E \) and we associate each summand \( E \) with a position in the Young diagram of \( \lambda \). In particular, if we write vectors in the boxes of \( \lambda \), then we obtain an element of \( E^{\times \lambda} \) and every element of \( E^{\times \lambda} \) is obtained in this way.

For a vector space \( E^{\times k} \) we define the linear map to \( E \otimes E \otimes \cdots \otimes E \) via \((\ell_1, \ldots, \ell_k) \mapsto \ell_1 \otimes \cdots \otimes \ell_k \). We can compose this with an antisymmetrization map and obtain a linear map \( E^{\times k} \to E^{\wedge k} \). We can tensor several of these maps to obtain the map \( \psi : E^{\times \lambda} \to \otimes_{i=1}^{\lambda_1} \wedge^{\mu_i} E \). Now

\[
E^\lambda := (\wedge^{\mu_1} E \otimes \cdots \otimes \wedge^{\mu_s} E)/Q^\lambda(E),
\]

where \( Q^\lambda(E) \) is the subrepresentation of \( \otimes_{i=1}^{\lambda_1} \wedge^{\mu_i} E \) generated by all elements of the form \( \psi(\vec{v}) - \sum \psi(\vec{w}) \), where for some fixed exchange type \( t \) the sum is over all \( \vec{w} \) obtained from \( \vec{v} \) by an exchange of type \( t \). \( E^\lambda \) is called a Schur module.

Suppose we have an ordered basis \( \{e_1, \ldots, e_m\} \) of \( E \). Then for any Young tableau of \( T \) of shape \( \lambda \) with entries in \( \{1, \ldots, m\} \) we get an element of \( E^{\times \lambda} \) by replacing every \( i \) in a box of \( T \) by the element \( e_i \). We call this element \( \hat{e}_T \). The image of this element in \( E^\lambda \) is denoted by \( e_T \).

One can now easily verify that the map \( \varphi : E^{\times \lambda} \to E^\lambda \) has the following three properties:

1. \( \varphi \) is multilinear
2. \( \varphi \) is alternating in the entries of any column of \( \lambda \)
3. For any \( \vec{v} \in E^{\times \lambda} \) and any exchange type \( t \) we have \( \varphi(\vec{v}) = \sum \varphi(\vec{w}) \), where the sum if over all \( \vec{w} \) obtained from \( \vec{v} \) by an exchange of the type \( t \).

17.3 A more explicit quotient space

17.3.1 Lemma. If \( e_1, \ldots, e_m \) is a basis of \( E \), then \( E^\lambda \simeq F/Q \), where \( F \) is the vector space whose basis is the set \( \hat{e}_T \) for all Young tableaux \( T \) of shape \( \lambda \) with entries from \( \{1, \ldots, m\} \) and \( Q \subseteq F \) is generated by the elements

(i) \( \hat{e}_T \) if \( T \) has two equal entries in a column,
(ii) \( \hat{e}_T + \hat{e}_{T'} \), where \( T' \) is obtained from \( T \) by interchanging two entries in a column,
(iii) \( \hat{e}_T - \sum S \hat{e}_S \), where for some exchange type \( t \) the sum is over all \( S \) obtained from \( T \) by an exchange of type \( t \).
Proof. For every Young tableau $T$ of shape $\lambda$ we get an element in $E^{\times \lambda}$ and these elements generate $E^{\times \lambda}$. Therefore their images $e_T$ generate $E^\lambda$, i.e., the map $F \to E^\lambda$ is surjective. Properties (2) and (3) imply that the generators of $Q$ map to zero, so $F/Q \to E^\lambda$ is surjective. We now routinely check that this is an isomorphism as follows. The vectors $\hat{e}$ and (3) imply that the generators of $Q$ then generate the vector space of relations (and the $\hat{e}$ that the $E$ that the fixed set of $\lambda$ columns of $N$ map to zero, so $F/Q \to E^\lambda$ is surjective. The lemma therefore follows from (17.2.1).

17.4 Sylvester’s lemma

A multilinear function $f : V^{\times d} \to \mathbb{C}$ is called alternating if $f(v_1, \ldots, v_i, v_{i+1}, \ldots, v_d) = -f(v_1, \ldots, v_i, v_{i+2}, \ldots, v_d)$.

17.4.1 Lemma. A multilinear function $f : V^{\times d} \to \mathbb{C}$ is alternating iff $f(v_1, \ldots, v_i, v_{i+1}, \ldots, v_d) = 0$ whenever $v_i = v_{i+1}$.

Proof. Clearly, if $f$ is alternating, then

$$f(v_1, \ldots, v_i, v_i, v_{i+2}, \ldots, v_d) = -f(v_1, \ldots, v_i, v_{i+2}, \ldots, v_d)$$

and thus $f(v_1, \ldots, v_i, v_i, v_{i+2}, \ldots, v_d) = 0$ (because $\operatorname{char}(\mathbb{C}) \neq 2$).

For the other direction,

$$f(v_1, \ldots, v_i, v_{i+1}, v_{i+2}, \ldots, v_d)$$

$$= f(v_1, \ldots, v_i, v_i, v_{i+2}, \ldots, v_d) + f(v_1, \ldots, v_i, v_{i+1} - v_i, v_{i+2}, \ldots, v_d)$$

$$= f(v_1, \ldots, v_i, v_i - v_i, v_{i+1} - v_i, v_{i+2}, \ldots, v_d) + f(v_1, \ldots, v_i, v_{i+1} - v_i, v_{i+2}, \ldots, v_d)$$

$$= -(f(v_1, \ldots, v_{i+1} - v_i, v_{i+1} - v_i, v_{i+2}, \ldots, v_d) + f(v_1, \ldots, v_{i+1} - v_i, v_{i+1} - v_i, v_{i+2}, \ldots, v_d))$$

$$= -(f(v_1, \ldots, v_i, v_i, v_{i+2}, \ldots, v_d))$$

$$= f(v_1, \ldots, v_i, v_i, v_{i+2}, \ldots, v_d).$$

17.4.2 Corollary. For $V = \mathbb{C}^p$ the only alternating multilinear function $V^{p+1} \to \mathbb{C}$ is the zero function.

Proof. If $f$ is alternating, to calculate $f(v)$ we express $v \in V^{p+1}$ over the standard basis. By the pigeonhole principle at least one standard vector appears at least twice. By Lemma 17.4.1 $f(v) = 0$.

For the explicit construction of the irreducibles we need the following lemma, proved by Sylvester in 1851.

17.4.3 Lemma. For any $p \times p$ matrices $M$ and $N$, and $1 \leq k \leq p$,

$$\det(M) \cdot \det(N) = \sum \det(M') \cdot \det(N'),$$

where the sum is over all pairs $(M', N')$ of matrices obtained from $M$ and $N$ by interchanging a fixed set of $k$ columns of $N$ with any $k$ columns of $M$, preserving the ordering of the columns.
Proof. By the alternating property of determinants, w.l.o.g. the fixed set of columns of $N$ are the first $k$ columns. For vectors $v_1,\ldots,v_p \in \mathbb{C}^p$ we write $\det(v_1 \cdots v_p)$ for the determinant of the matrix with these column vectors. We have to prove
\[
\det(v_1 \cdots v_p)\det(w_1 \cdots w_p) = \sum_{i_1 < \cdots < i_k} \det(v_1 \cdots w_1 \cdots w_k \cdots v_p)\det(v_{i_1} \cdots v_{i_k} w_{k+1} \cdots w_p),
\]
where in the sum the vectors $w_1,\ldots,w_k$ are interchanged with the vectors $v_1,\ldots,v_k$. It suffices to show that the difference of the two sides is an alternating function in the $p+1$ vectors $v_1,\ldots,v_p, w_1$, since any such function must vanish (see Cor. 17.4.2). For this it suffices to show (see Lemma 17.4.1) that the two sides are equal when two successive vectors $v_i$ and $v_{i+1}$ are equal (which is easy to see: The left hand side is zero and we can pair the nonzero summands on the right hand side such that each pair cancels out) and when $v_p = w_1$. In the latter case, fixing $v_p = w_1$, it suffices to show that the difference of the two sides is an alternating function of $v_1,\ldots,v_p, w_2$. Again, the case when $v_1 = v_{i+1}$ is immediate. This time $v_p = w_2$ means $w_1 = w_2$ and thus both sides vanish. \qed

Let $Z_{i,j}$ be variables, $1 \leq i \leq n$, $1 \leq j \leq m$. We write $\mathbb{C}[Z] := \mathbb{C}[Z_{1,1},\ldots,Z_{n,m}]$.

For each $p$-tuple $(i_1,\ldots,i_p)$ of integers from $\{1,\ldots,m\}$, with $p \leq n$, we define the symbolic determinant
\[
D_{i_1,i_2,\ldots,i_p} := \det \begin{pmatrix}
Z_{1,i_1} & \cdots & Z_{1,i_p} \\
\vdots & \ddots & \vdots \\
Z_{p,i_1} & \cdots & Z_{p,i_p}
\end{pmatrix}.
\]

For a Young tableau $T$ we take the product of the column determinants:
\[
D_T := \prod_{j=1}^{\lambda_1} D_{T(1,j),T(2,j),\ldots,T(\mu_j,j)},
\]
where $T(i,j)$ is the entry of $T$ in the $i$-th row and $j$-th column of $T$.

17.4.4 Lemma. There is a well-defined (canonical) homomorphism from $E^\lambda$ to $\mathbb{C}[Z]$ that maps $e_T$ to $D_T$ for all Young tableaux $T$.

Proof. Using Lemma 17.3.1, for well-definedness it suffices to show that the elements $D_T$ satisfy the corresponding properties (i)-(iii) of Lemma 17.3.1. Properties (i) and (ii) follows from the alternating property of determinants. Property (iii) follows from Sylvester’s lemma 17.4.3, applied to appropriate matrices. For this, suppose the two columns of $T$ in which the exchange takes place have entries $t_1,\ldots,t_p$ in the first and $j_1,\ldots,j_q$ in the second. Set
\[
M := \begin{pmatrix}
Z_{1,i_1} & \cdots & Z_{1,i_p} \\
\vdots & \ddots & \vdots \\
Z_{p,i_1} & \cdots & Z_{p,i_p}
\end{pmatrix}, \quad N := \begin{pmatrix}
Z_{1,j_1} & \cdots & Z_{1,j_q} & 0 \\
\vdots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \vdots & \Id_{p-q}
\end{pmatrix}
\]

Here the matrix $N$ has a lower right identity matrix of size $p-q$, and an upper right $q \times (p-q)$ block of zeros. Note that $\det N = \det \begin{pmatrix}
Z_{1,j_1} & \cdots & Z_{1,j_q} \\
\vdots & \ddots & \vdots \\
Z_{q,j_1} & \cdots & Z_{q,j_q}
\end{pmatrix}$. Sylvester’s lemma, applied to $M$ and $N$ and the fixed subset of columns in $N$ being specified by the subset of the right column of $T$ used in the exchange, translates precisely to the required equation. \qed
17.5 An explicit basis of the Schur module

17.5.1 Definition. A Young tableau is called semistandard if each row read from left to right is nondecreasing and each column read from top to bottom is strictly increasing.

Example of a semistandard Young tableau:

```
1 1 1 1 2 2 2 3
2 2 2 3 4
3 3 4 4
```

17.5.2 Theorem. If $e_1, \ldots, e_m$ is a basis of $E$, then $e_T$ is a basis of $E^\lambda$, where $T$ runs over all semistandard tableaux of $\lambda$ with entries from $\{1, \ldots, m\}$.

Proof. On the set of Young tableaux we define an ordering:

- $T' > T$ if in the right-most column which is different, the lowest box where they differ has a larger entry in $T'$.

We first prove that the $e_T$ generate $E^\lambda$. We also use $E^\lambda = F/Q$ from Lemma 17.3.1. We must show that, given any $T$ that is not semistandard, we can write $e_T$ as a linear combination of elements $e_S$ with $S > T$ and elements in $Q$, because then we can use this process recursively to express every $e_T$ as a linear combination of $e_S$ with $S$ semistandard and elements in $Q$.

We may assume that the entries in each column of $T$ are strictly increasing by using relations (i) and (ii); Note that by making the the columns strictly increasing in $T$ replaces $T$ by $T'$ with $T' > T$.

If the columns are strictly increasing, but $T$ is not semistandard, then suppose the $k$-th entry of the $j$-th column is strictly larger than the $k$-th entry of the $(j+1)$-st column. Then we have a relation $e_T = \sum S e_S$, the sum over all $S$ obtained from $T$ by exchanging the top $k$ entries of the $(j+1)$-st column with $k$ entries in the $j$-th column (preserving their order). Since each such $S$ has $S > T$, we proved that the $e_T$ for semistandard $T$ generate $E^\lambda$.

To prove that the $e_T$ are linearly independent, we use Lemma 17.4.4, so it suffices to prove that the $D_T$ are linearly independent as $T$ varies over all semistandard tableaux $T$. For this we order the variables $Z_{i,j}$ in the order: $Z_{i,j} < Z_{i',j'}$ if $i < i'$ or both $i = i'$ and $j < j'$. We order the monomials in these variables lexicographically: $M_1 < M_2$ if the smallest $Z_{i,j}$ that occurs to a different power occurs to a smaller power in $M_1$ than in $M_2$. Note that if $M_1 < M_2$ and $N_1 < N_2$, then $M_1 N_1 < M_2 N_2$. It follows immediately from this definition that the smallest monomial that appears in a determinant $D_{i_1, \ldots, i_p}$ if $i_1 < \cdots < i_p$ is the diagonal term $Z_{1,i_1} \cdots Z_{p,i_p}$. Therefore the smallest monomial occurring in $D_T$, if $T$ has increasing columns, is $\prod (Z_{i,j})^{m_T(i,j)}$, where $m_T(i,j)$ is the number of times $j$ occurs in the $i$-th row of $T$. This monomial occurs with coefficient 1.

Now order the semistandard tableaux by saying that $T < T'$ if the first row where they differ, and the first entry where they differ in that row, is smaller in $T$ than in $T'$. Equivalently, $T < T'$ if for the smallest $i$ for which there is a $j$ with $m_T(i,j) \neq m_{T'}(i,j)$ and for the smallest such $j$, we have $m_T(i,j) < m_{T'}(i,j)$. It follows that if $T < T'$, then the smallest monomial occurring in $D_T$ is smaller than the smallest monomial occurring in $D_{T'}$ and thus smaller than any monomial occurring in $D_{T'}$. From this the linear independence follows: If $\sum \alpha_T D_T = 0$, take $T$ minimal such that $\alpha_T \neq 0$, then the coefficient of $\prod (Z_{i,j})^{m_T(i,j)}$ in $\sum \alpha_T D_T$ is $\alpha_T$.

17.5.3 Remark. The proof of Theorem 17.5.2 provides an algorithm to express any $e_T$ for a tableau $T$ over the basis $(e_T)$ with $T'$ semistandard. This algorithm is called the straightening algorithm. Together with the upcoming Lemma 17.6.1 this gives an algorithmic way of computing the action of $GL_n$ in $E^\lambda$. 
17.6 Highest weight vectors

17.6.1 Lemma. Fix an arbitrary ordering on the set of boxes of a Young diagram $\lambda$. Let $T$ be a Young tableau and let $j_1, \ldots, j_n$ be the entries in its boxes. We write $T = \lambda(j_1, \ldots, j_n)$. If $g \in \text{GL}_n$, then

$$ge_T = \sum_{(i_1, \ldots, i_n) \in \{1, \ldots, m\}^n} g_{i_1, j_1} \cdots g_{i_n, j_n} e_{\lambda(i_1, \ldots, i_n)}.$$

Proof. This is not only true for $E^{\lambda}$, but it is already true for $E^{\otimes |\lambda|}$. More precisely, for a Young tableau $T$ consider the vector in $E^{\times \lambda}$ obtained from replacing each $i$ in $T$ by $e_i$. Let $\tilde{e}_T$ denote its image under the map $E^{\times \lambda} \to E^{\otimes |\lambda|}$. The action on $E^{\otimes |\lambda|}$ is given by

$$g(v_1 \otimes \cdots \otimes v_{|\lambda|}) = (gv_1) \otimes \cdots \otimes (gv_{|\lambda|}).$$

Thus by multilinearity of the tensor product we have

$$g\tilde{e}_T = \sum_{(i_1, \ldots, i_n) \in \{m\}^n} g_{i_1, j_1} \cdots g_{i_n, j_n} \tilde{e}_T(i_1, \ldots, i_n),$$

The claim follows by (17.2.1).

Using part (a') and (a) of Theorem 11.3.6, the following lemma finishes the explicit construction of the polynomial irreducible representations of the general linear group and proves part (c) of Theorem 11.3.6, where the $E^\lambda$ are the required irreducible representations.

17.6.2 Lemma. Up to multiplication with a nonzero scalar, the only highest weight vector in $E^\lambda$ is the vector $e_T$, where $T$ is the semistandard tableau of shape $\lambda$ whose $i$-th row contains only the integer $i$ as entries.

Proof. Let $T$ be the semistandard tableau of shape $\lambda$ whose $i$-th row contains only the integer $i$ as entries. Let $g \in U_n$ be an upper triangular matrix with 1s on the main diagonal, i.e., $g_{i,j} = 0$ if $i > j$. From Lemma 17.6.1 it follows that the only nonzero $e_T$ that can occur in $ge_T$ is $e_T$ itself and therefore $e_T$ is a HWV.

Similarly suppose that the $p$-th row of $T$ is the first row that contains an element larger than $p$. Let $q > p$ be the smallest such misplaced element in row $p$. Define $g$ to be the elementary matrix that has 1s on the main diagonal and an entry $\alpha$ in row $p$ and column $q$. We see that $ge_T = \sum \alpha^{\epsilon_T} e_T$, where the sum is over all tableaux $T'$ by exchanging some set (possibly empty) of the $qs$ appearing in $T$ to $qs$, and $e_T$ is the number of such exchanges. Considering this as a univariate polynomial in $\alpha$ and looking at the linear coefficient, this is a sum over $T'$ in which a single $q$ is switched to a $p$. Some of these $T'$ could have $p$ appear in a column twice, but the other $T'$ are pairwise distinct semistandard tableaux and there is at least one of them. Thus $ge_T \neq e_T$ and hence $e_T$ is not a HWV.

The explicit construction of the irreducible representations of the general linear group

This chapter finalizes our classification of the polynomial irreducible representations of $\text{GL}_n$. The irreducibles are indexed by partitions $\lambda$ that have at most $n$ parts. We will use the explicit quotient space from Lemma 17.3.1 for several constructions in Chapters 19 and 20, in particular we will use it to construct the irreducibles of the symmetric group (the so-called Specht modules).
Chapter 18

The algebraic Peter-Weyl theorem

In this chapter we prove the algebraic Peter-Weyl theorem. It will be used in Chapter 19 to prove the Schur-Weyl duality that is used to understand the highest weight vectors in \( \mathbb{C}[\mathbb{GL}_n \otimes \operatorname{det}_n] \). Moreover, we will see that it can directly be used to find upper bounds on the multiplicities for example in \( \mathbb{C}[\mathbb{GL}_n \otimes \operatorname{det}_n] \) by using so-called Kronecker coefficients. We will see how these coefficients can then be used in the multiplicity based approach outlined in section 12.4.

18.1 Regular functions

Let \( G = \mathbb{GL}_N \) and \( v \in A \), where \( A \) has a polynomial \( G \)-action. The coordinate ring \( \mathbb{C}[Gv] \) is a subring of the algebra \( \mathbb{C}[Gv] \) that we define next. The ring \( \mathbb{C}[Gv] \) will have a particularly nice representation theory.

18.1.1 Definition. A regular function \( f \) on a locally closed set \( X \subseteq A \) is a function \( f : X \to \mathbb{C} \) defined on the whole of \( X \) as follows: There exist finitely many fractions of polynomials \( \frac{f_i}{g_i} \) with \( f_i, g_i \in \mathbb{C}[A] \) such that

\[
\forall x \in X \exists i : g_i(x) \neq 0
\]

and

\[
\forall x \in X \forall i : \text{either } g_i(x) = 0 \text{ or } \frac{f_i(x)}{g_i(x)} = f(x).
\]

For a locally closed set \( X \) define \( \mathbb{C}[X] \) to be the algebra of regular functions on \( X \).

The following example is called the glued double cusp and was provided by Prof. Dr. Eike Lau.

18.1.2 Example. Let the closed set \( Y \subseteq \mathbb{C}^5 \) be cut out by the polynomials \( T_1^3 - T_2^2 \), \( T_1^3 - T_2^2 \), \( T_1T_3 - T_2^2 \), and \( T_2T_4 - T_3^2 \). Since \( \{0\} \subseteq Y \) is closed, the set \( X := Y \setminus \{0\} \) is a locally closed. One can check that \( X \) is parametrized by two variables as follows:

\[
X = \{(t_1, t_2, t_3, t_4, t_5) \mid \alpha, \beta \in \mathbb{C}, \alpha \neq 0 \text{ or } \beta \neq 0, t_1 = \alpha^2, t_2 = \alpha^3, t_3 = \beta^2, t_4 = \beta^3, t_5 = \alpha \beta\}.
\]

Now consider the following regular function defined by two fractions of polynomials:

\[
f = \frac{t_3t_1}{t_2} = \frac{t_4}{t_3},
\]

whose value is just \( \beta \) in the above syntax. Although \( f \) is defined on the whole \( X \), the two fractions of polynomials are not, because their denominators both have zeros in \( X \). In fact, one can show that \( f \) cannot be written as a single fraction of polynomials.
If \( Gv \) is a cone, then \( \mathbb{C}[Gv] \) is graded with the same argument as at the end of chapter 4.

18.1.3 Remark. If \( X \) in Definition 18.1.1 is Zariski-closed, then \( \mathbb{C}[X] \) coincides with our earlier definition of the coordinate ring, i.e., \( \mathbb{C}[X] = \mathbb{C}[\Lambda]/I(X) \).

Our main interest in \( \mathbb{C}[X] \) stems from the map
\[
\iota : \mathbb{C}[Gv]|_{\delta} \hookrightarrow \mathbb{C}[Gv]_{\delta}
\]
that is the restriction of functions. Clearly \( \iota \) is injective: If \( \iota(f) = 0 \), then \( f(w) = 0 \) for all \( w \in Gv \). Since \( f \) is continuous, \( f(w) = 0 \) for all \( w \in Gv \).

With Cor. 12.2.3 we obtain
\[
\text{mult}_{\lambda}(\mathbb{C}[Gv]|_{\delta}) \leq \text{mult}_{\lambda}(\mathbb{C}[Gv]_{\delta}). \tag{18.1.4}
\]

18.2 Invariants under the stabilizer

18.2.1 Definition. Let a group \( G \) act on a set \( S \). For \( v \in S \) define the stabilizer of \( v \) under the action of \( G \) as
\[
\text{stab}_G(v) := \{ g \in G \mid gv = v \}.
\]

18.2.2 Definition. Let \( H \) be a group. For an \( H \)-representation \( V \) define the set of \( H \)-invariants
\[
V^H := \{ v \in V \mid \forall g \in H : gv = v \}.
\]

For a group \( G \) the group algebra \( \mathbb{C}[G] \) is defined as the set of finite formal sums of group elements from \( G \). If \( G \) is finite, then \( \mathbb{C}[G] \cong \mathbb{C}^{|G|} \) as a vector space. The group \( G \times G \) acts on \( \mathbb{C}[G] \) via
\[
((g_1, g_2)f)(g) = f(g_1^{-1}g_2g).
\]

18.2.3 Theorem. Let \( G \) be a finite group, let \( V \) be a \( G \)-representation, and let \( v \in V \). Let \( \mathbb{C}[G]_{\text{stab}_G(v)} \) denote the set of right \( \text{stab}_G(v) \)-invariants, i.e., the elements fixed under the action of \( G \). Then the map
\[
\varphi : \mathbb{C}[Gv] \rightarrow \mathbb{C}[G]_{\text{stab}_G(v)}, \quad f \mapsto (g \mapsto f(gv))
\]
is an isomorphism of \( G \)-representations.

Proof. The finiteness of \( G \) implies that \( \mathbb{C}[Gv] \) is the vector space of all functions on \( Gv \) and \( \mathbb{C}[G] \) is the vector space of all functions on \( G \), without any additional constraints on the functions.

First of all, we verify that \( \varphi \) is well-defined, i.e., that \( \kappa : g \mapsto f(gv) \) is invariant under \( \text{stab}_G(v) \). Let \( g_2 \in \text{stab}_G(v) \). Then \( (g_2\kappa)(g) = \kappa(g_2g) = f(gg_2v) = f(gv) = \kappa(g) \), thus \( \kappa \) is fixed under \( g_2 \).

The map \( \varphi \) is clearly linear.

The inverse map \( \varphi^{-1} \) maps the \( \text{stab}_G(v) \)-invariant \( \psi \) to \( (gv \mapsto \psi(g)) \in \mathbb{C}[Gv] \). But \( \psi(g) \) depends on \( g \) and not just \( gv \), so we have to verify that this is well-defined: If \( g'v = gv \), then we have to show that \( \psi(g) \) coincides with \( \psi(g') \). We have \( v = g'^{-1}gv \), thus \( g'^{-1}g \in \text{stab}_G(v) \). Since \( \psi \) is \text{stab}_G(v)-invariant, \( \psi = (id, g'^{-1})\psi \) and therefore \( (id, g'g)v = (id, g)v \). In particular, if we evaluate both sides at the identity we obtain \( \psi(g') = \psi(g) \).

It is easy to verify that both maps are inverses of each other.

For the \( G \)-equivariance we have to show that \( g(\varphi(f)) = \varphi(gf) \).

We have \( \varphi(f) = (g' \mapsto f(g'v)) \) and hence \( g(\varphi(f)) = (g' \mapsto f((g^{-1}g')v)) \). Therefore \( \varphi(gf) = (g' \mapsto (gf)(g'v)) = (g' \mapsto f(g^{-1}g'v)) = g(\varphi(f)) \). \( \square \)

Without giving the proof we state that Theorem 18.2.3 also holds for large classes of groups, including \( GL_N \). This is particularly interesting because the representation theoretic structure of \( \mathbb{C}[G]_{\delta}^{\text{stab}_G(v)} \) can be obtained in Theorem 18.3.3 below.
18.3 Algebraic Peter-Weyl theorem

18.3.1 Proposition. Let $G$ and $H$ be groups and let $V$ be a $G$-representation and $W$ an $H$-representation. Then $V \otimes W$ is an irreducible $G \times H$-representation if both $V$ and $W$ are irreducible. Moreover, every irreducible $G \times H$-representation is isomorphic to some $V \otimes W$ for $V, W$ irreducible.

Proof. We omit the proof, but point to the beautiful exposition in Section 1.2 of http://www.math.toronto.edu/fiona/courses/mat445/ch1.pdf.

18.3.2 Definition. If $V$ is a $G$-representation, then the dual space $V^*$ is also a $G$-representation via $(gf)(v) := \ell(g^{-1}v)$ for all $\ell \in V^*$, $v \in V$. This representation is called the dual representation of $V$.

Clearly $\dim V = \dim V^*$. Moreover, $V$ is irreducible iff $V^*$ is irreducible.

If $V$ is a polynomial representation, then $V^*$ is not necessarily also a polynomial representation. Indeed, for $GL_N$ we have that both $V$ and $V^*$ are polynomial representation iff the type of $V$ is $(0, 0, \ldots, 0)$.

18.3.3 Theorem (Algebraic Peter-Weyl theorem for finite groups). Let $G$ be a finite group. On $\mathbb{C}[G]$ we have an action of $G \times G$ via $((g, g'))f)(\tilde{g}) := f(g^{-1}g')$ and we have

$$\mathbb{C}[G] = \bigoplus_{\lambda} \{\lambda\}^* \otimes \{\lambda\},$$

where $\lambda$ runs over all isomorphism types of irreducible $G$-representations and $\{\lambda\}$ denotes an irreducible $G$-representation of type $\lambda$.

Proof. We follow [Lan17, Thm. 8.6.4.3].

Let $V$ be a $G$-representation. Define the $G \times G$-equivariant linear map

$$i_V : V^* \otimes V \rightarrow \mathbb{C}[G], \quad i_V(\ell \otimes v)(g) = \ell(gv),$$

defined via linear continuation on all tensors.

We first show that if $V$ is irreducible, then $i_V$ is injective: $\ker(i_V)$ is a subrepresentation of $V^* \otimes V$. Since $i_V \neq 0$, $\ker(i_V) \neq V^* \otimes V$. $V^* \otimes V$ is an irreducible $G \times G$-representation (Pro. 18.3.1). Thus $\ker(i_V) = 0$. Hence $i_V$ is injective.

This already proves that the right-hand side is contained in the left-hand side.

To finish the proof we show that $i_V(V^* \otimes V)$ equals the isotypic component of type $V^*$ w.r.t. the action of $G \times \{1\}$.

Let $W^*$ be an irreducible $G \times \{1\}$-representation that is isomorphic to $V^*$. Let $j : W^* \rightarrow \mathbb{C}[G]$ be a $G \times \{1\}$-morphism. We need to show that $j(W^*) \subseteq i_V(V^* \otimes V)$.

We identify $W^*$ and $V^*$. Let $\ell \in V^*$ be arbitrary. Define $v \in V$ via $\ell(v) := j(\ell)(1_G)$. Note that $(g^{-1}\ell)(v) = (g^{-1}j(\ell))(1_G) = j(g^{-1}\ell)(1_G)$. Then $j(\ell) = i_V(\ell \otimes v)$ because:

$$j(\ell)(g) = (j(\ell))(g \cdot 1_G) = (g^{-1} \cdot j(\ell))(1_G) = (j(g^{-1} \cdot \ell))(1_G) = (g^{-1}\ell)(v) = \ell(gv) = (i_V(\ell \otimes v))(g).$$

The key fact is that Theorem 18.3.3 holds for all algebraic groups, in particular for $G = GL_N$. There the sum is over all types $\lambda$ of rational representations, which includes all polynomial representations. In particular the following corollary holds.

18.3.4 Corollary.

$$\text{mult}_\lambda(\mathbb{C}[Gv]) = \dim(\{\lambda^*\}^{stab_G}(v)).$$
CHAPTER 18. THE ALGEBRAIC PETER-WEYL THEOREM

Proof.  
\[ \text{mult}_\lambda(\mathbb{C}[Gv]) \overset{\text{Thm. 18.2.3}}{=} \text{mult}_\lambda(\mathbb{C}[G^\text{stab}(v)]) \overset{\text{Thm. 18.3.3}}{=} \text{mult}_\lambda(\bigoplus_{\lambda}(\lambda) \otimes (\lambda)^*)^{\text{stab}(v)}, \]  
where the last equality holds because \( \text{mult}_\lambda(\{\mu\}) = 1 \) iff \( \lambda = \mu \) and 0 otherwise. \( \square \)

Several variants of Theorem 18.3.3 are also true, with minor changes in the proof: On \( \mathbb{C}^{N \times N} \) we have the action of \( \text{GL}_N \times \text{GL}_N \) via \((g_1, g_2)M = g_1Mg_2^t\). The algebraic Peter-Weyl theorem implies the decomposition of the coordinate ring of the matrix space  

\[ \mathbb{C}[\mathbb{C}^{N \times N}]_d = \bigoplus_{\lambda \vdash N} \{\lambda\} \otimes \{\lambda\}. \quad (18.3.5) \]

This generalizes to a \( \text{GL}_a \times \text{GL}_b \) action on \( \mathbb{C}^{a \times b} \):  

\[ \mathbb{C}[\mathbb{C}^{a \times b}]_d = \bigoplus_{\lambda \vdash \min(a, b)} \{\lambda\}_a \otimes \{\lambda\}_b, \quad (18.3.6) \]

where \( \{\lambda\}_a \) denotes the irreducible \( \text{GL}_a \)-representation of type \( \lambda \).

18.4 The determinant and rectangular Kronecker coefficients

Combining (18.1.4) and Corollary 18.3.4 we see that  
\[ \text{mult}_\lambda(\mathbb{C}[Gv]) \overset{(18.1.4)}{\leq} \text{mult}_\lambda(\mathbb{C}[Gv]) \overset{\text{Cor. 18.3.4}}{=} \text{dim}(\{\lambda\}^{\text{stab}(v)}), \quad (18.4.1) \]

which could potentially be used to find multiplicity or even occurrence obstructions. If \( G = \text{GL}_n^2 \) and \( v = \det_n \), then \( \text{stab}(v) \) was determined by Frobenius in 1897 [Fro97]:  
\[ H := \text{stab}_{\text{GL}_n^2}(\det_n) = (\text{GL}_n \times \text{GL}_n)/\mathbb{C}^* \times \mathbb{Z}_2. \]

The multiplicities \( \text{dim}(\lambda)^H \) are known as rectangular symmetric Kronecker coefficients \( sk(\lambda, n \times d) \). We have seen in (18.4.1) that if \( a_\lambda(\delta, d) > 0 \) and \( sk(\lambda, n \times d) = 0 \), then the type \( \lambda \) occurs in the vanishing ideal \( I(\text{GL}_n^2 \det_n) \). We will see a working application of this approach in Chapter 22.

An upper bound for \( sk(\lambda, n \times d) \) is given by a similar coefficient: Given a partition \( \lambda \) of \( m \) with at most \( n^2 \) rows, we interpret it as a \( \text{GL}_n \times \text{GL}_n \)-representation via the map \((g, g') \mapsto g \otimes g'\), where the matrix \( g \otimes g' \) is the Kronecker product of matrices. Then \( \{\lambda\} \) decomposes into irreducible \( \text{GL}_n \times \text{GL}_n \)-representations and the Kronecker coefficient \( k(\lambda, \mu, \nu) \) is defined as the multiplicity of the irreducible \( \text{GL}_n \times \text{GL}_n \{\mu\} \otimes \{\nu\} \) in \( \{\lambda\} \). Mulmuley and Sohoni conjectured that the vanishing of the Kronecker coefficients should give enough elements in the vanishing ideal to separate \( \text{VP} \subseteq \text{VNP} \), but that was recently disproved [IP16] (and strengthened in [BIP16]).

Coordinate rings of orbits

The coordinate rings of orbit closures are not well understood. The coordinate rings of orbits are much better understood: Their multiplicities are dimensions of stabilizer-invariant subspaces. Moreover, we will use the Algebraic Peter-Weyl theorem in Chapter 19 to prove the Schur-Weyl duality.
Chapter 19

Explicit HWV constructions via Schur-Weyl duality

In this chapter we give an explicit interpretation of \( C[A]_d, A = C[X_1, \ldots, X_n] \), in terms of tensors. We understand the action of \( \text{GL}_n \) using the famous Schur-Weyl duality. We derive useful results on plethysm coefficients and potential candidates for partitions \( \lambda \) to separate the determinant from the padded permanent, and related orbit closure questions. In particular we will prove a result by Kadish and Landsberg that the first row of \( \lambda \) must be large and we will see that the degree \( d \) must be superpolynomially large.

As in Proposition 12.2.5, \( \text{HWV}_\lambda(W) \) denotes the vector space of highest weight vectors of weight \( \lambda \) in the \( \text{GL}_N \)-representation \( W \).

19.1 Specht modules

In this section we describe without proof the irreducible representations of the symmetric group. The irreducible representations of \( \mathfrak{S}_n \) are called \( \text{Specht modules} \). The Specht modules of \( \mathfrak{S}_n \) are indexed by partitions of \( n \), i.e., partitions \( \lambda \) with \( |\lambda| = n \). Note the difference to \( \text{GL}_n \), where the irreducible polynomial representations are indexed by partitions \( \lambda \) with \( \ell(\lambda) \leq n \), but where \( |\lambda| \) is arbitrary.

The construction of the Specht modules works as follows. For \( n \in \mathbb{N} \) let \( \{\lambda\} \) denote the irreducible \( \text{GL}_n \)-representation of type \( \lambda \). We assume \( |\lambda| = n \). Let \( \{\lambda\}^0 \) denote the weight space of \( \{\lambda\} \) of weight \( (1,1,\ldots,1) \in \mathbb{N}^n \). We embed \( \mathfrak{S}_n \subseteq \text{GL}_n \) via permutation matrices as in Lemma 11.3.1. Lemma 11.3.1 says that \( \mathfrak{S}_n \) acts on \( \{\lambda\}^0 \). It turns out (without proof) that

- \( \{\lambda\}^0 \) is irreducible as an \( \mathfrak{S}_n \)-representation,
- if \( \lambda \neq \mu \), then \( \{\lambda\}^0 \) and \( \{\mu\}^0 \) are non-isomorphic \( \mathfrak{S}_n \)-representations,
- all irreducible representations of \( \mathfrak{S}_n \) are obtained as \( \{\lambda\}^0 \) for some \( \lambda \) with \( |\lambda| = n \).

We denote by \([\lambda] := \{\lambda\}^0\) the Specht module of type \( \lambda \).

19.1.1 Example. 1. \( X^n_1 \) is a HWV in \( V = \mathbb{C}[X_1, \ldots, X_M]_n \) of weight \( \lambda = (n,0,\ldots,0) \). For \( n = M \) we have a unique line of weight \( (1,1,\ldots,1) = (1^n) \), which is \( \mathbb{C} \cdot X_1 X_2 \cdots X_n \). Thus \( \{(n)\}^0 = [(n)] \) is 1-dimensional. The action of \( \mathfrak{S}_n \) permutes the positions of the variables in the monomial \( X_1 X_2 \cdots X_n \), so the action of \( \mathfrak{S}_n \) is trivial: \([n]\) is the trivial representation.
2. Take $V = \mathbb{C}$, $\text{GL}_n$ acts on $V$ via $gv := \det(g)v$. The weight is $(1,1,\ldots,1) = (1^n)$. Thus $[(1^n)]$ is 1-dimensional. Moreover, if $g(\pi) \in \text{GL}_n$ denotes the permutation matrix of the permutation $\pi \in S_n$, then $\det(g(\pi)) = \text{sign}(\pi)$, so $[(1^n)]$ is the alternating representation: $\pi v = \text{sign}(\pi)v$.

A semistandard Young tableau with $n$ boxes in which each number $1,2,\ldots, n$ appears exactly once is called a standard Young tableau. A basis of $[\lambda]$ is given by $e_T$, where $T$ goes over the standard Young tableaux of shape $\lambda$. For example, $[(2,1)]$ is 2-dimensional and we have (when writing $T$ instead of $e_T$)

$$
\begin{pmatrix}
2 & 3 \\
3 & \end{pmatrix} = \begin{pmatrix}
1 & 2 \\
2 & \end{pmatrix}
$$

and

$$
\begin{pmatrix}
1 & 2 \\
3 & \end{pmatrix} = \begin{pmatrix}
2 & 1 \\
3 & \end{pmatrix} + \begin{pmatrix}
2 & 3 \\
1 & \end{pmatrix} = \begin{pmatrix}
1 & 2 \\
3 & \end{pmatrix} + \begin{pmatrix}
1 & 3 \\
2 & \end{pmatrix}
$$

19.2 Explicit Schur-Weyl duality

We write $\lambda \vdash d$ to denote that $\lambda$ is a partition of $d$ into at most $N$ parts.

The group $S_d \times \text{GL}_N$ acts on $\otimes^d \mathbb{C}^N$ via

$$(\pi,g)(v_1 \otimes \cdots \otimes v_N) := (gv_{\pi^{-1}(1)}) \otimes \cdots \otimes (gv_{\pi^{-1}(N)}).$$

Its decomposition into irreducibles is known as the Schur-Weyl duality.\[19.2.1\] Theorem (Schur-Weyl duality).

$$
\bigotimes^d \mathbb{C}^N \cong \bigoplus_{\lambda \vdash d} [\lambda] \otimes [\lambda].
$$

Proof sketch. We follow [Kmn17].

In (18.3.6), let $d = a$ and take the $\text{GL}_d$-weight $(1,1,\ldots,1)$ space:

$$
\mathbb{C}[C^{d \times b|0}_{1}] = \bigoplus_{\lambda \vdash d} [\lambda] \otimes [\lambda]_b.
$$

A partition with $d$ boxes cannot have more than $d$ rows, so $\ell(\lambda) \leq d$ is a void restriction. We obtain

$$
\mathbb{C}[C^{d \times b|0}_{1}] = \bigoplus_{\lambda \vdash d} [\lambda] \otimes [\lambda]_b.
$$

Degree $d$ polynomials on $\mathbb{C}^{d \times b}$ that have $\text{GL}_d$-weight $(1,1,\ldots,1)$ are linear combinations of the monomials $x_{j_1}x_{j_2}\cdots x_{j_d}$, $1 \leq j_i \leq b$. These are in bijection to the rank 1 tensors $x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_d}$, $1 \leq j_i \leq b$. This gives a canonical isomorphism

$$
\mathbb{C}[C^{d \times b|0}_{1}] \cong \otimes^d \mathbb{C}^b
$$

and the result follows.\]

The highest weight vectors in Theorem 19.2.1 can be described explicitly as follows.

We denote by $X_1,\ldots, X_N$ the standard basis vectors of $\mathbb{C}^N$. Let $\lambda \vdash D$ and $\mu$ denote the transpose of $\lambda$, so $\mu_i$ denotes the number of boxes in the $i$-th column of $\lambda$. For $j \leq N$ we note that $v_{j \times 1} := X_1 \wedge X_2 \wedge \cdots \wedge X_j$ is a highest weight vector of weight $j \times 1$: If we use the definition of $x_{iv}(\alpha)$ from the proof of Lemma 11.3.4, then for $i < i'$ we have

$$
x_{iv}X_1 \wedge X_2 \wedge \cdots \wedge X_j = X_1 \wedge X_2 \wedge \cdots \wedge X_j + X_1 \wedge \cdots \wedge X_{i'-1} \wedge X_i \wedge X_{i'+1} \wedge \cdots \wedge X_j,
$$

and

$$
x_{iv}X_i \wedge X_{i+1} \wedge \cdots \wedge X_j = X_i \wedge X_{i+1} \wedge \cdots \wedge X_j
$$

when $i < i'$, otherwise

$$
x_{iv}X_i \wedge X_{i+1} \wedge \cdots \wedge X_j = X_i \wedge X_{i+1} \wedge \cdots \wedge X_j + X_1 \wedge \cdots \wedge X_{i'-1} \wedge X_i \wedge X_{i'+1} \wedge \cdots \wedge X_j
$$

when $i < i'$.
We define now:

\[ v_{\lambda} := v_{\mu_1 \times 1} \otimes \cdots \otimes v_{\mu_{\lambda_1} \times 1} \in \bigotimes^D V. \]  

It is easy to check that \( v_{\lambda} \) is a nonzero highest weight vector of weight \( \lambda \).

**19.2.3 Proposition.** Let \( \lambda \in \mathcal{D}(V) \). Then the vector space \( \text{HWV}_\lambda(\bigotimes^D V) \) is spanned by the \( S_D \)-orbit of \( v_{\lambda} \).

**Proof.** Schur-Weyl duality provides a \( GL(V) \times S_D \)-isomorphism

\[ \bigotimes^D V \simeq \bigoplus_{\lambda \in \mathcal{D}(V)} \{ \lambda \} \otimes [\lambda]. \]

Recalling that \( \text{HWV}_\lambda(\{ \lambda \}) \) is one-dimensional, we see that \( \text{HWV}_\lambda(\bigotimes^D V) \) is isomorphic to \( [\lambda] \) as an \( S_D \)-module. From the irreducibility of \([\lambda]\) it follows that \( \text{HWV}_\lambda(\bigotimes^D V) \) is spanned by the \( S_D \)-orbit of any of its nonzero elements.

Note that Prop. 19.2.3 is even more explicit: The isomorphism \( \bigotimes^D V \simeq \bigoplus_{\lambda \in \mathcal{D}(V)} \{ \lambda \} \otimes [\lambda] \) maps \( v_{\lambda} \) to \( T \otimes S \), where \( T \) is the semistandard tableau with only letters \( i \) in row \( i \) and \( S \) is the so-called superstandard tableau: \( S \) contains the entries \( 1, \ldots, |\lambda| \) ordered columnwise from left to right, top to bottom. For example, \( v_{(3,2,1)} \) corresponds to

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & \otimes \\
3 & 3 \\
\end{array}
\]

This provides a new basis for \( \bigotimes^D V \) given by pairs \((T,S)\) of semistandard \((T)\) and standard \((S)\) tableaux. This is a special case of the so-called Robinson-Schensted-Knuth-correspondence.

### 19.3 Polynomials as symmetric tensors: Plethysms

We follow [BIP16].

In this section we establish the fundamental connection between tensors and the coordinate rings \( \mathbb{C}[X_1, \ldots, X_n] \). This leads to several results concerning the possible partitions \( \lambda \) that can be used to separate orbit closures.

#### 19.3(i) Wreath products and symmetric powers of symmetric powers

We have seen in Section 16.2 that the \( d \)th symmetric power \( \text{Sym}^d W \) of a vector space \( W \) can be defined as the \( S_d \)-invariant subspace of \( \bigotimes^d W \). This construction is easily seen to work for arbitrary \( GL_N \)-representations \( W \). In fact, if \( V = \mathbb{C}^N \) we can choose \( W = \text{Sym}^n V \) and define \( \text{Sym}^d \text{Sym}^n V := \text{Sym}^d(\text{Sym}^n V) \) as the space of \( S_d \)-invariants in \( \bigotimes^d(\text{Sym}^n V) \). This is a subrepresentation of \( \bigotimes^d(\bigotimes^n V) \) in a natural way that we want to understand now.

We partition the position set \([dn] := \{1, \ldots, dn\}\) into the blocks \( B_1, \ldots, B_d \), where \( B_u := \{(u-1)n+v \mid 1 \leq v \leq n\} \). The subgroup of \( S_{dn} \) of permutations that preserve the partition into blocks is called the **wreath product** \( S_n \wr S_d \). It is generated by the permutations leaving the blocks invariant, and the permutations of the form \((u-1)n+v \mapsto (\tau(u)-1)n+v\) with \( \tau \in S_d \), which simultaneously permute the blocks. Structurally, the wreath product is a semidirect product \( S_n \wr S_d \simeq (S_n)^d \rtimes S_d \). Note that its order equals \( d! n^d \). Symmetrizing over \( S_n \wr S_d \), we obtain the projection

\[ \Sigma_{d,n}(w) := \frac{1}{d! n^d} \sum_{\sigma \in S_n \wr S_d} \sigma(w), \]

onto the \( S_n \wr S_d \)-invariant subspace \( (\bigotimes^{dn} V)^{S_n \wr S_d} \subset \bigotimes^{dn} V \).
It is crucial and readily verified that
\[(\otimes^{d}V)^{S_{n}\times S_{d}} = \text{Sym}^{d} \text{Sym}^{n}V.\]  
(19.3.2)

19.3.3 Example. We give an example that will naturally lead to the connection with polynomials. First, it is easy to verify that
\[
\begin{array}{ccc}
1 & 2 & 2 \\
3 & 4 & 3 \\
\end{array}
\]
is invariant under $S_2 \wr S_2$. We write $a \wedge b := \frac{1}{2}(a \otimes b - b \otimes a)$ and $a \odot b := \frac{1}{2}(a \otimes b + b \otimes a)$. In particular $a \odot a = a \otimes a$. We write $x := e_1$ and $y := e_2$ and omit the tensor symbol between them.

\[
\begin{align*}
\begin{array}{ccc}
1 & 2 & 2 \\
3 & 1 & 3 \\
\end{array} + \begin{array}{ccc}
1 & 2 & 2 \\
3 & 1 & 3 \\
\end{array} &= \begin{array}{ccc}
1 & 2 & 2 \\
3 & 4 & 1 \\
\end{array} + \begin{array}{ccc}
1 & 2 & 2 \\
3 & 4 & 1 \\
\end{array} = \begin{array}{ccc}
2 & 1 & 2 \\
1 & 3 & 4 \\
\end{array} - \begin{array}{ccc}
1 & 3 & 4 \\
2 & 1 & 2 \\
\end{array} \\
&= 2(xxyy - yxxy - xyyx + yyxx) - (xyxy - yxxy - xyyx + yxyx) \\
&= 2xxyy - yxxy - xyyx + 2yyxx - xyyx - yxxy \\
&= 2(xxyy + yxxy) - (xyxy + xyyx + yxyx) \\
&= 2(xxyy + yxxy) - (xxyy + yxxy)\omega_{2} \\
&= 2(2(x^{2}) \odot (y^{2})) - (2x \odot y)\omega_{2} \\
&= 4((x^{2}) \odot (y^{2})) - 4(x \odot y)\omega_{2} \\
&= -4 \left( (x \odot y)\omega_{2} - ((x^{2}) \odot (y^{2})) \right),
\end{align*}
\]

which reminds of the discriminant function, but without the coefficients.

Why did we choose \[
\begin{array}{ccc}
1 & 2 & 3 \\
3 & 1 & 4 \\
\end{array}
\]
in the previous example? We will see this in the Remark 19.3.13.

19.3(ii) Polynomials as symmetric tensors

Let $W = \mathbb{C}^{N}$ with standard basis $X_{1}, \ldots, X_{N}$. For a list $I = (i_{1}, \ldots, i_{d}) \in \{1, \ldots, N\}^{d}$ we define

\[X_{I} := X_{i_{1}} \otimes X_{i_{2}} \otimes \cdots \otimes X_{i_{d}} \in \otimes^{d}W\]

Let $C_{N}(d)$ denote the set of all $\alpha \in \mathbb{N}^{N}$ with $|\alpha| = d$. These $\alpha$ are called compositions of $d$. For $I \in \{1, \ldots, N\}^{d}$ we define the type $\zeta(I) \in C_{N}(d)$, letting $\zeta(I)$ denote the number of appearances of $i$ in $I$. For example $\zeta(1, 2, 5, 3, 4, 2, 1, 3, 2) = (2, 3, 2, 1, 1)$. We associate with $\alpha \in C_{N}(d)$ the monomial

\[X^{\alpha} := \frac{1}{\alpha!} \sum_{\zeta(I) = \alpha} X_{I},\]  
(19.3.4)

where the sum is over all $\alpha \in C_{N}(d)$ such that $\zeta(I) = \alpha$ and $\binom{m}{\alpha} = \frac{m!}{\alpha_{1}! \alpha_{2}! \cdots \alpha_{N}!}$ is the multinomial coefficient.

This agrees with our definition:

\[X^{\alpha} = X_{\alpha_{1}} \odot \cdots \odot X_{\alpha_{d}} = \frac{1}{d!} \sum_{\pi \in \text{Sym}_{d}} \pi(X_{\alpha_{1}} \otimes \cdots \otimes X_{\alpha_{d}})\]

Note that $X^{\alpha}$ is a symmetric tensor: $X^{\alpha} \in \text{Sym}^{\alpha}$. 
Given a homogeneous degree $d$ polynomial $f$ we can interpret the evaluation at a point $p$ via

$$f(p) = \langle f, p^\otimes d \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product that is inherited from the standard inner product on $\mathbb{C}^N$: Eq. (19.3.5) is easily checked for monomials $f$, and since evaluation is a linear function, eq. (19.3.5) holds for all homogeneous polynomials $f$. The inner product of two tensors is called a tensor contraction.

We formalize this correspondence as follows.

19.3.6 Proposition. For a $\text{GL}_N$-representation $W$ there is a natural $\text{GL}_N$-isomorphism $\mathbb{C}[W]^d \simeq \text{Sym}^d W$ given by mapping monomials as in (19.3.4), which implies that evaluating polynomials is given by tensor contraction as in (19.3.5).

Iterating Proposition 19.3.6 twice we obtain the following corollary.

19.3.7 Corollary. There is a natural $\text{GL}_N$-isomorphism $\mathbb{C}[A]^d \simeq \text{Sym}^d \text{Sym}^n V$, where $A = \mathbb{C}[X_1, \ldots, X_{\dim V}]$.

19.3.8 Example. The tensor that corresponds (up to scale) to the discriminant via Corollary is the tensor from Example 19.3.3:

$$2xyyx + 2yxyy - xyxy - xyxy - yxyx.$$

We evaluate the discriminant at $aX^2 + bXY + cY^2$, which is a contraction with the tensor $(ax^2 + bx \odot y + cy^2)^\otimes 2$, according to (19.3.5). In order to do so, we expand first:

$$(ax^2 + bx \odot y + cy^2)^\otimes 2 = (axx + \frac{b}{2} xy + \frac{c}{2} yx + cyy)^2$$

$$= a^2 xxxx + \frac{ab}{2} xxxxy + \frac{ab}{2} xxyx + acxxyy$$

$$+ \frac{ac}{2} xyyx + \frac{bc}{2} yxxx + \frac{bc}{2} xxyy$$

$$+ \frac{ab}{2} yxxx + \frac{b^2}{4} yxyy + \frac{b^2}{4} yxyx + \frac{b}{2} cyxx$$

$$+ axxyyy + \frac{bc}{2} yyyx + \frac{bc}{2} yyyx + c^2 yyyy.$$

The contraction is color-coded and yields $4ac - b^2$, which is the evaluation of (the negative of) the discriminant at $aX^2 + bXY + cY^2$.

We study (19.3.5) in more detail in Section 20.1 in the situation of Corollary 19.3.7.

19.3(iii) Plethysm coefficients

In Section 12.4(i) we defined $a_\lambda(d,n)$ as the multiplicity of $\lambda$ in $\mathbb{C}[A]^d$ with $A = \mathbb{C}[X_1, \ldots, X_N]$. By Proposition 12.2.5 we know that

$$a_\lambda(d,n) = \dim \text{HWV}_\lambda(\mathbb{C}[A]^d).$$

With Corollary 19.3.7 we obtain

$$a_\lambda(d,n) = \dim \text{HWV}_\lambda(\text{Sym}^d \text{Sym}^n V).$$

19.3.9 Proposition. Formally $a_\lambda(d,n)$ depends on $m := \dim V$, i.e., we would need a symbol $a_{\lambda,m}(d,n)$. For $\ell(\lambda) > \dim V$ we have $a_{\lambda,\dim V}(d,n) := 0$, as there exists no $\text{GL}_{\dim V}$-representation of type $\lambda$. If $\ell(\lambda) \leq m_1$ and $\ell(\lambda) \leq m_2$, then $a_{\lambda,m_1}(d,n) = a_{\lambda,m_2}(d,n)$.

Proposition 19.3.9 justifies the notation $a_\lambda(d,n)$. This was claimed in Section 12.4(i).
CHAPTER 19. EXPLICIT HWV CONSTRUCTIONS VIA SCHUR-WEYL DUALITY

Proof of Proposition 19.3.9. Note that by Schur-Weyl duality and (19.3.2) we have

\[
\begin{align*}
\alpha_{\lambda}(d,n) &= \dim \text{HWV}_{\lambda}(\text{Sym}^d \text{Sym}^n V) \\
&= \dim \text{HWV}_{\lambda}(\otimes^{nd} V)^{\mathfrak{S}_n \wr \mathfrak{S}_d} \\
&= \dim \left( \bigoplus_{\mu} \text{HWV}_{\lambda}(\{\mu\}) \otimes (\{\mu\})^{\mathfrak{S}_n \wr \mathfrak{S}_d} \right) \\
&= \dim \left( \bigoplus_{\mu} \dim \text{HWV}_{\lambda}(\{\mu\}) \otimes (\{\mu\})^{\mathfrak{S}_n \wr \mathfrak{S}_d} \right) \\
&= \dim \lambda^{\mathfrak{S}_n \wr \mathfrak{S}_d},
\end{align*}
\]

provided \(\ell(\lambda) \leq \dim V\). ∎

19.3(iv) Semistandard tableaux again: Gay’s theorem

We follow [Ike12, Sec. 4.3(A)]. Since the HWVs of type \(\lambda\) in \(\text{Sym}^d \text{Sym}^n V\) are explicitly described by \(\mathfrak{S}_n \wr \mathfrak{S}_d\)-invariants in \(\lambda\) (see the proof of Proposition 19.3.9), we study those invariants now.

Recall that a standard tableau indexes a vector in \(\lambda = \{\lambda\}\) but via Schur-Weyl duality it also indexes a vector in \(\text{HWV}_{\lambda}(\otimes^{nd} V) \cong \lambda\). This can be confusing at first, but this beautiful correspondence makes things a lot easier.

19.3.10 Lemma. Let \(V\) be an \(\mathfrak{S}_n \wr \mathfrak{S}_d\)-representation. Then we have an action of \(\mathfrak{S}_d\) on the invariant space \(V^{\mathfrak{S}_n \wr \mathfrak{S}_d}\). Moreover, \(V^{\mathfrak{S}_n \wr \mathfrak{S}_d} = (V^{\mathfrak{S}_d})^{\mathfrak{S}_n \wr \mathfrak{S}_d}\).

Proof. Interchanging the block structure of invariant blocks keeps the blocks invariant, so we have an \(\mathfrak{S}_d\)-action. The second statement follows from our definition of \(\mathfrak{S}_n \wr \mathfrak{S}_d\). □

It turns out that the \(\mathfrak{S}_d^n\)-invariants of \(\lambda\) can be easily understood using semistandard tableaux. Let \(\{\lambda\}^{d \times n}\) denote the \(d \times n\) weight space in \(\lambda\) (recall that a basis of \(\{\lambda\}^{d \times n}\) is given by the \(e_T\) where \(T\) is semistandard of shape \(\lambda\) and content \(d \times n\)). Consider the map

\[
\varphi: \{\lambda\}^0 \rightarrow \{\lambda\}^{d \times n}
\]

that replaces each entry 1, 2, \ldots, \(n\) by 1, each entry \(n+1, n+2, \ldots, 2n\) by 2, and so on. This is an application of a matrix in \(\text{End}(\mathbb{C}[\lambda])\), for example:

\[
\begin{pmatrix}
2 & 1 & 3 & 5 & 8 \\
1 & 7 & 9 & 10 \\
6 & 12 \\
11
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 1 & 1 & 1 & 2 & 3 \\
2 & 3 & 3 & 4 \\
2 & 4 \\
4
\end{pmatrix}
\]

is obtained by applying the matrix

\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
2 & 1 & 3 & 5 & 8 \\
1 & 7 & 9 & 10 \\
6 & 12 \\
11
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 1 & 1 & 2 & 3 \\
2 & 3 & 3 & 4 \\
2 & 4 \\
4
\end{pmatrix}
\]

For a standard tableau \(T\) there are two cases:
1. $\varphi(T)$ is semistandard or
2. $\varphi(T)$ has a column with a repeated entry.

Using the relations in the columns, we see that only the semistandard tableaux correspond to nonzero vectors. Moreover, distinct semistandard tableaux correspond to linearly independent vectors in $\{\lambda\}^{d\times n}$.

The map $\varphi$ gives a way of representing the $S_d^n$-invariants:

19.3.12 Lemma (Gay’s theorem). Let $S_d$ act on $[\lambda]$ by permuting the $d$ blocks, i.e., for example the transposition $(1\ 2)$ switches 1 with $n+1$, switches 2 with $n+2$, . . . , and switches $n$ with $2n$. The map $\varphi$ is $S_d$-equivariant. Moreover, if we restrict $\varphi$ to the invariant space $[\lambda]^{S_d^n}$, then it becomes an isomorphism of $S_d$-representations $[\lambda]^{S_d^n}$ and $\{\lambda\}^{d\times n}$.

**Proof.** Permuting blocks and then setting the whole block to a number has the same effect as setting the whole block to a number and then permuting the blocks. Thus $\varphi$ is $S_d$-equivariant.

The inverse function $\varphi^{-1}$ is given (up to scale) by summing over all ways of replacing each entry $i$ in the semistandard tableau by all entries $i(n-1)+j$, $1 \leq j \leq n$. For example

$$\varphi^{-1}(1\ 1\ 2\ 2) \mapsto 1\ 2 + 2\ 1 + 1\ 2 + 2\ 1$$

\hfill $\square$

19.3.13 Remark. Note that

$$1\ 2 + 2\ 1 + 1\ 2 + 2\ 1 = 2(1\ 2 + 2\ 1),$$

which explains the choice at the end of Section 19.3(i).

Taking $S_d$-invariants in Lemma 19.3.12 we obtain:

19.3.14 Corollary. $a_\lambda(d,n) = \dim([\lambda]^{d\times n})^{S_d}$.

The following example shows how we can calculate plethysm coefficients using tableaux.

19.3.15 Example. For $\lambda = (2,2)$ there is a unique semistandard tableau of shape $\lambda$ with rectangular content $2 \times 2$:

$$1\ 1\ 2\ 2$$

Thus $\dim[\lambda]^{S_2^2} = 1$. Let us consider the action of $S_2$:

$$\frac{1}{2} (1\ 1\ 2\ 2 + (1\ 2)) = 1\ 1\ 2\ 2$$

and thus $\dim[\lambda]^{S_2^2} = 1 = a_{(2,2)}(2,2)$. This invariant tableau can therefore now be used with the above constructions to find the unique HWV of weight $(2,2)$: The discriminant, as it was done in Example 19.3.3.

19.3.16 Corollary. Via Schur-Weyl duality we identify tableaux $S$ of shape $\lambda$ that have content $(1,1,\ldots,1)$ with vectors $v_S \in \text{HWV}_\lambda(S^d \otimes \mathcal{G}_n V)$. The vector space $\text{HWV}_\lambda(S^d \otimes \mathcal{G}_n V)$ has as a basis the highest weight vectors $\varphi^{-1}(T)$, where $T$ ranges over all semistandard tableaux of shape $\lambda$ with content $d \times n$. Moreover, the vector space $\text{HWV}_\lambda(S^d \otimes \mathcal{G}_n V)$ is spanned by the highest weight vectors $\sum_{\pi \in S_d} \varphi^{-1}(\pi T)$, where $T$ ranges over all semistandard tableaux of shape $\lambda$ with content $d \times n$. 


19.3.17 Example.
\[ \text{Sym}^2 \text{Sym}^2 \mathbb{C}^m = \{(2, 2)\} \oplus \{(4)\} \text{ for all } m \geq 2. \]

Proof. 
- There is a unique semistandard tableau of shape \((2, 2)\) and rectangular content \(2 \times 2\):
  \[
  \begin{array}{cc}
  1 & 1 \\
  2 & 2
  \end{array}
  \]

  It is invariant under \(S_2\).

- There is a unique semistandard tableau of shape \((4, 0)\) and rectangular content \(2 \times 2\):
  \[
  \begin{array}{cccc}
  1 & 1 & 2 & 2
  \end{array}
  \]

  It is invariant under \(S_2\).

- There is a unique semistandard tableau of shape \((3, 1)\) and rectangular content \(2 \times 2\):

\[
\begin{array}{ccc}
1 & 1 & 2 \\
2 & &
\end{array}
\]

(19.3.18)

It vanishes under symmetrization over \(S_2\).

- There are no semistandard tableaux of shape \((2, 1, 1)\) or \((1, 1, 1, 1)\) with rectangular content \(2 \times 2\).

Some more small examples:

19.3.19 Theorem (Howe’s theorem). Let \(d \geq 1\). Let \(d \times n := (n, n, \ldots, n)\) denote the partition of \(nd\) whose Young diagram is rectangular with \(n\) columns and \(d\) rows.

\[ a_{d \times n}(d, n) = \begin{cases} 
1 & \text{if } n \text{ is even,} \\
0 & \text{otherwise.}
\end{cases} \]

Proof. There is exactly 1 semistandard tableau of shape \(d \times n\) and rectangular content \(d \times n\). For example:

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4
\end{array}
\]

Applying a transposition \((i j) \in S_d\) gives a sign change for each column. So the tableau is invariant if and only if \(n\) is even.

19.3.20 Proposition. A partition \(\lambda\) is called a nontrivial hook is \(\lambda = (k, 1^m) := (k, 1, 1, \ldots, 1)\) with \(k \geq 1\) and \(m \geq 1\). If \(\lambda\) is a partition of \(nd\) and \(\lambda\) is a nontrivial hook, then the plethysm coefficient \(a_{\lambda}(d, n)\) is zero.

Proof. This generalizes the vanishing of eq. (19.3.18) under symmetrization.

Let \(\lambda \vdash nd\) be a nontrivial hook. Pick any Young tableau \(T\) of shape \(\lambda\) with rectangular content \(d \times n\). Then \(T\) will have at least two numbers \(a\) and \(b\) appearing in the first column, for example

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 \\
2 & 2 & 2 & 2 & 3 & 3 & 4 & 4 \\
3 & & & & & & &
\end{array}
\]

\[
\begin{array}{cc}
2 & 3 \\
3 &
\end{array}
\]
Then pick two numbers $a$ and $b$ that appear in the first column. Then $T + (a b)T = 0$ and thus $T$ vanishes when symmetrizing over a cardinality 2 subgroup of $\mathfrak{S}_d$, therefore $T$ also vanishes when symmetrizing over the whole $\mathfrak{S}_d$.

Explicit construction of HWVs

The explicit construction of highest weight vectors gives us first results about plethysm coefficients. In Chapter 20 we will lift these techniques to multiplicities in coordinate rings of orbit closures.
Chapter 20

Tensor contraction

In this chapter we describe a combinatorial interpretation of the contraction in (19.3.5) and give several applications, including strong restrictions on the shape of partitions \( \lambda \) that can potentially serve as obstructions. We will see that those \( \lambda \) require a very long first part.

20.1 Contracting highest weight vectors in plethysms with rank one tensors

In this section we give a combinatorial interpretation of the contraction in (19.3.5). This will enable us to deduce restrictions on the possible \( \lambda \) that can serve as obstructions.

Let \( V = \mathbb{C}^N \) and let \( s : \{1, \ldots, dn\} \rightarrow \mathbb{C}^N \).

We view \( \lambda \vdash dn \) as a Young diagram and, for convenience, denote by \( \lambda \) also the set of boxes of the diagram. Recall the map \( \varphi \) from (19.3.11).

20.1.1 Definition. Let \( T \) be a tableau of shape \( \lambda \) with content \( d \times n \) and \( \vartheta : \lambda \rightarrow [dn] \) be a bijection. This results in a tableau \( S_\vartheta \) with content \((1, \ldots, 1)\). We say that \( \vartheta \) respects \( T \) iff \( \varphi(S_\vartheta) = T \).

Clearly for a given \( T \) there are \((n!)^d d!\) maps \( \vartheta \) that respect \( T \).

Pictorially, the composition \( s \circ \vartheta \) puts vectors in the tableau cells.

Let \( j = (j_1, \ldots, j_k) \) be a list of vectors in \( \mathbb{C}^N \). We define

\[
\det(j_1, \ldots, j_k) := \langle e_1 \wedge e_2 \wedge \cdots \wedge e_k, j_1 \otimes j_2 \otimes \cdots \otimes j_k \rangle,
\]

which is the determinant of the top \( k \times k \) submatrix of the \( N \times k \) matrix \( j \).

Suppose \( \vartheta : \lambda \rightarrow [dn] \) respects the tableau \( T \) of shape \( \lambda \) with content \( d \times n \), and take a map \( s : \{1, \ldots, dn\} \rightarrow \mathbb{C}^N \). We define the value \( \text{val}_\vartheta(s) \) of \( \vartheta \) at \( s : \{1, \ldots, dn\} \rightarrow \mathbb{C}^N \) by

\[
\text{val}_\vartheta(s) := \prod_{\text{column } c \text{ of } \lambda} \det(s(\vartheta(1, c)), \ldots, s(\vartheta(\mu, c))),
\]

where \( \mu = \lambda^t \). This is natural in the following sense:

\[
\text{val}_\vartheta(s) = \langle e_1 \wedge \cdots \wedge e_{\mu_1} \otimes \cdots \otimes e_1 \wedge \cdots \wedge e_{\mu_{\lambda_1}}, s(\vartheta(1, 1)) \otimes s(\vartheta(2, 1)) \otimes \cdots \otimes s(\vartheta(\mu_{\lambda_1}, \lambda_1)) \rangle
\]

Pictorially, \( s \circ \vartheta \) places vectors in the tableau and \( \text{val}_\vartheta(s) \) is the product over the column determinants.
20.1.4 Theorem. Let $T$ be a tableau of shape $\lambda \vdash d \times n$ and let $\pi_T$ be the corresponding HWV in $\text{HWV}_\lambda(\bigotimes^d \text{Sym}^n V)$ (Cor. 19.3.16). Let $\tilde{\nu}_T := \frac{1}{d!} \sum_{\pi \in \mathfrak{S}_d} \pi \pi_T$ be the HWV in $\text{HWV}_\lambda(\text{Sym}^d \text{Sym}^n V)$. Let $s: \{1, \ldots, dn\} \to \mathbb{C}^N$ be a map. Then

$$\langle \tilde{\nu}_T, s(1) \otimes \ldots \otimes s(dn) \rangle = \frac{1}{d! n!} \sum_\vartheta \text{val}_\vartheta(s),$$

where the sum is over all bijections $\vartheta: \lambda \to \{1, \ldots, dn\}$ respecting $T$.

Proof. Let $S$ be a tableau with content $(1, 1, \ldots, 1)$ such that $\varphi(S) = T$. Let $\pi \in \mathfrak{S}_{dn}$ any permutation such that $\pi S_0 = S$, where $S_0$ is the standard tableau that is ordered columnwise from left to right, top to bottom.

$$\langle \tilde{\nu}_T, s(1) \otimes \ldots \otimes s(dn) \rangle = \frac{1}{d! n!} \sum_{\sigma \in \mathfrak{S}_d} \langle \sigma v_S, s(1) \otimes \ldots \otimes s(dn) \rangle$$

$$= \frac{1}{d! n!} \sum_{\sigma \in \mathfrak{S}_d} \langle \nu_S, \sigma(s(1) \otimes \ldots \otimes s(dn)) \rangle$$

$$= \frac{1}{d! n!} \sum_{\sigma \in \mathfrak{S}_d} \langle \nu v_S, \sigma(s(1) \otimes \ldots \otimes s(dn)) \rangle$$

$$= \frac{1}{d! n!} \sum_{\sigma \in \mathfrak{S}_d} \langle \nu v_S, \pi^{-1} \sigma(s(1) \otimes \ldots \otimes s(dn)) \rangle$$

$$= \frac{1}{d! n!} \sum_{\sigma \in \mathfrak{S}_d} \text{val}_\sigma(s),$$

where the last equality follows from $v_{s_0} = e_1 \wedge \cdots \wedge e_{\mu_1} \otimes \cdots \otimes e_1 \wedge \cdots \wedge e_{\mu_1}$.

20.2 Applications: Waring rank and a proof of Weintraub’s conjecture

20.2(i) The discriminant

We let

$$T := \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

and aim to evaluate $\tilde{\nu}_T$ at a tensor of Waring rank 1: $\ell^2$, where $\ell \in \mathbb{C}^M$.

We observe $(\ell^2)^{\otimes 2} = \ell \otimes \ell \otimes \ell \otimes \ell$ and set $s = (\ell, \ell, \ell, \ell)$.

$$\langle \tilde{\nu}_T, (\ell^2)^{\otimes 2} \rangle = \frac{1}{4} \sum_{\vartheta} \text{val}_\vartheta(s)$$

But every summand $\text{val}_\vartheta(s)$ is zero, because it is a product of $2 \times 2$ determinants in which both columns are $\ell$. Therefore the discriminant vanishes on Waring rank 1 polynomials.

20.2(ii) Too many rows

The following observation generalizes the discriminant.

20.2.1 Observation. Let $T$ be a semistandard tableau of shape $\lambda$ and content $d \times n$ with more than $k$ rows. Then $\tilde{\nu}_T$ vanishes on all points of border Waring rank $\leq k$. 

Proof. 
\[
\langle \hat{\nu}_T, h^\otimes d \rangle = \sum_{1 \leq a_1, \ldots, a_d \leq k} \langle \hat{\nu}_T, \ell_{a_1}^3 \otimes \cdots \otimes \ell_{a_d}^3 \rangle \\
= \frac{1}{d(d-1)^3} \sum_{1 \leq a_1, \ldots, a_d \leq k} \sum_{\vartheta} \text{val}_\vartheta(\ell_{a_1}, \ldots, \ell_{a_1}, \ell_{a_2}, \ldots, \ell_{a_2}, \ldots, \ell_{a_d}, \ldots, \ell_{a_d}) 
\]
As for the discriminant, each summand vanishes independently. Indeed, for \( s = (\ell_{a_1}, \ldots, \ell_{a_1}, \ell_{a_2}, \ldots, \ell_{a_2}, \ldots, \ell_{a_d}, \ldots, \ell_{a_d}) \), the map \( s \circ \vartheta \) places the \( \ell_i \) in the tableau such that each position that has the same number in \( T \) gets the same \( \ell_i \). But the first column of \( T \) has more than \( k \) different numbers. So by the pigeonhole principle \( s \circ \vartheta \) puts at least two coinciding \( \ell_i \) in the first column. Thus \( \text{val}_\vartheta(s) = 0 \) because a determinant with a repeating column is zero. \( \square \)

20.2(iii) Aronhold’s invariant

We can do a little bit better, i.e., use less rows, as Aronhold’s invariant in \( \text{Sym}^4 \text{Sym}^3 V \) shows (and this can be generalized):

\[
T := \begin{pmatrix}
1 & 1 & 1 & 2 \\
2 & 2 & 3 & 3 \\
3 & 4 & 4 & 4
\end{pmatrix}
\]

Let \( h \in \text{Sym}^3 V \) be of Waring rank 3, \( h = \ell_1^3 + \ell_2^3 + \ell_3^3 \). Then

\[
h^\otimes 4 = \ell_1^3 \otimes \ell_1^3 \otimes \ell_1^3 \otimes \ell_1^3 + \ell_1^3 \otimes \ell_1^3 \otimes \ell_2^3 \otimes \ell_2^3 + \cdots + \ell_3^3 \otimes \ell_3^3 \otimes \ell_3^3 \otimes \ell_3^3.
\]

\[
\langle \hat{\nu}_T, h^\otimes 4 \rangle = \sum_{1 \leq a, b, c, d \leq 3} \langle \hat{\nu}_T, \ell_a^3 \otimes \ell_b^3 \otimes \ell_c^3 \otimes \ell_d^3 \rangle \\
= \frac{1}{d(d-1)^3 d(d-1)^2} \sum_{1 \leq a, b, c, d \leq 3} \sum_{\vartheta} \text{val}_\vartheta(\ell_a, \ell_a, \ell_b, \ell_b, \ell_c, \ell_c, \ell_d, \ell_d, \ell_d)
\]

Again each summand vanishes independently. Indeed, for \( s = (\ell_a, \ell_a, \ell_a, \ell_b, \ell_b, \ell_c, \ell_c, \ell_d, \ell_d, \ell_d) \), the map \( s \circ \vartheta \) places the \( \ell_i \) in the tableau such that each position that has the same number in \( T \) gets the same \( \ell_i \). For a nonzero summand it is required that \( \vartheta \) puts different vectors \( \ell_i \) on the numbers 1, 2, 3 because of the first column. But in \( T \) the number 4 shares columns with each 1, 2, 3 and there is no 4th different vector \( \ell_i \). Thus for every \( \vartheta \) there is at least one column in which the determinant vanishes because of a repeated column.

20.2(iv) Proof of Weintraub’s conjecture

With one additional idea it is now straightforward to prove Weintraub’s conjecture that allows us to create nonzero \( \hat{\nu}_T \):

20.2 Theorem ([BC11]). Let \( n \) be even. Given a partition \( \lambda \) of \( dn \) into at most \( d \) parts such that all \( \lambda_i \) are even. Then \( a_\lambda(d, n) > 0 \).

Proof. If \( \lambda \) is even, \( \ell(\lambda) \leq d \), then we can fill it with content \( d \times n \) such that each column appears twice (for example in a greedy fashion, taking the column pair with the most empty rows first). For example

\[
T = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 4 & 4 & 2 & 2 & 3 & 3 & 4 & 4 \\
2 & 2 & 2 & 2 & 1 & 1 \\
3 & 3 & 3 & 3 \\
4 & 4
\end{pmatrix}.
\]
CHAPTER 20. TENSOR CONTRACTION

Now choose a homogeneous polynomial \( h \) as the sum of \( d \) many homogeneous degree \( n \) linear forms with real coefficients:

\[
h = \ell_1^n + \cdots + \ell_d^n,
\]

where each \( \ell_i \in \mathbb{R}^d \). Here we choose each \( \ell_i \) in a “generic” way, i.e., its entries should be algebraically independent or at least all top \( k \times k \) determinants of all subsets of \( k \) vectors \( \ell_i \) should be nonzero. We will see that

\[
\langle \tilde{\nu}_T, h^{\otimes d} \rangle > 0.
\] (20.2.3)

Indeed,

\[
\langle \tilde{\nu}_T, h^{\otimes d} \rangle = \frac{1}{\mathfrak{S}_n(n)!} \sum_{1 \leq a_1, a_2, \ldots, a_d \leq d} \sum_{\vartheta} \text{val}_\vartheta(\ell_{a_1}, \ldots, \ell_{a_d}, \ldots, \ell_{a_d})
\]

When expanding we get a sum of products of determinants, but in each product each determinant appears an even number of times. Therefore we sum over squares of real numbers! Since squaring nonzero real numbers results in positive real numbers, (20.2.3) will be positive. It remains to show that there is at least one positive summand, but that is given for example for the summand \( \vartheta \) in which \( \ell_i \) is placed on \( i \).

\[\square\]

20.3 Application: Obstructions require long first rows

For a partition \( \lambda \) we define \( \bar{\lambda} \) to be its body, i.e., \( \bar{\lambda} \) is obtained from \( \lambda \) by removing its first row.

The following insight is due to Kadish and Landsberg [KL14]. It puts a strong restriction on the types \( \lambda \) that can be used to separate the determinant from the padded permanent.

20.3.1 Proposition ([KL14]). Let \( \Omega_{n,m} := \mathbb{GL}_n^\perm(X_{1,1}^{n-m} \perm_m) \). If \( \lambda \vdash nd \) occurs in \( \mathbb{C}[\Omega_{n,m}]_d \), then \( \ell(\lambda) \leq m^2 + 1 \) and \( |\bar{\lambda}| \leq md \).

Proof. Suppose that \( \lambda \vdash nd \) satisfies \( \ell(\lambda) > m^2 + 1 \) or \( |\bar{\lambda}| > md \). We need to show that \( \lambda \) does not occur in \( \mathbb{C}[\Omega_{n,m}]_d \). We will show that \( \langle \tilde{\nu}_T, h^{\otimes d} \rangle = 0 \) for every semistandard tableau \( T \) of shape \( \lambda \) with content \( d \times n \) and every \( h \in \Omega_{n,m} \). So fix a tableau \( T \) of shape \( \lambda \) with content \( d \times n \) and fix a point \( h \in \Omega_{n,m} \).

Assume first that \( \ell := \ell(\lambda) > m^2 + 1 \). We prove that \( \langle \tilde{\nu}_T, h^{\otimes d} \rangle = 0 \). We use an argument that is very similar to Observation 20.2.1. For this, it suffices to show that \( \langle \tilde{\nu}_T, t \rangle = 0 \) for all tensors \( t = s(1) \otimes \cdots \otimes s(dn) \), where \( s: \{1, \ldots, dn\} \to \{e_1, \ldots, e_{m^2+1}\} \) and \( e_i \) are the standard basis vectors of \( \mathbb{C}^{m^2+1} \). Indeed, for every \( \vartheta \) we have \( \text{val}_\vartheta(s) = 0 \) because the determinant that corresponds to the first column is zero: It is a determinant of a \( \ell \times \ell \) matrix, \( \ell > m^2 + 1 \), whose last \( \ell - (m^2 + 1) \) rows are zero. Thus \( \langle \tilde{\nu}_T, h^{\otimes d} \rangle = 0 \).

Assume now \( |\bar{\lambda}| > md \), so that \( \lambda_1 < (n-m)d \). For \( g \in \mathbb{GL}_{dn} \) let \( Z := gX_{1,1} \) so that \( Z \cdot g \perm_m \in \mathbb{GL}_n^\perm(X_{1,1}^{n-m} \perm_m) \). Let \( q := Z^{n-m} \perm_m \). We can express \( q^{\otimes d} \) as a linear combination of tensors \( t = s(1) \otimes \cdots \otimes s(dn) \), where \( s: \{1, \ldots, dn\} \to \mathbb{C}^M \) maps at least \( (n-m)d \) elements to the vector \( Z \). Fix such a tensor \( t \). It suffices to show that \( \langle \tilde{\nu}_T, t \rangle = 0 \). Indeed, each summand of \( \sum_{\vartheta} \text{val}_\vartheta(s) \) vanishes independently. This can be seen as follows. Since \( \lambda_1 < (n-m)d \), the partition \( \lambda \) has less than \( (n-m)d \) columns. By the pigeonhole principle, there is a column \( c \) in which \( s \circ \vartheta \) puts a vector \( Z \) in at least two boxes. Thus the determinant corresponding to this column vanishes. \(\square\)

Explicit construction of HWVs

Evaluation of highest weight vectors can be defined via tensor contraction. In several cases this tensor contraction can be fully understood, so that nontrivial results about multiplicities in coordinate rings of orbit closures can be deduced, for example equations for Waring rank.
Chapter 21

Good occurrence obstructions for determinant vs padded permanent do not exist

In this chapter we will see that showing \( (X_1)^{n-m}_{\text{per}_m} \notin GL_{n^2}\text{det}_n \) for superpolynomially large \( n \) cannot be achieved with occurrence obstructions. Multiplicity obstructions might still work, as well as occurrence obstructions in models of computation where no padding is involved. For example homogeneous iterated matrix multiplication.

We roughly follow [BIP16].

21.1 The degree lower bound

In this section we prove a lower bound on the degree \( d \) that (polynomial) obstructions must have: \( d > \sqrt{n} \). In particular if we want to prove superpolynomial lower bounds on the border determinantal complexity of the permanent, we need superpolynomially high degree.

21.1(i) Padded low rank embedding

We will apply the following theorem with \( s = md \).

21.1.1 Theorem. Let \( n, s, d \) be positive integers such that \( n \geq sd \) and \( Z, v_1, 1, \ldots, v_1, s, v_2, 1, \ldots, v_d, s \in V \). Then we have \( Z^{n-s}(v_1, 1 v_1, 2 + \cdots + v_d, 1 v_d, 2 + \cdots + v_d, s) \in GL_{n^2}\text{det}_n \).

Proof. Let \( X_1, \ldots, X_{d,s}, \ldots, X_n \) denote the standard basis of \( V \) (that last \( n - ds \) variables are indexed by just one integer). Writing the polynomial \( X_{1,1} X_{1,2} \cdots X_{1,s} + \cdots + X_{d,1} X_{d,2} \cdots X_{d,s} \) as a formula requires at most \( (s-1)d + d - 1 = sd - 1 \) many additions and multiplications. Valiant’s construction [Val79] implies that \( X_{1,1} X_{1,2} \cdots X_{1,s} + \cdots + X_{d,1} X_{d,2} \cdots X_{d,s} \) has the determinantal complexity at most \( sd \leq n \), i.e., it can be written as the determinant of an \( n \times n \)-matrix with affine linear entries in \( X_{1,1}, \ldots, X_{d,s} \). The determinantal complexity is invariant under invertible linear transformations. Hence the determinantal complexity of \( v_1, 1 v_1, 2 + \cdots + v_d, 1 v_d, 2 + \cdots + v_d, s \) is at most \( n \), for any linearly independent system \( v_1, 1, \ldots, v_d, s \) of linear combinations of \( X_{1,1}, \ldots, X_{n,2} \). By homogenizing with respect to a new variable \( Y \) and then substituting \( Y \) by \( Z \in V \), we see that \( Z^{n-s}(v_1, 1 v_1, 2 + \cdots + v_d, 1 v_d, 2 + \cdots + v_d, s) \in GL_{n^2}\text{det}_n \). Since \( GL_{n^2}\text{det}_n \) is closed, we can go over to the limit and drop the assumption that the \( v_{i,j} \) are linearly independent.
21.1(ii) Low rank evaluation

We present now a useful lemma on the evaluation of polynomials at "points of low rank".

21.1.2 Lemma. Let \( f \in \text{Sym}^d W \) be such that \( f(\sum_{j=1}^r v_j) \neq 0 \) for some \( v_1, \ldots, v_r \in W \). Then there exists \( S \subseteq \{1, \ldots, r\} \) with \( |S| \leq d \) and \( f(\sum_{j \in S} v_j) \neq 0 \).

Proof. Let \( [r] := \{1, \ldots, r\} \). For a subset \( S \subseteq [r] \) we write \( v_S := v_{S_1} + v_{S_2} + \cdots + v_{S_d} \). For a map \( \sigma : [d] \to [r] \) we write \( v_\sigma := v_{\sigma(1)} \odot v_{\sigma(2)} \odot \cdots \odot v_{\sigma(d)} \). In the following inclusion/exclusion calculation we can assume all sets \( S_i \) to have at most \( i \) elements. A two-headed arrow \("\twoheadrightarrow\"\) indicates a surjective map.

\[
\begin{align*}
\langle f, v^d \rangle &= \sum_{\sigma : [d] \twoheadrightarrow [r]} \langle f, v_\sigma \rangle \\
&= \sum_{S \subset [r]} \sum_{\sigma : [d] \twoheadrightarrow S} \langle f, v_\sigma \rangle \\
&= \sum_{S \subset [r]} \left( f(v_S) - \sum_{\sigma : [d] \twoheadrightarrow S, \sigma \text{ not surj}} \langle f, v_\sigma \rangle \right) \\
&= \sum_{S \subset [r]} \left( f(v_S) - \sum_{S_d \triangleleft S} \sum_{\sigma : [d] \twoheadrightarrow S_{d-1}} \langle f, v_\sigma \rangle \right) \\
&= \sum_{S \subset [r]} \left( f(v_S) - \sum_{S_{d-1} \triangleleft S_d} \left( f(v_{S_{d-1}}) - \sum_{S_{d-2} \triangleleft S_{d-1}} \sum_{\sigma : [d] \twoheadrightarrow S_{d-2}} \langle f, v_\sigma \rangle \right) \right) \\
&= \sum_{S \subset [r]} \left( f(v_S) - \sum_{S_{d-1} \triangleleft S_d} \left( f(v_{S_{d-1}}) - \sum_{S_{d-2} \triangleleft S_{d-1}} \left( \cdots \sum_{S_1 \triangleleft S_2} \sum_{\sigma : [d] \twoheadrightarrow S_1} \langle f, v_\sigma \rangle \right) \right) \right) \
\end{align*}
\]

Hence we see that \( f(v) \) can be expressed as a linear combination of evaluations \( f(v_S) \), where \( |S| \leq d \). Since \( f(v) \neq 0 \) there exists \( S \) such that \( f(v_S) \neq 0 \).

A direct corollary of Lemma 21.1.2 (using \( v_j \) to be the monomial basis) is the following.

21.1.3 Corollary. Let \( V \) be a finite dimensional \( \mathbb{C} \)-vector space and \( d, n \geq 1 \). If \( f \in \text{Sym}^d \text{Sym}^n V \) is nonzero, then there exists a polynomial \( h \in \text{Sym}^n V \) that has at most \( d \) nonzero coefficients such that \( f(h) \neq 0 \).

21.1(iii) The inner degree tableau lifting

Given a tableau with shape \( \lambda \) and content \( d \times n \) we define the lifted tableau to be the tableau of shape \( \lambda + (d) \) where we append the entries \( 1, \ldots, d \) to the first row. For example:

\[
\begin{array}{cccccc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \hline
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}
\]

If \( \tilde{T} \) is a HWV in \( \text{Sym}^d \text{Sym}^n V \), then \( \tilde{T}' \) is a HWV in \( \text{Sym}^{d+1} \text{Sym}^{n+1} V \), where \( T' \) is the lifted tableau of \( T \). The lifting is an injective map \( \text{HWV}_\lambda(\text{Sym}^d \text{Sym}^n V) \hookrightarrow \text{HWV}_{\lambda+(d)}(\text{Sym}^{d+1} \text{Sym}^{n+1} V) \). The proof can be readily obtained from the semigroup property of the partitions in the coordinate ring \( \mathbb{C}[\text{GL}_d(X_1 X_2 \cdots X_d)] \), but we omit the details.
We can apply the lifting several times and also call the result a lifted tableau.

Recall the monomial
\[ X^\alpha = \frac{1}{d!} \sum_{\pi \in \Phi_d} \pi(X_{\alpha_1} \otimes \cdots \otimes X_{\alpha_d}) \]

We define a linear map \( \text{Sym}^n V \to \text{Sym}^{n+1} V \) that is suited to this lifting (and which is basically a rescaled multiplication with \( X \))

\[
X_1 \boxtimes \left( \sum_{\pi \in \Phi_d} \pi(X_{\alpha_1} \otimes \cdots \otimes X_{\alpha_d}) \right) := \frac{1}{k} \sum_{\pi \in \Phi_{d+1}} \pi(X_{\alpha_1} \otimes \cdots \otimes X_{\alpha_d} \otimes X_1) 
\]

where \( k = |\{j \mid i_j = 1\}| + 1 \) is the \( X_1 \)-degree of the right-hand side. The crucial property is that the summands on the left-hand side are in bijection (taking into account the rescaling factor \( \frac{1}{k} \)) to the summands on the right-hand side which have \( X_1 \) as their last tensor factor. Using this property we see that

\[
\langle \tilde{v}_T, (p)^{\otimes d} \rangle = \langle \tilde{v}_{T'}, (X_1 \boxtimes p)^{\otimes d} \rangle \tag{21.1.4}
\]

by comparing summands in the tensor contractions. Applying \( X_1 \boxtimes h \) to a polynomial preserves the number of monomials that have a nonzero coefficient, but it changes the coefficients individually, depending on the number of occurrences of \( X_1 \) in each monomial. We write \( (X_1)^{n-m} \boxtimes p := X_1 \boxtimes (X_1 \boxtimes (\cdots \boxtimes p)) \).

**21.1.5 Corollary.** Let \( n, s, d \) be positive integers such that \( n \geq sd \) and \( Z, v_{1,1}, \ldots, v_{1,s}, v_{2,1}, \ldots, v_{d,s} \in V \). Then we have \( Z^{n-s} \boxtimes (v_{1,1}v_{1,2} \cdots v_{1,s} + \cdots + v_{d,1}v_{d,2} \cdots v_{d,s}) \in \mathcal{G}_{n^2 \det n} \).

**Proof.** This is an immediate consequence of Theorem 21.1.1 and the fact that rescaling coefficients does not increase the number of nonzero coefficients.

**21.1.6 Corollary.** If \( a_{\lambda}(d, n) > 0 \), then \( a_{\lambda \delta n}(d, N) > 0 \) for all \( N \geq n \).

**Proof.** If \( a_{\lambda}(d, n) > 0 \), then there is \( \tilde{v}_T \in \text{HWV}_\lambda(\text{Sym}^d \text{Sym}^n V) \) and \( h \in \text{Sym}^n V \) such that \( \tilde{v}_T(h) \neq 0 \). We obtain \( \tilde{v}_{T'} \) that satisfies \( \tilde{v}_{T'}(X_1^{N-n} \boxtimes h) = \tilde{v}_T(h) \neq 0 \) according to \( (21.1.4) \). Therefore \( a_{\lambda \delta n}(d, N) > 0 \).

We will use the following proposition with \( M = md \).

**21.1.7 Proposition.** Suppose that \( \lambda \vdash nd \) satisfies \( \lambda_2 \leq M \) and \( \lambda_2 + |\lambda| \leq Md \). Then every HWV of weight \( \lambda \) in \( \text{Sym}^d \text{Sym}^n V \) is obtained by lifting a HWV in \( \text{Sym}^d \text{Sym}^M V \) of weight \( \mu \), where \( \mu \vdash Md \) such that \( \mu = \lambda \).

**Proof.** Note that \( \lambda_2 + |\lambda| \leq Md \) is the number of boxes of \( \lambda \) that appear in columns that are not singleton columns. We can therefore shorten the given \( \lambda \) to a partition \( \mu \vdash Md \) by removing singleton columns. Indeed, if \( T \) is semistandard of shape \( \lambda \) with content \( d \times n \), then each number can appear in nonsingleton columns at most \( M \) times, because \( \lambda_2 \leq M \). Therefore shortening \( T \) to shape \( \mu \vdash Md \) can be done by removing \( (n-M) \) of each number 1, \ldots, \( d \) from singleton columns, which gives the HWV we searched for.

**21.1(iv) The degree lower bound**

If we want to separate with polynomials, then the following proposition gives a lower bound on the possible degree.
CHAPTER 21. GOOD OCC. OBSTRUCTIONS FOR DET VS PER DO NOT EXIST

21.1.8 Proposition. Let $\lambda \vdash nd$ be such that there exists a positive integer $m$ satisfying $|\lambda| \leq md$ and $d \leq \sqrt{n/m}$. Then every nonzero HWV of weight $\lambda$ in $\text{Sym}^d(\text{Sym}^nV)$ does not vanish on $\mathbb{C}[\text{GL}_n, \text{det}_n]$.

In particular, to show superpolynomial lower bounds on the border determinantal complexity of $\text{per}_m$, we need superpolynomially high degree $d$.

Proof. The case $d = 1$ is trivial as $(n)$ occurs in $\mathbb{C}[\text{GL}_n, \text{det}_n]$. So suppose $d \geq 2$.

Let $F \in \text{HWV}_\lambda(\text{Sym}^d\text{Sym}^nV)$. We have $\lambda_2 \leq |\lambda| \leq md$ and $\lambda_2 + |\lambda| \leq 2|\lambda| \leq 2md \leq md \cdot d$. Therefore, we are in the setting of Proposition 21.1.7 with respect to the lifting $\text{Sym}^d\text{Sym}^mV \rightarrow \text{Sym}^d\text{Sym}^nV$. We conclude that $F$ arises by an inner degree lifting from a HWV $f \in \text{Sym}^d\text{Sym}^mV$ of weight $\lambda - (d(n - m))$.

By Corollary 21.1.3, there are $v_1, \ldots, v_{d,m} \in V$ such that $f$ does not vanish on

$$p := v_1v_2 \cdots v_{d,m} + \cdots + v_{d,1}v_{d,2} \cdots v_{d,m}.$$

Using (21.1.4) we see that $F$ does not vanish on $q := X_1^{n-m} \boxtimes p$. By Corollary 21.1.5 we have $q \in \mathbb{C}[\text{GL}_n, \text{det}_n]$ since $n \geq md \cdot d$ (i.e., $d \leq \sqrt{n/m}$). Therefore, $F$ does not vanish on $\mathbb{C}[\text{GL}_n, \text{det}_n]$. $\square$

21.1(v) A first row that is too long

In a mostly analogous way one can prove the following proposition that takes care of cases with a huge $\lambda_1$. We omit the details.

21.1.9 Proposition. Let $\lambda \vdash nd$ and assume there exist positive integers $s, m$ such that $\ell(\lambda) \leq m^2$, $\lambda_2 \leq s$, $m^2s^2 \leq n$, and $m^2s \leq d$. Then every nonzero $h \in \text{HWV}_\lambda(\text{Sym}^d\text{Sym}^nV)$ of weight $\lambda$ does not vanish on $\mathbb{C}[\text{GL}_n, \text{det}_n]$.

21.2 No occurrence obstructions

In this section we prove that occurrence obstructions cannot prove superpolynomial lower bounds on $\text{dc}(\text{per}_m)$.

21.2.1 Theorem. Let $n, d, m$ be positive integers with $n \geq m^{25}$ and $\lambda \vdash nd$. If $\lambda$ occurs in $\mathbb{C}[\text{Z}^{n-m}\text{per}_m]$, then $\lambda$ also occurs in $\mathbb{C}[\text{GL}_n, \text{det}_n]$. In particular occurrence obstructions cannot show superpolynomial lower bounds on $\text{dc}(\text{per}_m)$.

Proof. We may assume that $m \geq 2$, as the case $m = 1$ is trivial. Suppose that $\lambda \vdash nd$ occurs in $\mathbb{C}[\text{Z}^{n-m}\text{per}_m]$ and $n \geq m^{25}$. Proposition 20.3.1 implies that $|\lambda| \leq md$ and $\ell(\lambda) \leq m^2$.

In the case of “small degree”, where $n \geq md^2$, Proposition 21.1.8 implies that $\lambda$ occurs in $\mathbb{C}[\text{GL}_n, \text{det}_n]$.

So we may assume that $d > \sqrt{n/m}$. In this case we have $d \geq \sqrt{m^{25}/m} = m^{12}$. We conclude by two further case distinctions.

If $|\lambda| < m^{10}$, we can apply Proposition 21.1.9 with $s := m^{10}$ since $\lambda_2 \leq |\lambda| \leq s$, $m^2s^2 = m^{22} \leq n$, and $m^2s = m^{12} \leq d$. Thus $\lambda$ occurs in $\mathbb{C}[\text{GL}_n, \text{det}_n]_{d}$.

Finally, if $|\lambda| \geq m^{10}$, then an explicit construction (Corollary 21.2.13) tells us that $\lambda$ occurs in $\mathbb{C}[\text{GL}_n, \text{det}_n]_{d}$. $\square$

The explicit construction mentioned in the proof crucially uses the so-called semigroup property that we introduce in the next section.
21.2(i) The semigroup property

To have an explicit construction of HWVs in $\mathbb{C}[\text{GL}_n \times \text{det}_n]$ we use the semigroup property, which allows us to construct HWVs as products of HWVs of smaller degrees.

Let $\mathbb{A} := \mathbb{C}^N$. A Zariski-closed subset $Z \subseteq \mathbb{A}$ is called an irreducible subvariety if $Z$ is not a union of two distinct proper Zariski-closed subsets of $Z$. Moreover, if $S$ is a multiplicity $m$ ring of $\mathbb{A}$, then $S$ is irreducible, i.e., $S$ has no zero divisors.

21.2.2 Lemma. The affine space $\mathbb{A} := \mathbb{C}^N$ is irreducible.

Proof. Let $V(f_1, \ldots, f_r)$ denote the simultaneous vanishing set of $f_1, \ldots, f_r$, where $f_i \in \mathbb{C}[\mathbb{A}]$. If $\mathbb{A}$ is reducible, then $\mathbb{A} = V(f_1, \ldots, f_r) \cup V(g_1, \ldots, g_s)$, where $f_i, g_j \in \mathbb{C}[\mathbb{A}]$, $f_i \neq 0$. In particular $\mathbb{A} \subseteq V(f_1) \cup V(g_1) = V(f_1g_1)$. Since $f_1g_1 \neq 0$ there exists a point $x \in \mathbb{A}$ with $(f_1g_1)(x) \neq 0$ (Lemma 1.3.2). But this means that $\mathbb{A} \not\subseteq V(f_1g_1)$, which is a contradiction.

21.2.3 Lemma. The orbit closure $\mathbb{C}[\text{GL}_n \times \text{det}_n] \subseteq \mathbb{C}[X_1, \ldots, X_n^2]_m$ is an irreducible subvariety.

Proof. By Lemma 21.2.2 the orbit matrix space $\mathbb{C}^{n \times n^2} = \text{End}_{n^2}$ is irreducible. Thus $\text{End}_{n^2}$ is irreducible.

We now prove that the orbit $\text{End}_{n^2}$ is irreducible. Indeed, every image $f(X)$ of an irreducible set $X$ under a continuous map $f$ is irreducible: $f(X) = f(Y) = f(Z)$ with nontrivial distinct Zariski-closed subsets $Y_1$ and $Y_2$. Thus $X = (f^{-1}(Y_1) \cup f^{-1}(Y_2))$. Moreover, if $f^{-1}(Y_1) = X$, then $f(X) = Y_1$, in contradiction to $Y_1 \not\subseteq f(X)$. Analogously it holds $f^{-1}(Y_2) \neq X$. Thus $X$ is a union of nontrivial closed subsets, in contradiction to $X$ being irreducible.

Now we prove that the closure $\overline{Y}$ of every irreducible set $Y$ is irreducible.

Assume that $\overline{Y}$ is not irreducible, i.e., $\overline{Y} = S \cup T$ with closed subsets $S \subseteq \overline{Y}$ and $T \subseteq \overline{Y}$, and $S, T \neq \overline{Y}$. We have $Y \subseteq \overline{Y} \subseteq S \cup T$. Thus $Y = (S \cap Y) \cap (T \cap Y)$, where both $(S \cap Y)$ and $(T \cap Y)$ are closed in $Y$ (subspace topology). The decomposition is nontrivial: if $S \cap Y = Y$, then $Y \subseteq S$, and therefore $\overline{Y} \subseteq \overline{S} = S$ and thus $\overline{Y} = S$, a contradiction. Analogously for $T$. Therefore $Y$ is not irreducible.

21.2.4 Claim. In a domain (i.e., a ring without zero divisors) we have:

$$ax = ay \text{ with } a \neq 0 \Rightarrow x = y.$$

Proof.

$$ax = ay \Rightarrow a(x - y) = 0 \Rightarrow a = 0 \text{ or } x = y.$$

21.2.5 Lemma. For an irreducible subvariety $Z$ the coordinate ring $\mathbb{C}[Z]$ has no zero divisors.

Proof. Let $f, g$ with $fg = 0$ in $\mathbb{C}[Z]$, i.e., $fg(z) = 0$ for all $z \in Z$. Since $fg$ vanishes on $Z$, $Z = V(fg) \cap Z = V(f) \cap Z \cup V(g) \cap Z$. Since $Z$ is irreducible, $V(f) \cap Z$ and $V(g) \cap Z$ cannot be both proper subsets of $Z$. Therefore either $V(f) \cap Z = Z$ or $V(g) \cap Z = Z$. W.l.o.g. $V(f) \cap Z = Z$, thus $f = 0$ in $\mathbb{C}[Z]$.

21.2.6 Proposition (The semigroup property). Let $G = \text{GL}_n$ act polynomially on $\mathbb{A}$. Let the cone $Z \subseteq \mathbb{A}$ be an irreducible subvariety that is closed under the action of $G$. Then the coordinate ring of $Z$ in degree $d$ is a $G$-representation and we have the following:

If the type $\lambda$ occurs with positive multiplicity $m_1$ in $\mathbb{C}[Z]_{d_1}$ and the type $\mu$ occurs with positive multiplicity $m_2$ in $\mathbb{C}[Z]_{d_2}$, then the type $\lambda + \mu$ occurs with multiplicity at least $\max(m_1, m_2)$ in $\mathbb{C}[Z]_{d_1 + d_2}$. 
21.2.10 Proposition. Given a partition \( \lambda \) with \( |\lambda| = nd \) such that there exists \( m \geq 2 \) with \( \ell(\lambda) \leq m^2, m^{10} \leq |\lambda| \leq md, n \geq 24m^6, \) and \( d > 4m^6 \). Then \( \lambda \) occurs in \( \mathbb{C}[GL_n, \text{det}_n]_d \).
Proof. Let $L := \ell(\lambda)$ and $c_k$ denote the number of columns of length $k$ in $\lambda$ for $1 \leq k \leq L$. Let $K$ be the index $k \geq 2$, for which $c_k$ is maximal, i.e., $c_K = \max(c_k; k = 2, \ldots, L)$. By assumption, we have $2 \leq K \leq m^2$ and

$$m^{10} \leq |\lambda| = \sum_{k=2}^{L} (k-1)c_k \leq c_k \sum_{k=2}^{L} (k-1) \leq c_K \frac{L^2}{2} \leq c_K \frac{m^4}{2},$$

hence $c_K \geq 2m^6$.

The columns of odd length of $\lambda$ need a special treatment: let $S$ denote the set of integers $k \in \{2, \ldots, L\}$ for which $c_k$ is odd. For $k \in S$ we define the partition

$$\omega_k := k \times 1 + K \times i_k,$$

where the even integer $i_k \leq 2m^4$ is taken from Theorem 21.2.9, so that $\omega_k^{3\lambda m^4}$ occurs in $C[GL_{\omega_2^{\lambda}} \det_{\omega_3^n}]^{3m^4}$. (Here we have used the assumption $n \geq 24m^6$.)

Assume first that $K \notin S$, that is, $c_K$ is even. Then we can split $\lambda$ vertically in rectangles as follows:

$$\lambda = 1 \times c_1 + \sum_{k \in U \cup \{K\}} k \times c_k + \sum_{k \in S} k \times c_k + K \times c_K$$

$$= 1 \times c_1 + \sum_{k \in U \cup \{K\}} k \times c_k + \sum_{k \in S} k \times (c_k - 1) + \sum_{k \in S} \omega_k + K \times \left(c_K - \sum_{k \in S} i_k\right).$$

If, for $k \leq L$, we set $d_k := c_k$ if $k \notin S \cup \{K\}$ and $d_k := c_k - 1$ if $k \in S$, and define $d_K := c_K - \sum_{k \in S} i_k$, then the above can be briefly written as

$$\lambda = 1 \times c_1 + \sum_{k \in S} k \times d_k + \sum_{k \in S} \omega_k.$$  \hspace{1cm} (21.2.11)

By construction, all $d_k$ are even. It is crucial to note that, using $i_k \leq 2m^4$,

$$d_K = c_K - \sum_{k \in S} i_k \geq c_K - (m^2 - 1) \cdot 2m^4 \geq c_K - 2m^6 \geq 0.$$  

The last inequality is due to our observation at the beginning of the proof.

In the case where $K \in S$, we achieve the same decomposition as in (21.2.11) with the modified definition $d_K := c_K - 1 - \sum_{k \in S} i_k$. Here, as well $d_K \geq 0$ and all $d_k$ are even.

We need to round down rational numbers to the next even number, so for $a \in \mathbb{Q}$ we define $\|a\| := 2 \lfloor a/2 \rfloor$. Note that $\|a\| \geq a - 2$ for all $a \in \mathbb{Q}$. Hence $\|n/k\| \geq n/k - 2 \geq 2$ for all $2 \leq k \leq m^2$, since $n \geq 4m^2$.

Using division with remainder, let us write $d_k = q_k \|n/k\| + r_k$ with $0 \leq r_k < \|n/k\|$. Then we split $k \times d_k = q_k (k \times \|n/k\|) + k \times r_k$. Since $d_k$ is even and $\|n/k\|$ is even, $r_k$ is even as well. From (21.2.11) we obtain that the partition

$$\mu := \sum_{k=2}^{L} q_k ((k \times \|n/k\|)^{2nk}) + \sum_{k=2}^{L} (k \times r_k)^{2nk} + \sum_{k \in S} \omega_k^{3\lambda m^4}$$  \hspace{1cm} (21.2.12)

coincides with $\lambda$ in all but possibly the first row.

Since $\|n/k\| \leq n/k$, $r_k \leq n/k$, and both $\|n/k\|$ and $r_k$ are even, Proposition 21.2.8 implies that $(k \times \|n/k\|)^{2nk}$ and $(k \times r_k)^{2nk}$ occur as highest weights in $C[GL_{\omega_2^{\lambda}} \det_{\omega_3^n}]$. Moreover, Theorem 21.2.9 tells us that $\omega_k^{3\lambda m^4}$ occurs as a highest weight in $C[GL_{\omega_2^{\lambda}} \det_{\omega_3^n}]^{3m^4}$. The semigroup property implies that $\mu$ occurs in $C[GL_{\omega_2^{\lambda}} \det_{\omega_3^n}]$. 

**\( \Box \)**
CHAPTER 21. GOOD OCC. OBSTRUCTIONS FOR DET VS PER DO NOT EXIST

Claim. $|\mu| \leq dn$.

Let us finish the proof assuming the claim. If $|\mu| \leq dn$, we can obtain $\lambda$ from $\mu$ by adding boxes to the first row of $\mu$. Note that $|\lambda| - |\mu|$ is a multiple of $n$. Since $(n) \in \mathbb{C}[GL_n,\det_n]$, the semigroup property implies that $\lambda$ occurs in $\mathbb{C}[GL_n,\det_n]_d$.

It remains to verify the claim. From (21.2.12) we get

$$|\mu| \leq \sum_{k=2}^{L} (q_k nk + nk + 3nm^4).$$

We have, using $\|n\| \geq a - 2$,

$$q_k \leq \frac{d_k}{[n/k]} \leq \frac{kd_k}{n - 2k}.$$ 

This implies

$$|\mu| \leq n \sum_{k=2}^{L} \left( \frac{k^2d_k}{n - 2k} + k + 3m^4 \right).$$

Using $d_k \leq c_k$ and $L \leq m^2$, we get

$$|\mu| \leq n \sum_{k=2}^{L} \frac{m^2}{n - 2m^2}kc_k + n \sum_{k=2}^{m^2} k + 3nm^4(m^2 - 1).$$

Noting that $\sum_{k=2}^{L} kc_k = |\bar{\lambda}| + \lambda_2 \leq 2|\bar{\lambda}|$, we continue with

$$|\mu| \leq \frac{nm^2}{n - 2m^2} \cdot 2|\bar{\lambda}| + n \left( \frac{m^2(m^2 + 1)}{2} + 3m^4(m^2 - 1) \right)$$

$$\leq \frac{nm^2}{12m^6 - m^2} \cdot |\bar{\lambda}| + n \left( 3m^6 - \frac{5}{2}m^4 + \frac{1}{2}m^2 \right),$$

where we have used $n > 24m^6$ for the second inequality. Plugging in the assumptions $|\bar{\lambda}| \leq dm$ and $d > 4m^6$, we obtain

$$|\mu| \leq \frac{dnm^3}{11m^6} + 3nm^6 \leq \frac{dn}{11} + 3nm^6 \leq \frac{dn}{11} + \frac{3dn}{4} \leq dn,$$

which shows the claim and completes the proof. \hfill \square

21.2.13 Corollary. Let $m \geq 2$, $n \geq m^{25}$, $\lambda \vdash nd$, $|\bar{\lambda}| \leq md$, $\ell(\lambda) \leq m^2$, $d > \sqrt{n/m}$, $|\bar{\lambda}| \geq m^{10}$. Then $\lambda$ occurs in $\mathbb{C}[GL_n,\det_n]_d$.

Proof. To apply Proposition 21.2.10 we need to ensure that $n \geq 24m^6$ and $d > 4m^6$. Indeed, $n \geq m^{25} \geq 32m^{30} \geq 24m^6$ and $d > \sqrt{n/m} \geq \sqrt{m^{24}} = m^{12} \geq 4m^{10} \geq 4m^6$. \hfill \square

At this point, to finish the proof of Theorem 21.2.1 it just remains to prove Theorem 21.2.9.

21.2(iii) Explicit constructions of tableaux and positivity of plethysms

The goal of this section is to provide the proof of Theorem 21.2.9, which finishes the proof of Theorem 21.2.1. We achieve this by a direct construction of a HWV of type $\lambda$. The first Proposition 21.2.14 treats a simple case, while Proposition 21.2.15 covers the full generalization. Since the degree is low enough, we can then use the methods from Section 21.1 to show that $\lambda$ occurs in $\mathbb{C}[GL_n,\det_n]$. 


21.2.14 Proposition. Let \( t \geq r, i \geq 2t + 3 \) be positive integers and let \( n \geq i \) and \( d \geq 2t + i + 1 \). Let \( \nu = (t + 1) \times i + (r + 1) \times 1 + (j) \), where \( j = dn - (t + 1)i - (r + 1) \). Then \( a_{\nu}(d[n]) > 0 \).

Proof. We may assume that \( n = i \) and \( d = 2t + i + 1 \) (see Lemma 21.2.7 for \( d > 2t + i + 1 \) and Corollary 21.1.6 if \( n > i \)).

Let \( T \) be a tableau of shape \( \nu \) labeled with the integers \( 1, 2, 3, \ldots, d \), each appearing \( n \) times, as explained in Figure 21.1 for the case \( t = 5, r = 3 \) and \( i = 13 \). Formally, if \( 1 \leq k \leq r \), the row \( \begin{array}{ccccccccccccccccccc}
11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 1 & 1 & 1 \vdots \\
1 & 1 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\
2 & 2 & 2 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
3 & 3 & 3 & 3 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
4 & 4 & 4 & 4 & 4 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
5 & 5 & 5 & 5 & 5 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
\end{array} \)

Figure 21.1: Prop. 21.2.15: \( t = 5, r = 3, i = 13, d = 24, n = 13, D = 10, dn = 312, j = 230 \).

\( k + 1 \) of \( T \) has \( i + 1 \) boxes: \( k + 1 \) boxes are labeled \( k \), and the remaining \( i - k \) boxes are labeled \( 2t + 1 - k \). If \( r < k \leq t \), then the row \( k + 1 \) of \( T \) has \( i \) boxes: \( k + 1 \) boxes are labeled \( k \) and the remaining \( i - k - 1 \) boxes labeled \( 2t + 1 - k \). The first row of \( T \) starts with the first \( i + 1 \) boxes labeled with \( 2t + 1, \ldots, d = 2t + i + 1 \), respectively, and all the remaining \( j \) labels are put in the singleton columns of \( T \) such that each integer in \( 1, \ldots, d \) appears exactly \( n \) times. Note that each integer \( 1, \ldots, d \) appears in at least one singleton column, since \( n \geq i \geq 2t + 3 \).

Put \( D := 2t \). By construction, for any \( 1 \leq u \leq D \) in \( T \), \( u \) appears in row \( 1 \) and in a unique row \( k_u + 1 \) for some \( 1 \leq k_u \leq t \). Let \( \beta(u) \) denote the number of occurrences of \( u \) in row \( k_u + 1 \). Note that \( 2 \leq \beta(1) < \beta(2) < \ldots < \beta(D) \) by construction.

We consider now the tensor

\[
\Phi := \bigotimes_{u=1}^{D} (e_{k_u+1}^u \otimes e_1^{(n-\beta(u))}) \otimes \bigotimes_{u=D+1}^{d} e_1^n.
\]

which, more precisely, is defined by the map, \( s: [dn] \to \mathbb{C}^N \),

\[
s_{(u-1)n+v} = \begin{cases} 
  e_{k_u+1}^u & \text{if } 1 \leq u \leq D \text{ and } 1 \leq v \leq \beta(u) \\
  e_1 & \text{otherwise.}
\end{cases}
\]

Since \( \Phi \) is of rank 1, the tensor contraction \( \tilde{\nu}_T, \Phi \) from Section 20.1 simplifies: Since the \( \beta(u) \) are all distinct, the only nonzero summands in the expansion of \( \tilde{\nu}_T, \Phi \) satisfy \( (s \circ \vartheta)(\emptyset) = e_{\text{row}(\emptyset)} \).

These summands have \( \text{val}_0(s) = 1 \), which makes the overall contraction nonzero. \( \square \)

By generalizing this construction in the proof, we can show the following.

21.2.15 Proposition. Let \( t, r \) be positive integers, \( i \in \left\lceil \frac{(r+2)t^2}{2t}, \frac{(r+2)t^2}{2t} + r + t + 1 \right\rceil \), and let \( n > 6t + 2r \) and \( d > r + 2t + i \). Let \( \nu = (t + 1) \times i + (r + 1) \times 1 + (j) \), where \( j = dn - (r + 1) - (t + 1)i \). Then \( a_{\nu}(d[n]) > 0 \).

Proof. If \( r < t \) then we can directly apply Proposition 21.2.14, noticing that

\[
2t + 2 < \frac{(1 + 2t)^2}{2t} \leq \frac{(r + 2t)^2}{2t} \leq i \leq \frac{(t + 2t)^2}{2t} + r + t + 1 \leq \frac{11}{2} t + r + 1 \leq 6t + r \leq n.
\]
Let now \( r \geq t \). The proof is similar to the proof of Proposition 21.2.14, so we describe a more general construction which applies in the case \( r < t \) as well. Define \( e := 2\left\lfloor \frac{(r-1)}{(2t)} \right\rfloor + 1 \), so that \( r \leq te \leq r + 2t - 1 \) and \( e \) is even. Put

\[
i' := (te + 1)\frac{e}{2} \leq (r + 2t)\frac{e}{2} \leq (r + 2t)\left\lfloor \frac{r-1}{2t} \right\rfloor + 1 \leq (r + 2t)\frac{r + 2t - 1}{2t} \leq i.
\]

We will prove the statement for \( i = i' \). When \( i > i' \), the tableau construction below can be modified by increasing the number of appearances of the \( t \) largest labels by \( i - i' \leq r + t \) in the subtableau \( T' \) as defined below. By assumption, \( n > 6t + 2r \geq te + 2 \) and \( d > r + 2t + i \geq te + i + 1 \). Indeed, we will prove the statement for the more general case in which we do not require \( n > 6t + 2r \) and \( d > r + 2t + i \), but only \( n \geq te + 2 \) and \( d \geq te + i + 1 \). It suffices to prove the statement with \( n = te + 2 \) and \( d = te + i + 1 \).

Let \( T \) be a tableau of shape \( \nu \) filled with the labels \( 1, 2, 3, \ldots, d = te + i + 1 \), each number appearing \( n = te + 2 \) times, as in Figure 21.2 for the case \( t = 2, r = 8, e = 4, i = 18, n = 10, d = 27 \).

![Figure 21.2](image_url)

In the first row and in the first \( i + 1 \) columns we have the labels \( te + 1, \ldots, te + i + 1 \). In the first column and in the rows \( 2 \) to \( r + 1 \) we have the labels \( te, te - 1, \ldots, te - r + 1 \). The remaining rectangular \( t \times i \) subshape of \( T \), denoted \( T' \), consisting of the columns \( 2 \) to \( i + 1 \) and the rows \( 2 \) to \( t + 1 \), is filled with the remaining labels \( 1, \ldots, te \), so that each label appears a different number of times. More precisely, for each \( 1 \leq s \leq te \), let the label \( s \) appear in \( T' \) exactly \( s \) times and only in row \( \min\{\ell, 2t - \ell + 1\} \), where \( s \equiv \ell \) (mod \( 2t \)), \( 1 \leq \ell \leq 2t \). (Note that the first row in \( T' \), which we are referring to, is actually the second row in \( T \).) So the row \( k \) of \( T' \) contains the \( e \) different labels \( k, 2t + 1 - k, 2t + k, 4t + 1 - k, \ldots, t(e - 2) + k, te + 1 - k \), each appearing that many times, adding up to the row length of

\[
\sum_{\alpha=1}^{e/2} \left( 2(\alpha - 1) + k \right) = (te + 1)\frac{e}{2} = i.
\]

The remaining labels of each kind are then put in the singleton boxes of \( T \).

As in Proposition 21.2.14, we show that the corresponding highest-weight vector \( \tilde{v}_\nu \) in \( \text{HWV}_e(Sym^d Sym^n V) \) is nonzero by contracting it with a particular monomial tensor \( \Phi \). For each label \( u \), \( 1 \leq u \leq d \), let the associated monomial be

\[
m_u = \bigotimes_{\square \in T, \text{row}(\square) = u} X_{\text{row}(\square)},
\]

where the product goes over all boxes of \( T \) labeled \( u \) and for each such box we take the variable \( X \) whose index is the row the box is in. Again, let \( \Phi := \otimes_{u=1}^d m_u \) be the tensor. The nonvanishing of the contraction \( \langle \nu \tau, \Phi \rangle \) can be seen analogously as in Proposition 21.2.14.
Finally we can complete the proof of the promised technical result.

**Proof of Theorem 21.2.9.** We apply Proposition 21.2.15 with 
\[ r = b - 1 \leq m^2 - 1 \] and 
\[ t = c - 1 \leq m^2 - 1. \] We have
\[
\frac{(r + 2t)^2}{2t} = \frac{(b + 2c - 3)^2}{2(c - 1)} \leq \max \left( \frac{(b + 1)^2}{2}, \frac{(b + 2m^2 - 3)^2}{2(m^2 - 1)} \right) \leq m^4,
\]
where we use the fact that \((b + 2c - 3)^2/(2(c - 1))\) is a convex function of \(c\) and so attains its maximum at the end points of the interval \([2, m^2]\). We can then find an even integer \(i\) in the interval 
\[
\left[ \frac{(r + 2t)^2}{2t}, \frac{(r + 2t)^2}{2t} + r + t + 1 \right] \subseteq [1, m^4 + 2m^2].
\]
By Proposition 21.2.15, there exists a highest weight vector \(f\) of weight \(\nu = b \times 1 + c \times i + 1 \times j'\) in \(\text{Sym}^d \text{Sym}^N V\) for
\[
d := 3m^4 > 3m^2 + 2m^2 + m^4 \geq r + 2t + i, \quad N := 8m^2 > 6t + 2r.
\]

By Proposition 21.1.3 we have \((f, h^d) \neq 0\) for a generic polynomial \(h \in \text{Sym}^N V\) that has at most \(d\) nonzero coefficients. Moreover, by Corollary 21.1.5, \(q := X_1^{n-N} \boxplus p\) is contained in \(\mathfrak{GL}_{n^2 \det n}\) for all \(n \geq dN\), in particular for \(n \geq 24m^6\). Consider the lifting \(F \in \text{Sym}^d \text{Sym}^n V\) of \(f\); it has the weight \(\lambda = \nu^{dn}\) with \(dn = 3m^4n\). By (21.1.4) we see that \(F(X_1^{n-m} \boxplus h) = f(h) \neq 0\). Therefore, \(\lambda\) occurs in \(C[\mathfrak{GL}_{n^2 \det n}]_{3m^4}\).

**No occurrence obstructions**

By an explicit construction of HWVs in \(C[\mathfrak{GL}_{n^2 \det n}]\) we showed that occurrence obstructions are too weak to separate \(Z^{n-m} \boxplus \text{per}_n\) from \(\mathfrak{GL}_{n^2 \det n}\) for \(n\) being superpolynomially large in \(m\).

The proof relied heavily on the padding of the permanent (i.e., the partition having a very long first part). Other algebraic computational models do not have that property. Multiplicity obstructions could also still be an option to prove Valiant’s conjecture.
Chapter 22

Occurrence obstructions for matrix multiplication

In this last chapter we show that obstructions can indeed show complexity lower bounds. In most parts we follow [BI11], [BI13], and [Ike12].

We will construct occurrence obstructions that show the border rank lower bound
\[ R(\langle m,m,m \rangle) \geq \frac{3}{2} m^2 - 2. \]

Recall from the end of Chapter 14 that
\[ R(\langle m,m,m \rangle) \leq n \text{ iff } \langle m,m,m \rangle \in \text{GL}_n^3 \langle n \rangle, \]
which is equivalent to \( \text{GL}_n^3 \langle m,m,m \rangle \subseteq \text{GL}_n^3 \langle n \rangle \) (see Lemma 3.4.3). By Proposition 18.3.1 the irreducible polynomial representations of \( \text{GL}_n^3 \langle n \rangle \) are given by triples \( \lambda = (\lambda(1), \lambda(2), \lambda(3)) \) of partitions into at most \( n \) parts each. For odd \( m \) and \( n \geq \frac{3}{2} m^2 - 1 \) let \( \kappa := \frac{m^2 - 1}{2} \) and set \( \lambda(1) = \lambda(2) = \lambda(3) = ((2\kappa + 1) \times 1) + (1 \times \kappa) \) and \( d = |\lambda| = 3\kappa + 1 \). We will show that
\[ \text{mult}_\lambda(\mathbb{C}[\text{GL}_n^3 \langle n \rangle]_d) = 0 \] (22.0.1)
\[ < 1 = \text{mult}_\lambda(\mathbb{C}[\text{GL}_n^3 \langle m,m,m \rangle]_d) \] (22.0.2)

The irreducible representation in the coordinate ring of the ambient space is unique: \( \text{mult}_\lambda(\mathbb{C}[\otimes^3 \mathbb{C}^n]) = 1 \), which is proved in [Rem89, Thm. 2.1], see also [Ros01, Thm. 3(4.)]. We construct the HWV of weight \( \lambda \) and evaluate it at a point in \( \text{GL}_n^3 \langle m,m,m \rangle \) where the evaluation is nonzero, which proves (22.0.2), see Section 22.3. For (22.0.1) we present two proofs. In the first one we examine the HWV that we constructed (see Section 22.2), while in the second one (see Section 22.4) we use the approach presented in (18.4.1), i.e., we study
\[ \text{mult}_{\lambda,\lambda',\lambda''}(\mathbb{C}[\text{GL}_n^3 \langle n \rangle]) \leq \text{mult}_{\lambda,\lambda',\lambda''}(\mathbb{C}[\text{GL}_n^3 \langle n \rangle]) = \dim(\{\lambda\} \otimes \{\lambda'\} \otimes \{\lambda''\})^{\text{stab}(n)}. \] (22.0.3)

22.1 Highest weight vectors in the tensor setting

In this section we describe the HWVs in \( \otimes^d(\otimes^3 \mathbb{C}^n) \), because each HWV in \( \text{Sym}^d(\otimes^3 \mathbb{C}^n) \) arises from one in \( \otimes^d(\otimes^3 \mathbb{C}^n) \) via symmetrization over \( \mathfrak{S}_d \). This is analogous to studying semistandard tableaux with rectangular content in Section 19.3. The natural equivalent to semistandard tableaux with rectangular content are triples of standard tableaux, but we want to use a more graphical description via hypergraphs.
22.1(i) Highest Weight Vectors

For a partition \( \lambda \vdash d \) recall the definition of \( v_\lambda \) from (19.2.2). We want to use the convenient bra-ket notation

\[
\langle \lambda \rangle := v_\lambda. \tag{22.1.1}
\]

We write \( \lambda \vdash d \) to denote that \( \lambda = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) \) with \( \lambda^{(1)} \vdash d \), \( \lambda^{(2)} \vdash d \), and \( \lambda^{(3)} \vdash d \). For a partition triple \( \lambda \vdash d \) we define

\[
\langle \lambda \rangle := \text{reorder}_{3,0}(\langle \lambda^{(1)} \rangle \otimes \langle \lambda^{(2)} \rangle \otimes \langle \lambda^{(3)} \rangle),
\]

where for \( a, b \in \mathbb{N} \) the linear isomorphism \( \text{reorder}_{a,b} : \otimes^a \otimes^b \mathbb{C}^n \rightarrow \otimes^a \otimes^b \mathbb{C}^n \) is defined on rank 1 tensors as follows:

\[
\begin{pmatrix}
  a \\
  b \\
  v_{ij}
\end{pmatrix}_{i=1}^{a} \quad \begin{pmatrix}
  b \\
  a \\
  v_{ij}
\end{pmatrix}_{j=1}^{b} \mapsto \begin{pmatrix}
  a \\
  b \\
  v_{ij}
\end{pmatrix}_{i=1}^{a} \begin{pmatrix}
  b \\
  a \\
  v_{ij}
\end{pmatrix}_{j=1}^{b}, \quad v_{ij} \in \mathbb{C}^n. \tag{22.1.3}
\]

Since \( \langle \lambda \rangle \) is a HWV of weight \( \lambda \), \( \langle \lambda \rangle \) is a highest weight vector of weight \( \lambda \) in \( \otimes^d \otimes^3 \mathbb{C}^n \). Moreover, since the \( \langle \lambda \rangle, \pi, \pi, \pi \in \mathcal{S}_d \) generate \( \text{HWV}_{\lambda}(\otimes^d \otimes^3 \mathbb{C}^n) \) as a vector space, we also see following:

**22.1.4 Claim.** The highest weight vector space \( \text{HWV}_{\lambda}(\otimes^d \otimes^3 \mathbb{C}^n) \) is generated by \( \langle \lambda \rangle \pi \) with \( \pi \in \mathcal{S}_d \).

Embed \( \mathcal{S}_d \hookrightarrow \mathcal{S}_d^3, \pi \mapsto (\pi, \pi, \pi) \). Let \( \mathcal{P}_d : \otimes^d \otimes^3 \mathbb{C}^n \rightarrow \text{Sym}^d \otimes^3 \mathbb{C}^n \) denote the symmetrization over \( \mathcal{S}_d \). Since the actions of \( \text{GL}_d \) and \( \mathcal{S}_d^3 \) commute, we draw the following important conclusion.

**22.1.5 Claim.** The tensors \( \langle \lambda \rangle \pi \mathcal{P}_d \) with \( \pi \in \mathcal{S}_d^3 \) generate \( \text{HWV}_{\lambda}(\text{Sym}^d \otimes^3 \mathbb{C}^n) \).

22.1(ii) Set Partitions

In this subsection we start deriving a more graphical interpretation of the highest weight vectors. Let \( \varphi(S) \) denote the powerset of a finite set \( S \), i.e., the set of all subsets of \( S \). Given a set \( S \), we call a subset \( \Lambda \subseteq \varphi(S) \) of the powerset \( \varphi(S) \) a set partition of \( S \), if for all \( s \in S \) there exists exactly one set \( e_s \in \Lambda \) with \( s \in e_s \). We call \( e_s \) the hyperedge corresponding to \( s \). If \( |e_s| = 1 \), then \( e_s \) is called a singleton hyperedge. The type of a set partition \( \Lambda \) is defined as the partition \( \mu \vdash |S| \) obtained from sorting the multisets \( \{e | e \in \Lambda \} \) and transposing the partition afterwards. Let \( V(\Lambda) := \bigcup_{e \in \Lambda} e = S \) denote the ground set.

For a given partition \( \lambda \vdash d \), we can define a canonical set partition \( \Lambda \) of \( \{1, \ldots, d\} \) as follows, where \( \mu \) is the transpose of \( \lambda \):

\[
\omega_i := \sum_{j=1}^{i} |\mu_j| \quad \text{be the number of boxes in the first } i \text{ columns of } \lambda.
\]

We define the disjoint hyperedges

\[
e_i := \{\omega_{i-1} + 1, \omega_{i-1} + 2, \ldots, \omega_i\}
\]

and set \( \Lambda := \{e_i | 1 \leq i \leq \lambda_1\} \). For example, if \( \lambda = (4,3,1) \), then \( \mu = (3,2,2,1) \) and \( e_1 = \{1,2,3\}, e_2 = \{4,5\}, e_3 = \{6,7\}, e_4 = \{8\} \), corresponding to the superstandard tableau \( T_\lambda \) (see Section 19.2) of shape \( \lambda \), see Figure 22.1.

Analogously, from any Young tableau \( T \) of \( \lambda \) we obtain a set partition \( \Lambda_T \) of \( \{1, \ldots, d\} \) with type \( \lambda \) by grouping together each column in a hyperedge.

The map \( T \mapsto \Lambda_T \) is not injective in general. Our aim is to classify the fibers. Two Young tableaux \( T \) and \( T' \) are mapped to the same \( \Lambda_T = \Lambda_{T'} \), iff \( T' \) can be obtained from \( T \) by permuting entries inside of columns and by permuting whole columns of the same length. This observation gives rise to the following definition (see the right hand side of Figure 22.2 for an example).
22.1.6 Definition. An ordered set partition Λ of a vertex set \( V(Λ) := \{1, \ldots, d\} \) of type \( λ \) is a set partition of \( V(Λ) \) of type \( λ \) endowed with (1) linear orderings on each hyperedge \( e ∈ Λ \) and (2) for each length \( 1 ≤ ℓ ≤ ℓ(λ) \) a linear ordering on the set \( \{e ∈ Λ : |e| = ℓ\} \) of hyperedges with the same cardinality \( ℓ \).

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
7 & 8 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{pmatrix}
\]

Figure 22.2: The bijections between permutations, Young tableaux, and ordered set partitions of type \( λ = (4, 3, 1) \). The orderings are shown with arrows pointing from the smaller element to the bigger.

The above discussion gives an explicit bijection between the set of Young tableaux \( T \) of \( λ \) and the set of ordered set partitions of type \( λ \) by grouping together each column of \( T \) in one hyperedge, ordered from top to bottom, and ordering hyperedges of equal length by their appearance in \( T \) from left to right, see Figure 22.2 for an example. The columns of \( T \) are ordered from left to right and this induces an additional linear ordering on the set of hyperedges of an ordered set partition, which is consistent with the single linear orderings of hyperedges of the same length. In particular we can speak of the \( i \)th hyperedge of \( Λ \). Additionally, since the hyperedges are linearly ordered, we have a linear order on the set of vertices \( V(Λ) \) and hence we can write \( V(Λ)_i \) for the \( i \)th element of \( V(Λ) \).

The permutations \( π ∈ S_d \) are in bijection to the Young tableaux of \( λ \) via replacing the entry \( i \) in \( T_λ \) with the integer \( π^{-1}(i) \). Hence we get an explicit bijection between \( S_d \) and the set of ordered set partitions of type \( λ \). For a given ordered set partition \( Λ \) of type \( λ \) we denote by \( π_λ \) the corresponding permutation. We can state a first observation:

\[
π_λ(V(Λ)_i) = i. \quad (22.1.7)
\]

The crucial property of our bijection is shown in the upcoming Claim 22.1.8, for whose statement we introduce some notation.

The group \( S_d \) acts naturally on \( (C^n)^d \) by permuting the positions as follows:

\[
π(ζ_1, ζ_2, \ldots, ζ_d) := (ζ_{π^{-1}(1)}, ζ_{π^{-1}(2)}, \ldots, ζ_{π^{-1}(d)}).
\]

Given a linearly ordered subset \( (e, \prec) ⊆ \{1, \ldots, d\} \) with \( ℓ \) elements, \( ℓ ≤ d \), where the order \( \prec \) is not necessarily consistent with the natural order on \( \{1, \ldots, d\} \), we define the list elements \( e^1, \ldots, e^ℓ \)
for a fixed partition $\lambda \vdash d$ and let $e_i$ denote the $i$th column of $T_\lambda$. Let $\Lambda$ be an ordered set partition. Then, for the $i$th hyperedge $e$ of $\Lambda$, we have

$$(\zeta_1, \zeta_2, \ldots, \zeta_d)_e = (\pi(\zeta_1, \zeta_2, \ldots, \zeta_d))_{e_i}$$

for all $\zeta \in (\mathbb{C}^n)^d$.

**Proof.** Note that $\pi(e^i) = e^{i_j}$ according to (22.1.7). Now the proof is straightforward as follows:

$$(\zeta_1, \zeta_2, \ldots, \zeta_d)_e = (e^{i_1}, \ldots, e^{i_d}) = (\zeta_{\pi^{-1}(e^{i_1})}, \ldots, \zeta_{\pi^{-1}(e^{i_d})})$$

$$(\pi(\zeta_1, \zeta_2, \ldots, \zeta_d))_{e_i}$$

$\square$

**From set partition triples to highest weight vectors** Our main motivation for looking at set partitions is the construction of highest weight vectors. For each ordered set partition $\Lambda$ of type $\lambda \vdash d$ we have $\pi_\lambda \in S_d$ and hence obtain a nonzero highest weight vector

$$f_\lambda := \pi_\lambda(\hat{\lambda})$$

of weight $\lambda$, provided $n \geq \ell(\lambda)$. We conveniently write $f_\lambda = (\hat{\lambda})|\pi_\lambda$. This roughly corresponds to $v_T$ in Chapter 20.

We next want to determine the *projective stabilizer* $Y_\lambda \subseteq S_d$ of $\langle \hat{\lambda} \rangle$, which is defined as

$$Y_\lambda := \{ \tau \in S_d : \langle \hat{\lambda} \rangle|\tau = \pm \langle \hat{\lambda} \rangle \}. \quad (22.1.9)$$

Consider the Young subgroup $Y_\lambda^{\text{inner}} := S(e_1) \times \cdots \times S(e_{\lambda_1})$, where we recall that $e_i$ denotes the $i$th column of $T_\lambda$. For $\tau \in Y_\lambda^{\text{inner}}$ we have $\langle \hat{\lambda} \rangle|\tau = \text{sgn}(\tau)|\langle \hat{\lambda} \rangle$, hence $Y_\lambda^{\text{inner}} \subseteq Y_\lambda$. Let $Y_\lambda^{\text{outer}}$ denote the group that interchanges columns of the same length in $T_\lambda$ while preserving the order in each column. For $\tau \in Y_\lambda^{\text{outer}}$ we have $\langle \hat{\lambda} \rangle|\tau = \langle \hat{\lambda} \rangle$, hence $Y_\lambda^{\text{outer}} \subseteq Y_\lambda$. One can prove that the projective stabilizer $Y_\lambda$ is the group generated by $Y_\lambda^{\text{inner}}$ and $Y_\lambda^{\text{outer}}$.

We are interested in classifying the left cosets of $Y_\lambda \subseteq S_d$. The ordered set partition corresponding to $\pi$ and the ordered set partition corresponding to $\tau \pi$ for $\tau \in Y_\lambda^{\text{inner}}$ are the same up to reordering the elements in each hyperedge. For $\tau \in Y_\lambda^{\text{outer}}$ the ordered set partitions corresponding to $\pi$ and $\tau \pi$ are the same up to reordering the hyperedges. All reorderings can be obtained by applying elements of $Y_\lambda$. If we forget about the orderings of ordered set partitions, we obtain the following claim.

**22.1.10 Claim.** For a fixed partition $\lambda \vdash d$ there is a bijection between the left cosets of $Y_\lambda \subseteq S_d$ and the set of set partitions of type $\lambda$.

Hence a set partition $\Lambda$ of type $\lambda \vdash d$ uniquely determines a highest weight vector of weight $\lambda$

$$f_\lambda := (\hat{\lambda})|\pi_\lambda \in \otimes^d \mathbb{C}^n$$

up to a sign, where $\pi_\lambda$ is the permutation corresponding to some ordering of $\Lambda$.
22.1.12 Definition. An ordered set partition triple \( \mathcal{H} \) consists of a vertex set \( V(\mathcal{H}) = \{1, \ldots, d\} \) and three ordered set partitions \( E^{(k)} := E^{(k)}(\mathcal{H}), k \in \{1, 2, 3\} \), of \( V(\mathcal{H}) \). The elements of each \( E^{(k)} \) are called hyperedges.

The ordered set partition triple \( \mathcal{H} \) is said to have type \( \lambda \), where \( \lambda \vdash d \) is a partition triple, if the set partition \( E^{(k)} \) has type \( \lambda^{(k)} \) for all \( 1 \leq k \leq 3 \).

Via our explicit bijections between \( \mathcal{S}_d \) and the set of set partitions of a fixed type, we get an explicit bijection between \( \mathcal{S}_d^3 \) and the set of ordered set partition triples of type \( \lambda \), see Figure 22.3.

The permutation triple corresponding to an ordered set partition \( \mathcal{H} \) is denoted by \( \pi_\mathcal{H} \).
CHAPTER 22. OCCURRENCE OBSTRUCTIONS FOR MATRIX MULTIPLICATION

128

\[ \sigma^{(1)} = \left( \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 4 & 3 \end{array} \right) \]
\[ \sigma^{(2)} = \left( \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 4 & 2 \end{array} \right) \]
\[ \sigma^{(3)} = \left( \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 4 & 3 \end{array} \right) \]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure22.3.png}
\caption{A set partition triple \( \mathcal{H} \) of type \((\lambda, \lambda, \lambda)\), where \( \lambda = (2, 1, 1) \) (and thus \( \mu = (3, 1) \)), and its corresponding permutation triple \( \pi_{\sigma} \) with inverse \( \sigma \). The solid lines represent the first hyperedge set partition, the dashed lines represent the second one, and the dotted lines represent the third one. To simplify the picture, the hyperedge orderings respect the natural ordering on the natural numbers and are not depicted.}
\end{figure}

Analogously to (22.1.9), for partition triples \( \lambda \vdash d \) we define the projective stabilizer \( Y_\lambda \) of \( (\lambda) = \text{reorder}_3, n \left( (\lambda^{(1)}) \otimes (\lambda^{(2)}) \otimes (\lambda^{(3)}) \right) \), see (22.1.2), as
\[ Y_\lambda := \{ \tau \in S^3_d : (\lambda)\tau = \pm (\lambda) \}. \]
One can show that \( Y_\lambda \) = \( Y_{\lambda^{(1)}} \times Y_{\lambda^{(2)}} \times Y_{\lambda^{(3)}} \), where \( Y_{\lambda^{(k)}} \) is the projective stabilizer defined in (22.1.9). Therefore we can again forget about the orderings and arrive at the following definition.

22.1.13 Definition. A set partition triple \( \mathcal{H} \) consists of a vertex set \( V(\mathcal{H}) \) = \( \{1, \ldots, d\} \) and three set partitions \( E^{(k)}, k \in \{1, 2, 3\} \) of \( V(\mathcal{H}) \).

The above discussion implies the following claim, analogously to Claim 22.1.10.

22.1.14 Claim. For a fixed partition triple \( \lambda \vdash d \) there is a bijection between the left cosets of \( Y_\lambda \subseteq S_d \) and the set of set partition triples of type \( \lambda \).

So each set partition triple \( \mathcal{H} \) defines (up to sign) the highest weight vector
\[ f_{\mathcal{H}} := \pm (\lambda)\pi_{\sigma} \in \bigotimes^3 d C^n \]
of weight \( \lambda^* \), where \( \lambda \vdash d \) denotes the type of \( \mathcal{H} \).

Triple Contraction A finite sequence of vectors in \((C^n)^3\) shall be called a triple list. Given a triple list \( \zeta \) containing \( d \) triples, we write
\[ \zeta = \left( \begin{array}{c}
\zeta^{(1)} \\
\zeta^{(2)} \\
\zeta^{(3)}
\end{array} \right) \]
Moreover, we write \( \zeta^{(k)} := (\zeta_1^{(k)}, \ldots, \zeta_d^{(k)}) \) and \( \zeta_i := (\zeta_1^{(1)}, \zeta_2^{(2)}, \zeta_3^{(3)}) \), and we write \( |\zeta| := \text{reorder}_3, d ((|\zeta^{(1)}|) \otimes |\zeta^{(2)}|) \otimes |\zeta^{(3)}|) ) \in \bigotimes^3 d C^n \), where \( \text{reorder}_3, d \) is the linear map defined in (22.1.3).

We want to analyze how the tensor contraction \( (\lambda)\pi_{\sigma} |\zeta| \) can be interpreted combinatorially using set partitions. For an ordered subset \( e \subseteq V(\mathcal{H}) \) of vertices we identify triple lists \( (\zeta_1, \ldots, \zeta_e) \) with labelings on \( e \) whose codomain is \((C^n)^3\).

We define the evaluation function for ordered set partition triples as follows: Given a labeling \( \zeta : V(\mathcal{H}) \rightarrow (C^n)^3 \), we set
\[ \text{val}_{\mathcal{H}}(\zeta) := \text{val}_{E^{(1)}(\mathcal{H})}(\zeta^{(1)}) \cdot \text{val}_{E^{(2)}(\mathcal{H})}(\zeta^{(2)}) \cdot \text{val}_{E^{(3)}(\mathcal{H})}(\zeta^{(3)}). \]
22.1.15 Proposition. Let $\mathcal{H}$ be an ordered set partition triple of type $\lambda$. Let $\zeta : V(\mathcal{H}) \to (\mathbb{C}^n)^3$ be a labeling. We have
\[
\text{val}_H(\zeta) = \langle \hat{\lambda} | \pi \rangle.
\]
Proof. By (22.1.2) we have \( \langle \hat{\lambda} | \pi \rangle = \text{reorder}_{\lambda,d}(\langle \hat{\lambda}^{(1)} | \otimes \langle \hat{\lambda}^{(2)} | \otimes \langle \hat{\lambda}^{(3)} | \rangle). \) The claim follows with Proposition 22.1.11. \( \square \)

Symmetrization. Let $\mathcal{P}_d : \bigotimes^d \mathbb{C}^n \to \text{Sym}^d \mathbb{C}^n$ denote the symmetrization over $\mathfrak{S}_d$. Since $\tau \mathcal{P}_d = \mathcal{P}_d$ for all $\tau \in \mathfrak{S}_d$, we get $\langle \lambda | \pi \rangle \mathcal{P}_d = \langle \lambda | \pi \tau \rangle \mathcal{P}_d$ for all $\tau \in \mathfrak{S}_d$. Hence the polynomial described by a set partition triple is independent of the numbering of its vertices. This explains the following definition.

22.1.16 Definition. An obstruction predesign is defined to be an equivalence class of set partition triples under renumbering of the vertices. When depicting obstruction predesigns, we omit the vertex numbering of the corresponding set partition triple.

So each obstruction predesign describes some polynomial $\langle \lambda | \pi \rangle \mathcal{P}_d \in \text{Sym}^d \bigotimes^3 \mathbb{C}^n$ of degree $d$ up to sign. Since we do not care about the sign, we abuse notation in the following way: For every obstruction predesign $\mathcal{H}$ we implicitly fix an ordered set partition triple $\mathcal{H}'$ in a way such that $\mathcal{H}$ is obtained from $\mathcal{H}'$ by forgetting about orderings and vertex numbers. Then we define $\text{val}_H(\zeta) := \text{val}_{H'}(\zeta)$.

22.1.17 Corollary. Let $\mathcal{H}$ be an ordered set partition triple with $d$ vertices. Let $\zeta : V(\mathcal{H}) \to (\mathbb{C}^n)^3$ be a labeling. We have
\[
\frac{1}{d!} \sum_{\zeta \in \mathfrak{S}_d \xi} \text{val}_H(\zeta) = \langle \hat{\lambda} | \pi \rangle \mathcal{P}_d \langle \zeta \rangle.
\]
Proof. Follows from Proposition 22.1.15 and the definition of the symmetrization $\mathcal{P}_d$. \( \square \)

It is straightforward to verify (see e.g. [Ike12, Lemma 7.2.7]) that the polynomials described by obstructions predesigns are the zero function if they are not obstruction designs:

22.1.18 Definition. An obstruction design $H$ is an obstruction predesign $\mathcal{H}$ which satisfies
\[
|e_1 \cap e_2 \cap e_3| \leq 1 \text{ for all hyperedge triples } (e_1, e_2, e_3) \in E^{(1)} \times E^{(2)} \times E^{(3)}.
\]

22.1.19 Proposition. For a partition triple $\lambda$ we have
\[
\text{HWV}_\lambda(\text{Sym}^d \bigotimes^3 \mathbb{C}^n) = \text{span}\{f_H : \mathcal{H} \text{ is an obstruction design of type } \lambda\}.
\]
In particular
\[
k(\lambda) = \dim \text{span}\{f_H : \mathcal{H} \text{ is an obstruction design of type } \lambda\}.
\]
Proof. According to Claim 22.1.5, for $\lambda \vdash d$, we have that
\[
\text{HWV}_\lambda(\text{Sym}^d \bigotimes^3 \mathbb{C}^n) = \text{span}\{\langle \hat{\lambda} | \pi \rangle \mathcal{P}_d : \pi \in \mathfrak{S}_d \}.
\]
But since obstruction designs $\mathcal{H}$ determine $f_H = \langle \hat{\lambda} | \pi \rangle \mathcal{P}_d$ up to a sign, the first assertion follows. The rest of the proposition follows from the fact that $k(\lambda) = \dim \text{HWV}_\lambda(\text{Sym}^d \bigotimes^3 \mathbb{C}^n)$. \( \square \)
CHAPTER 22. OCCURRENCE OBSTRUCTIONS FOR MATRIX MULTIPLICATION

Polynomial Evaluation

We now describe how to evaluate the polynomial \( f_H \) corresponding to an obstruction design \( H \) at a point \(|w\rangle = \sum_{i=1}^{r} |w_i^{(1)}\rangle \otimes |w_i^{(2)}\rangle \otimes |w_i^{(3)}\rangle \in \otimes^3 \mathbb{C}^n \). We calculate

\[
f_H(w) = \langle \tilde{\lambda} | \pi \lambda_d | w \otimes d \rangle = \sum_{J \in \{1, \ldots, r\}^d} \langle \tilde{\lambda} | \pi | w_{J_1} w_{J_2} \cdots w_{J_d} \rangle
\]

We can interpret \( \varsigma := (w_{J_1}, w_{J_2}, \ldots, w_{J_d}) \) as a vertex labeling \( \varsigma : V(H) \to (\mathbb{C}^n)^3 \) and see that the sum in (22.1.15) is over all vertex labelings \( \varsigma : V(H) \to (\mathbb{C}^n)^3 \) with \( \varsigma(y) \in \{w_i \mid 1 \leq i \leq r\} \) for all \( y \in V(H) \).

22.1(iv) The obstruction design for the hook triple

The obstruction design \( H := H_\kappa \) consists of \( d \) vertices divided into disjoint sets \( V^{(1)} \cup V^{(2)} \cup V^{(3)} \cup \{y_0\} \), where \( |V^{(k)}| = \kappa \) for all \( 1 \leq k \leq 3 \). There are only three hyperedges of size larger than 1, called \( e^{(k)} \) for \( 1 \leq k \leq 3 \). We set \( e^{(k)} := V^{(k+1)} \cup V^{(k+2)} \cup \{y_0\} \), where \( V^{(k)} := V^{(k-3)} \) for \( k > 3 \). The obstruction design \( H \) is depicted in Figure 22.4.

![Figure 22.4: The family of obstruction designs corresponding to the hook partition triple.](image)

22.2 Vanishing at low rank points

In this subsection we prove (22.0.1), relying on our precise knowledge of the obstruction design \( H \) defined in Section 22.1(iv). A second proof that only uses the invariance properties of the unit tensor is presented in Section 22.4.

Let \( A \in (\mathbb{C}^{3\kappa \times 3\kappa})^3 \) be arbitrary. We define the triple list

\[
w := ((A^{(1)}|1), (A^{(2)}|1), (A^{(3)}|1)), \ldots, (A^{(1)}|3\kappa), (A^{(2)}|3\kappa), (A^{(3)}|3\kappa))
\]

According to (22.1.20) we have

\[
f_H(3\kappa) = \sum_{J \in \{1, \ldots, 3\kappa\}^{3\kappa+1}} \text{val}_H(Aw_{J_1}, \ldots, Aw_{J_{3\kappa+1}}).
\]
CHAPTER 22. OCCURRENCE OBSTRUCTIONS FOR MATRIX MULTIPLICATION

The crucial property of $\mathcal{H}$ is that for each pair of vertices $\{y_1, y_2\}$ there exists a hyperedge $e$ of $\mathcal{H}$ containing both $y_1$ and $y_2$. By the pigeon-hole principle, for each labeling $J: V(\mathcal{H}) \to \{1, \ldots, 3s\}$ there exists a pair of vertices $\{y_1, y_2\}$ such that $J(y_1) = J(y_2)$. The crucial property of $\mathcal{H}$ implies that $y_1$ and $y_2$ lie in a common hyperedge $e$. Hence $\val_e((Aw_{J_1}, \ldots, Aw_{J_{3s+1}})) = 0$, because it is the determinant of a matrix with two equal columns. Therefore, each summand in $(\ast)$ vanishes, which proves (22.0.1).

This argument is analogous to Section 20.2(iii).

22.3 Nonvanishing at the matrix multiplication tensor

In this rather technical section we prove (22.0.2) by an explicit construction of a matrix triple $A = (A^{(1)}, A^{(2)}, A^{(3)})$ consisting of maps $A^{(k)}: \mathbb{C}^{m \times m} \to \mathbb{C}^{m^2}$. We make use of the fact that $m$ is odd.

For notational convenience, we define the triples

$$t_{ijl} := \left(\left([ij)], [jl]), ([li])\right)\right) \in (\mathbb{C}^{m \times m})^3 \quad (22.3.1)$$

and the triple list $w$ of length $m^3$ obtained by concatenating all $t_{ijl}$ for $1 \leq i, j, l \leq m$ in any order. Recall that

$$(m, m, m) = \sum_{i,j,l} t_{ijl}^{(1)} \otimes t_{ijl}^{(2)} \otimes t_{ijl}^{(3)}.$$

We put

$$\mathcal{T} := \{t_{ijl} \mid 1 \leq i, j, l \leq m\}.$$

According to (22.1.20) we have

$$f_\mathcal{H}(A(m, m, m)) = \sum_{J \in \{1, \ldots, m^3\}^d} \val_\mathcal{H}(Aw_{J_1}, \ldots, Aw_{J_{3s}}). \quad (\ast)$$

Consider the polynomial ring $\Gamma = \mathbb{C}[X_1, \ldots, X_N]$, where $X_i$ are indeterminates. According to Lemma 1.3.2, if a function $f \in \Gamma$ is nonzero, then there exist values $\alpha_i \in \mathbb{C}$, $1 \leq i \leq N$, such that $f(\alpha_1, \ldots, \alpha_N) \neq 0$. We will define the $m^2 \times m^2$ matrix triple $A$ with matrix entries being affine linear in the indeterminates $X_i$. Hence we write the sum $f_\mathcal{H}(A(m, m, m))$ as an element of $\Gamma$. We will provide a monomial of $f_\mathcal{H}(A(m, m, m))$ in the $X_i$ with nonzero coefficient in (22.3.3).

Invariance in each $V^{(k)}$. We use the short notation $\val_e(\zeta) := \val_e(\zeta^{(k)}_e)$ for a hyperedge $e \in E^{(k)}$ and a triple labeling $\zeta$. We start out with the following easy claim.

22.3.2 Claim. Let $\sigma: V(\mathcal{H}) \to V(\mathcal{H})$ be a bijection satisfying $\sigma(V^{(k)}) = V^{(k)}$ for all $1 \leq k \leq 3$. For every triple labeling $\zeta: V(\mathcal{H}) \to (\mathbb{C}^{m^2})^3$ we have

$$\val_\mathcal{H}(\zeta) = \val_\mathcal{H}(\zeta \circ \sigma).$$

Proof. It suffices to show the claim for a transposition $\tau = \sigma$ exchanging two elements of $V^{(1)}$, because the situation for $V^{(2)}$ and $V^{(3)}$ is completely symmetric. We have $\prod_{e \in E^{(1)}} \val_e(\zeta) = \prod_{e \in E^{(1)}} \val_e(\zeta \circ \tau)$, because up to reordering both products have the same factors. For $2 \leq k \leq 3$ we have $\val_e(\zeta) = \val_e(\zeta \circ \tau)$ for every singleton hyperedge $e \in E^{(k)}$ and $\val_e(\zeta) = -\val_e(\zeta \circ \tau)$. Therefore $\prod_{e \in E^{(k)}} \val_e(\zeta) = -\prod_{e \in E^{(k)}} \val_e(\zeta \circ \tau)$. As a result we get $\val_\mathcal{H}(\zeta) = (-1)^2 \val_\mathcal{H}(\zeta \circ \tau)$.
**CHAPTER 22. OCCURRENCE OBSTRUCTIONS FOR MATRIX MULTIPLICATION**

**Special structure of the matrix triple**

Recall that $m$ is odd and $\kappa = \frac{m^2 - 1}{2}$. Let $a := \frac{m + 1}{2}$. Define the set $O_m := \{1, \ldots, m\} \times \{1, \ldots, m\} \setminus \{(a, a)\}$ consisting of $m^2 - 1$ pairs. Fix an arbitrary bijection

$$\varphi : O_m \to \{2, \ldots, m^2\}.$$ 

Let $\tilde{i} := m + 1 - i$ for $1 \leq i \leq m$. (We may think of the map $i \mapsto \tilde{i}$ as a reflection at $a$; note $\tilde{a} = a$.) Let

$$\Gamma := \mathbb{C}[\{X_i^{(k)} : 1 \leq k \leq 3, 1 \leq i \leq m\}]$$

denote the polynomial ring in $3m$ variables. For each $1 \leq k \leq 3$ we define the linear map $A^{(k)} : \mathbb{C}^{m \times m} \to \mathbb{C}^{m^2}$ by

$$A^{(k)}((ij)) := \begin{cases} X^{(k)}_a & \text{if } i = j = a \\ |\varphi(i, \tilde{i})| & \text{if } i \neq j \text{ and } j = \tilde{i} \\ |\varphi(i, j)| & \text{if } j \neq \tilde{i} \end{cases}.$$ 

Hence $A^{(k)}$ looks as follows:

$$\begin{pmatrix} X^{(k)}_a & X^{(k)}_1 & \cdots & X^{(k)}_{a-1} & X^{(k)}_{a+1} & \cdots & X^{(k)}_{m-1} & X^{(k)}_m \\ 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} X^{(k)}_2 \\ \vdots \\ X^{(k)}_{m} \\ 0 \end{pmatrix}, \quad (**)$$

where we arranged the rows and columns as follows: The left $m$ columns correspond to the vectors $|(ii)|$, where the leftmost one corresponds to $|(a, a)|$. The top row corresponds to the vector $|1|$, and the following $m - 1$ rows correspond to the vectors $|\varphi(i, \tilde{i})|$. Recall that $f_{\mathcal{H}}(A(m, m, m))$ is a sum of products of determinants of submatrices of $A^{(k)}$.

The sum $f_{\mathcal{H}}(A(m, m, m))$ is an element of $\Gamma$ and we are interested in its coefficient of the monomial $X$, where

$$X := \prod_{k=1}^3 \prod_{i=1}^m X^{(k)}_i (X^{(k)}_i)^{|i - \tilde{i}|}.$$  \hfill (22.3.3)

We remark that the degree of $X$ is $3(1 + \sum_{i=1}^m |i - \tilde{i}|)$. It is readily checked that $\sum_{i=1}^m |i - \tilde{i}| = \kappa$.

Fix any numbering of the vertices of $\mathcal{H}$. For $J \in \{1, \ldots, m^3\}^d$ we abuse notation and define the map $J : V(\mathcal{H}) \to T$ via $J(y) := w_{J_y}$. With this notation, $(*)$ becomes

$$\sum_J \text{val}_{\mathcal{H}}(AJ(1), \ldots, AJ(d)),$$

or $\sum_J \text{val}_{\mathcal{H}}(AJ)$ in short notation. We call a triple labeling $J : V(\mathcal{H}) \to T$ nonzero, if the coefficient of $X$ in the polynomial $\text{val}_{\mathcal{H}}(AJ)$ is nonzero. Note that the sum of the evaluations of all nonzero triple labelings is the coefficient of $X$ in the polynomial $f_{\mathcal{H}}(A(m, m, m))$. We will count and classify all nonzero triple labelings and show that they evaluate to the same nonzero value. This implies that the coefficient of $X$ in $f_{\mathcal{H}}(A(m, m, m))$ is a sum without cancellations and is hence nonzero.
Proof.

For each nonzero triple labeling \( J \) and fix \( 1 \leq k \leq 3 \). For all \( y \in V^{(k)} \) we have \( J^{(k)}(y) = \{(ii)\} \) for some \( 1 \leq i \leq m \).

**Proof.** Let \( y \in V^{(k)} \). Since \( \{y\} \in E^{(k)} \) we have \( \{A^{(k)}|J^{(k)}(y)\} \neq 0 \). From the definition of \( A \) it follows that \( J^{(k)}(y) = \{(ii)\} \) and the third case \( j \neq i \) is excluded. Hence \( j = i \).

**22.3.4 Claim.** Fix a nonzero triple labeling \( J \) and fix \( 1 \leq k \leq 3 \). For all \( \lambda \in \mathbb{V}^{(k)} \) we have \( J^{(k)}(y) = \{(aa)\} \).

Proof. Let \( y \in V^{(k)} \). Since \( \{y\} \in E^{(k)} \) we have \( \{A^{(k)}|J^{(k)}(y)\} \neq 0 \). From the definition of \( A \) it follows that \( J^{(k)}(y) = \{(aa)\} \) and the third case \( j \neq \lambda \) is excluded. Hence \( j = \lambda \).

**22.3.5 Claim.** For every nonzero triple labeling \( J \) we have \( J(y^0) = \{(aa),(aa),(aa)\} \).

Proof. Let \( J \) be a nonzero triple labeling. Hence the coefficient of \( \mathcal{X} \) in \( \text{val}_{A}(AJ(1),\ldots,AJ(d)) \) is nonzero. For the following argument it is important to keep the structure of the matrix \( A^{(k)} \) in mind, cf. (**) . Recall that \( f_{\mathcal{H}}(A(m,m,m)) \) is a sum of products of certain subdeterminants of \( A^{(k)} \) that are determined by the hyperedges in \( E^{(k)}(\mathcal{H}) \). Since the degree of \( X_{a}^{(k)} \) in \( \mathcal{X} \) is 1, we have that for all \( 1 \leq k \leq 3 \) there is exactly one vertex \( y_k \in V(\mathcal{H}) \) with \( J^{(k)}(y_k) = \{(aa)\} \).

Recall that the hyperedge \( e^{(k)} \) has size \( 2\kappa + 1 = m^2 \). Since \( J \) is a nonzero triple labeling, \( J^{(k)} \) is injective on hyperedges and hence \( \{(J^{(k)}(y) : y \in e^{(k)})\} = \{e^{(k)}\} \). But since the image \( J^{(k)}(V(\mathcal{H})) \) has cardinality at most \( m^3 \), \( J^{(k)} \) is actually bijective on \( e^{(k)} \). Since there is only one vertex \( y \) satisfying \( J^{(k)}(y) = \{(aa)\} \), namely the vertex \( y = y_k \), it follows \( y_k \in e^{(k)} \). Since \( e^{(2)} \cap e^{(3)} \cap e^{(3)} = \{y^0\} \), it remains to show that \( y_1 = y_2 = y_3 \).

The structure of the matrix multiplication tensor implies that \( J(y_1) = \{(aa),(aa),(aa)\} \) for some \( 1 \leq i \leq m \). If \( a = i \), then, by definition of \( y_2 \) and \( y_3 \) and uniqueness, we have \( y_1 = y_2 = y_3 \) and we are done.

Now assume \( a \neq i \) and \( y_1 \neq y^0 \). W.l.o.g. \( y_1 \in V^{(3)} \). Using Claim 22.3.4 we conclude that \( J^{(3)}(y_1) = \{(ii)\} \) for some \( 1 \leq i \leq m \). Hence \( i = a \) contradicting \( i \neq a \). Thus we have shown that \( y_1 = y^0 \). Similarly, we show that \( y_2 = y_3 = y^0 \).

**22.3.6 Claim.** For each nonzero triple labeling \( J \) we have \( J^{(k)}(V^{(k)}) = \{|(ii)| 1 \leq i \leq m \} \setminus \{|(aa)\} \), where the preimage of each \( \{(ii)\} \) under \( J^{(k)} \) has size \( |i - \lambda| \).

Proof. According to Claim 22.3.5 we have \( J(y^0) = \{(aa),(aa),(aa)\} \). For the following look again at the structure of \( A^{(k)} \), cf. (**) . Since \( A^{(k)}(\{(aa)\}) \) is a multiple of \( \{1\} \), we have that \( \text{val}_{\mathcal{X}^{(k)}}(J^{(k)}) \) is a multiple of \( X_{a}^{(k)} \). Moreover, for \( \lambda 
eq a \), the variable \( X^{(k)} \) does not appear in the expansion of \( \text{val}_{\mathcal{X}^{(k)}}(J^{(k)}) \).

Since for a fixed \( 1 \leq k \leq 3 \) there is \( \kappa = \sum_{i=1}^{m} |i - \lambda| \) many contributions of a factor \( X_{\lambda}^{(k)} \) in the monomial \( \mathcal{X} \), these factors must be contributed at vertices in \( V^{(k)} \). Since \( |V^{(k)}| = \kappa \), the only possibility is that all \( y \in V^{(k)} \) satisfy \( J^{(k)}(y) = \{(ii)\} \) for some \( 1 \leq i \leq m , \lambda = a \). The specific requirement for the number of factors \( X_{\lambda}^{(k)} \) which are encoded in \( \mathcal{X} \) in (22.3.3) finishes the proof.

**Coupling the Analysis of the Three Layers**

Define the bijective map

\[ \tau : O_m \rightarrow O_m, \quad \tau(ij) = (ji), \]

which corresponds to the rotation by 90°. Clearly, \( \tau^4 = \text{id} \). The map \( \tau \) induces a map \( \varphi(O_m) \rightarrow \varphi(O_m) \) on the powerset, which we also call \( \tau \). Define the involution (taking the complement)

\[ \iota : \varphi(O_m) \rightarrow \varphi(O_m), \quad S \mapsto O_m \setminus S. \]

Clearly, we have \( \tau \circ \iota = \iota \circ \tau \). We will only be interested in subsets \( S \subseteq O_m \) with exactly \( |O_m|/2 = \kappa \) many elements and their images under \( \tau \) and \( \iota \). The subsets \( S \subseteq O_m \) that satisfy \( \iota(S) = \tau(S) \) will be of special interest. Geometrically, these are the sets that get inverted when rotating by 90°, see Figure 22.5 for examples.
In the following we identify the sets $J^{(k)}(V^{(k)})$, for $1 \leq k, k' \leq 3$, with their corresponding subsets of $O_m$.

In Claim 22.3.6 we analyzed the labels $J^{(k)}(V^k)$. In the next claim we turn to $J^{(k)}(V^{k'})$, where $k \neq k'$.

22.3.7 Claim. Every nonzero triple labeling $J$ is completely determined by the image $J^{(1)}(V^{(3)})$ (up to permutations in the $V^{(k)}$, see Claim 22.3.2) as follows.

- $J^{(2)}(V^{(3)}) = \tau(J^{(1)}(V^{(3)}))$,
- $J^{(2)}(V^{(1)}) = \iota(J^{(2)}(V^{(3)}))$,
- $J^{(3)}(V^{(1)}) = \tau(J^{(2)}(V^{(1)}))$,
- $J^{(3)}(V^{(2)}) = \iota(J^{(3)}(V^{(1)}))$,
- $J^{(1)}(V^{(2)}) = \tau(J^{(3)}(V^{(2)}))$.

Moreover, $\tau(J^{(1)}(V^{(3)})) = \iota(J^{(1)}(V^{(3)}))$.

Proof. According to Claim 22.3.6 we have that each vertex $y \in V^{(3)}$ satisfies

$$J(y) = (\{ij\}, |(\tau(ij))|, |(\tilde{ii})|)$$

for some $1 \leq i, j \leq m$, $i \neq a$. In particular, using that $\tau$ is injective, we have

$$\tau(J^{(1)}(V^{(3)})) = J^{(2)}(V^{(3)}).$$

Since $J$ is nonzero, $J^{(2)}$ is injective on $e^{(2)}$. We even have that $J^{(2)}$ is bijective on $e^{(2)}$, because $|e^{(2)}| = m^2$. Using that $e^{(2)} = V^{(1)} \cup V^{(2)} \cup \{|y\}$ we see that

$$J^{(2)}(V^{(3)}) = O_m \setminus J^{(2)}(V^{(3)}) = \iota(J^{(2)}(V^{(3)})).$$

For the same reason, we can deduce $J^{(3)}(V^{(1)}) = \tau(J^{(2)}(V^{(1)}))$ and $J^{(3)}(V^{(2)}) = \iota(J^{(3)}(V^{(1)}))$. And applying these arguments one more time we get $J^{(1)}(V^{(2)}) = \tau(J^{(3)}(V^{(2)}))$ and $J^{(1)}(V^{(3)}) = \tau(J^{(1)}(V^{(3)}))$. Summarizing (recall $\tau \circ \iota = \iota \circ \tau$) we have

$$J^{(1)}(V^{(3)}) = \tau^3 \iota^3 J^{(1)}(V^{(3)}) = \tau^{-1} \iota J^{(1)}(V^{(3)}),$$

which is equivalent to $\tau(J^{(1)}(V^{(3)})) = \iota(J^{(1)}(V^{(3)})).$ \hfill \square

Additionally to the constraint $\tau(J^{(1)}(V^{(3)})) = \iota(J^{(1)}(V^{(3)}))$ given in Claim 22.3.7, Claim 22.3.6 implies that in $J^{(1)}(V^{(3)})$ there are $|i - \bar{i}|$ many elements of the form $|(i\bar{i})|$ for each $1 \leq i \leq m$.

This motivates the following definition.

22.3.8 Definition. A subset $S \subseteq O_m$ is called valid, if

1. $|S| = \frac{m^2 - 1}{2} = \kappa$,
2. $\tau(S) = \iota(S)$,
3. $|p^{-1}(i)| = |i - \bar{i}|$ for all $1 \leq i \leq m$

where $p : S \rightarrow \{1, \ldots, m\}$ is the projection to the first component, see Figure 22.5 for an example.

22.3.9 Proposition. For all nonzero triple labelings $J$ we have that $J^{(1)}(V^{(3)})$ is a valid set. On the other hand, for every valid set $S$ there exists exactly one nonzero triple labeling $J$ with $J^{(1)}(V^{(3)}) = S$ up to permutations in the $V^{(k)}$.

Proof. For the first statement, property (2) of Definition 22.3.8 follows from Claim 22.3.7 and property (3) of Definition 22.3.8 follows from Claim 22.3.6. The second statement can be readily checked with Claim 22.3.5 and Claim 22.3.7. \hfill \square
**CHAPTER 22. OCCURRENCE OBSTRUCTIONS FOR MATRIX MULTIPLICATION**

135

The next claim classifies all valid sets.

**22.3.10 Lemma.** A set \( S \subseteq O_m \) is valid iff the following conditions are all satisfied (see Figure 22.6 for an illustration):

1. \( \{(ij) \mid (i < j \text{ and } i < j) \text{ or } (i > j \text{ and } i > j)\} \subseteq S \), represented by solid vertices in Figure 22.6.

2. \( \{(ij) \mid (i < j \text{ and } i < j) \text{ or } (i > j \text{ and } i > j)\} \cap S = \emptyset \), represented by dotted vertices in Figure 22.6.

3. For all \( 1 \leq i \leq \frac{m-1}{2} \) there are two mutually exclusive cases, (a) and (b), represented by the two vertices \( x_i \) and the two vertices \( \overline{x_i} \), respectively, in Figure 22.6.

- (a) \( \{(ii), (ii)\} \subseteq S \) and \( \{(ii), (ii)\} \cap S = \emptyset \),

- (b) \( \{(ii), (ii)\} \subseteq S \) and \( \{(ii), (ii)\} \cap S = \emptyset \).

These choices result in \( 2^{\frac{m-1}{2}} \) valid sets.

**Proof.** As indicated in Figure 22.6, for each tuple \((ij)\) we call \( i \) the row of \((ij)\). For \( S \) to be valid, according to Definition 22.3.8(3), \( S \) must contain \(|i-i|\) elements in row \( i \) and according to Definition 22.3.8(2), \( \tau(s) \notin S \) for all \( s \in S \).

In particular, \( S \) must contain \( m-1 \) elements in row 1. If \((11) \in S\), then \((1m) \notin S\), because \( \tau(11) = (1m) \). Hence there are only two possibilities: (a): \( \{(1j) \mid 1 \leq j < m\} \subseteq S \) or (b): \( \{(1j) \mid 1 < j \leq m\} \subseteq S \). By symmetry, for row \( m \) we get (a'): \( \{(mj) \mid 1 \leq j < m\} \subseteq S \) or (b'): \( \{(mj) \mid 1 < j \leq m\} \subseteq S \). But since \( \tau(1m) = (m2) \) and \( \tau(m1) = (1l) \), the fact \( \tau(S) = \tau(S) \) implies that (a) iff (b') and that (a') iff (b). We are left with the two possibilities ((a) and (b')) or ((a') and (b)).

Now consider row 2. We have \( \tau(21) = (1, m-1) \in S \) and hence \( (21) \notin S \). In the same manner we see \( (2m) \notin S \). We are left to choose \( m-3 \) elements from the \( m-2 \) remaining elements in row 2. The same argument as for row 1 gives two possibilities: (a): \( \{(2j) \mid 2 \leq j < m-1\} \subseteq S \) or (b'): \( \{(2j) \mid 2 < j \leq m-1\} \subseteq S \). Analogously for row \( m-1 \) we have (a): \( \{((m-1), j) \mid 2 \leq j < m-1\} \subseteq S \) or (b'): \( \{((m-1), j) \mid 2 < j \leq m-1\} \subseteq S \). With the same reasoning as for the rows 1 and \( m \) we get (a) iff (b') and that (a') iff (b). Again we are left with the two possibilities ((a) and (b')) or ((a') and (b')).

Continuing these arguments we end up with \( 2^{\frac{m-1}{2}} \) possibilities. It is easy to see that each of these possibilities gives a valid set. 

\[\square\]
CHAPTER 22. OCCURRENCE OBSTRUCTIONS FOR MATRIX MULTIPLICATION

Remark that $\sigma$

22.3.11 Claim. Figure 22.6:

of the two symmetric cases, namely for $i$.

Take two nonzero triple labelings $J$ and $J'$. According to Proposition 22.3.9, both sets $J^{(1)}(V^{(3)})$ and $J'^{(1)}(V^{(3)})$ are valid sets. Because of Lemma 22.3.10, it suffices to consider only the case where $J^{(1)}(V^{(3)})$ and $J'^{(1)}(V^{(3)})$ differ by a single involution $\sigma: O_m \to O_m$, where for some fixed $1 \leq i \leq \frac{m-1}{2}$ we have $\sigma(i) = (i)$ and $\sigma(ii) = (ii)$, and $\sigma$ is constant on all other pairs. We remark that $\sigma$ restricted to the four pairs $\{(ii), (ii), (ii), (ii)\}$ corresponds to a reflection in the second component.

We analyze the labels that are affected by this reflection. We only perform the analysis for one of the two symmetric cases, namely for $\{(ii), (ii), (ii), (ii)\} \subseteq J^{(1)}(V^{(3)})$. Note that this implies

$$\{(i(i)), (i(ii)), (i(iii)), (i(iv))\} \subseteq J(V^{(3)}),$$

according to Claim 22.3.6. We adapt the notation from (22.3.1) to our special situation and write $t_{000} := t_{i\bar{i}i}$, $t_{001} := t_{i\bar{i}i}$, $\ldots$, $t_{111} := t_{i\bar{i}i}$. Using this notation, $(\dagger)$ reads as follows: $\{t_{110}, t_{001}\} \subseteq J(V^{(3)})$. Using Claim 22.3.7 we get

$$\{t_{110}, t_{001}\} \subseteq J(V^{(3)}), \quad \{t_{101}, t_{010}\} \subseteq J(V^{(2)}), \quad \{t_{011}, t_{100}\} \subseteq J(V^{(1)}).$$

Applying the involution $\sigma$ to $J^{(1)}(V^{(3)})$, we can use Claim 22.3.6 again to get

$$\{(i(i)), (i(ii)), (i(iii)), (i(iv))\} \subseteq J'(V^{(3)}).$$

Applying Claim 22.3.7 and using our short syntax, we get:

$$\{t_{100}, t_{011}\} \subseteq J'(V^{(3)}), \quad \{t_{001}, t_{110}\} \subseteq J'(V^{(2)}), \quad \{t_{010}, t_{101}\} \subseteq J'(V^{(1)}).$$
We see that exactly the same triples occur in \( J(V(\mathcal{H})) \) as in \( J'(V(\mathcal{H})) \). We focus now on \( J^{(1)} \) and \( J'^{(1)} \) and see the following:

\[
\{(ii), (i\bar{i})\} \subseteq J^{(1)}(V(3)) \quad \text{and} \quad \{(i\bar{i}), (ii)\} \subseteq J^{(1)}(V(2))
\]

and

\[
\{(i\bar{i}), (ii)\} \subseteq J'^{(1)}(V(3)) \quad \text{and} \quad \{(i\bar{i}), (ii)\} \subseteq J'^{(1)}(V(2)).
\]

This gives exactly two switches of positions in \( e^{(1)} = V(2) \cup V(3) \cup \{y^0\} \), hence

\[
\text{val}_{e^{(1)}}(AJ) = (-1)^2 \text{val}_{e^{(1)}}(AJ') = \text{val}_{e^{(1)}}(AJ').
\]

Analogously we can prove that \( \text{val}_{e^{(k)}}(AJ) = \text{val}_{e^{(k)}}(AJ') \) for all \( 2 \leq k \leq 3 \) and therefore \( \text{val}_H(AJ) = \text{val}_H(AJ') \).

(22.0.2) is completely proved.

### 22.4 The coordinate ring of the unit tensor orbit

In this section we determine the multiplicities in the coordinate ring of the unit tensor. This proves (22.0.1) via (22.0.3). As a warm-up we study the closely related case of the power sum polynomial.

#### 22.4(i) Warm-up: The coordinate ring of the power sum orbit

We use the notation from [BI11]: \( m \) is now the number of variables and \( D \) is the degree. The formulas in this subsection are unpublished calculations by Ikenmeyer and Panova. Parts also appear in the unpublished preprint [Nis].

The power sum is the polynomial \( x_1^\mu + \cdots + x_m^\mu \). Let \( H := \mathbb{Z}_D^m \rtimes \mathfrak{S}_m \) denote its stabilizer (see e.g. [CKW11, Ch. 2]). Let \( \lambda \vdash Dd \).

If \( \varrho \vdash d \) is a partition, then the frequency notation \( \lambda \in \mathbb{N}^m \) is defined via

\[
\kappa_i = |\{ j \mid \varrho_j = i \}|.
\]

E.g., the frequency notation of \( \varrho = (3, 3, 2, 0) \) is \( (0, 1, 2, 0) \). We observe that \( |\varrho| = \sum_i i\kappa_i \).

We group \( \mathfrak{S}_m \) acts on \( \mathbb{N}^m \) by permuting the positions. Note that under this action we have \( \text{stab}_\varrho = \mathfrak{S}_{\kappa_1} \times \mathfrak{S}_{\kappa_2} \times \cdots \times \mathfrak{S}_{\kappa_m} \).

#### 22.4.1 Theorem

\[
\dim \{\lambda\}^H = \sum_{\varrho \vdash D} \sum_{\mu_1, \mu_2, \ldots} c_{\mu_1, \mu_2, \ldots}^{\lambda} \prod_{\gamma=1}^d a_{\mu_\gamma}(\kappa_\gamma, iD), \quad \text{where} \quad \kappa \in \text{is the frequency notation of} \ \varrho, \quad \text{and} \quad c_{\mu_1, \mu_2, \ldots}^{\lambda} \text{is the multi-Littlewood-Richardson coefficient that denotes the multiplicity of} \ \{\lambda\} \text{ in the tensor product} \ \{\mu_1\} \otimes \cdots \otimes \{\mu_d\}.
\]

**Proof.**

\[
\{\lambda\}^H = \left( \{\lambda\}^D \right)^{\mathfrak{S}_m} = \left( \bigoplus_{\gamma=1}^d \left[ \lambda \right]^{G_\gamma} \right) \right)^{\mathfrak{S}_m}
\]

where for \( \gamma \in \mathbb{N}^m, |\gamma| = d, G_\gamma \subseteq \mathfrak{S}_{dD} \) is defined as the Young subgroup \( \mathfrak{S}_{\gamma_1D} \times \cdots \times \mathfrak{S}_{\gamma_mD} \). The last equality can be seen using the tableau bases on both sides.

For a partition \( \varrho \vdash d \) let \( \mathfrak{S}_m \varrho \subseteq \mathbb{N}^m \) denote the orbit of \( \varrho \). Note that \( \varrho \) is the only partition in its orbit, while the other lists are not in the correct order. Grouping the right-hand side in the previous equation we obtain

\[
\bigoplus_{\varrho \vdash d} \left( \bigoplus_{\gamma \in \mathfrak{S}_m \varrho} \left[ \lambda \right]^{G_\gamma} \right) \right)^{\mathfrak{S}_m},
\]
so we can study each \( \rho \) independently.

Let \( \text{stab}_\rho \leq \mathfrak{S}_m \) denote the stabilizer of \( \rho \).

22.4.2 Claim. \( \dim \left( \bigoplus_{\gamma \in \mathfrak{S}_m} [\lambda]^{\mathfrak{G}_\gamma} \right)^{\mathfrak{S}_m} = \dim \left( [\lambda]^{\mathfrak{G}_\rho} \right)^{\text{stab}_\rho} \).

Proof. We construct an isomorphism of vector spaces.

Let \( W^\rho := [\lambda]^{\mathfrak{G}_\rho} \) and \( W_\rho := \bigoplus_{\gamma \in \mathfrak{S}_m} W^\gamma \). Let \( \pi_1, \ldots, \pi_r \) be a system of representatives of left cosets for \( \text{stab}_\rho \leq \mathfrak{S}_m \) with \( \pi_1 = \text{id} \), i.e., \( \mathfrak{S}_m = \pi_1 \text{stab}_\rho \cup \cdots \cup \pi_r \text{stab}_\rho \) and we have \( \mathfrak{S}_m := \{\pi_1 \rho, \ldots, \pi_r \rho\} \). Therefore we have the decomposition

\[
W_\rho = \bigoplus_{j=1}^r \pi_j W^\rho.
\]

Let \( p : W^\rho \to W^\rho \) be the projection according to this decomposition. We claim that the restriction

\[
p : (W^\rho)^{\mathfrak{S}_m} \to (W^\rho)^{\text{stab}_\rho}
\]

is an isomorphism of vector spaces. This then finishes the proof. We verify well-definedness, injectivity, and surjectivity of \( p \).

Well-definedness: The spaces \( \pi_1 W^\rho, \ldots, \pi_r W^\rho \) are permuted by \( \mathfrak{S}_m \). Every \( \sigma \in \text{stab}_\rho \) fixes \( W^\rho \), thus \( \sigma v = v \) if \( v \in W^\rho \). Thus the map \( v = \sum_{j=1}^r v_j \pi_j \) maps \( W^\rho \) to \( (W^\rho)^{\text{stab}_\rho} \).

Injectivity: If \( v \in (W^\rho)^{\mathfrak{S}_m} \), then \( v = \sum_j \pi_j v_j \). Therefore \( v_j = \pi_j v_1 \). If \( p(v) = 0 \), then \( v_1 = 0 \), thus all \( v_j = 0 \), which proves injectivity.

Surjectivity: Let \( v_1 \in (W^\rho)^{\text{stab}_\rho} \). Set \( v_j := \pi_j v_1 \) and put \( v := \sum_j v_j \). Clearly \( p(v) = v_1 \). It remains to verify that \( v \) is \( \mathfrak{S}_m \)-invariant. \( \square \)

We are left with determining \( \dim \left( [\lambda]^{\mathfrak{G}_\rho} \right)^{\text{stab}_\rho} \).

\[
\dim \left( [\lambda]^{\mathfrak{G}_\rho} \right)^{\text{stab}_\rho} = \dim \text{HWV}_\lambda([\lambda] \otimes ([\lambda]^{\mathfrak{G}_\rho})^{\text{stab}_\rho}) = \dim \text{HWV}_\lambda((\otimes D V)^{\mathfrak{G}_\rho})^{\text{stab}_\rho})
\]

\[
(\otimes D V)^{\mathfrak{G}_\rho}^{\text{stab}_\rho} = (\text{Sym}^{D_1} V \otimes \cdots \otimes \text{Sym}^{D_m} V)^{\text{stab}_\rho} = (\bigotimes_{\kappa_1} \text{Sym}^{D_1} V \otimes \bigotimes_{\kappa_2} \text{Sym}^{D_2} V \otimes \cdots \bigotimes_{\kappa_n} \text{Sym}^{D_n} V)^{\text{stab}_\rho} = \bigoplus_{\mu} c^{\lambda}_{\mu} \text{Sym}^{D_1} V \otimes \cdots \otimes \text{Sym}^{D_n} V \quad (\text{(i)})
\]

where \( \kappa \) is the frequency notation of \( \rho \). The multiplicity of \( \{\mu^i\} \) in \( \text{Sym}^{\kappa_1} \text{Sym}^{i D V} \) is \( a_{\mu^i}(\kappa_1, iD) \). Let \( c^{\lambda}_{\mu_1, \mu_2, \ldots, \mu_n} \) denote the multiplicity of \( \{\lambda\} \) in the tensor product \( \{\mu^1\} \otimes \cdots \otimes \{\mu^d\} \). Using distributivity we obtain that the multiplicity of \( \{\lambda\} \) in the representation \( \text{(i)} \) equals

\[
\sum_{\mu_1, \mu_2, \ldots, \mu_n} c^{\lambda}_{\mu_1, \mu_2, \ldots, \mu_n} \prod_{i=1}^d a_{\mu_i}(\kappa_i, iD)
\]
We conclude
\[
\dim\{\lambda\}^H = \sum_{\varrho} \sum_{\mu^1, \mu^2, \ldots, \mu^d} c^\lambda_{\mu^1, \mu^2, \ldots, \mu^d} \prod_{i=1}^{d} a_{\mu^i}(\kappa_i, iD).
\]

22.4(ii) From the power sum to the unit tensor

The stabilizer of \(\langle n \rangle\) in \(\text{GL}_n^3\) is \(H := D_m \times S_m\), where
\[
D_m := \{(\text{diag}(\alpha_1^{(1)}, \ldots, \alpha_m^{(1)}), \ldots, \text{diag}(\alpha_1^{(3)}, \ldots, \alpha_m^{(3)})) \mid \forall i : \alpha_i^{(1)} \alpha_i^{(2)} \alpha_i^{(3)} = 1\},
\]
see [BL11, Prop. 4.1].

As a straightforward generalization of Gay’s theorem (Lemma 19.3.12) we define the generalized plethysm coefficient \(a_\lambda(\mu, k)\) for a partition \(\lambda \vdash mk\), a partition \(\mu \vdash m\) and a natural number \(k\) via the decomposition
\[
\{\lambda\}^{m \times k} = \bigoplus_{\mu \vdash m} \gamma^k(\mu \lambda).
\]

We obtain the classical plethysm coefficients \(a_\lambda(m, k)\) when \(\mu = (m)\) is a single row.

**22.4.3 Theorem.** \(\dim \{\lambda, \lambda', \lambda''\}^H = \sum_{\varrho} c_{\rho}^{\lambda} \prod_{i=1}^{d} a_{\rho^i}(\beta^i, \gamma),\) where for \(\varrho\) being the frequency notation of \(\rho\)

- the sum for \(\beta\) is over all lists of partitions such that \(\beta^i \vdash \kappa_i\) and analogously for \(\beta'\) and \(\beta''\),
- and
\[
\beta, \rho(\lambda) := \sum_{\nu_1, \ldots, \nu_m} c^\lambda_{\nu_1, \ldots, \nu_m} (\prod_{i=1}^{m} a_{\nu_i}(\beta^i, \gamma)),
\]

**Proof.** \(\{\lambda, \lambda', \lambda''\} = \{\lambda\} \otimes \{\lambda'\} \otimes \{\lambda''\}\).

\(\{\lambda, \lambda', \lambda''\}\) has a basis given by triples of tableaux and \(D_m\) rescales basis vectors. Thus a vector is invariant if all basis vectors in its support are invariant.

\(D_m\) contains the subgroup
\[
\{(\text{diag}(\alpha, 1, 1, \ldots, 1), \text{diag}(\alpha^{-1}, 1, 1, \ldots, 1), \text{id})\}
\]
and all other such subgroups where \(\alpha\) and \(\alpha^{-1}\) are both on position \(i\) on two different diagonals.

A basis vector is invariant under these groups if all three tableaux have the same content. Since \(D_m\) is generated by these groups, this precisely characterizes the invariants: \(\{\lambda, \lambda', \lambda''\}^{D_m}\) has as a basis those triples of tableaux in which all three tableaux share the same content \(\gamma \in \mathbb{N}^m\), \(|\gamma| = d\):
\[
\{\lambda, \lambda', \lambda''\}^{D_m} = \bigoplus_{\gamma \in \mathbb{N}^m, |\gamma| = d} \{\lambda\}^\gamma \otimes \{\lambda'\}^\gamma \otimes \{\lambda''\}^\gamma,
\]
where \(\{\lambda\}^\gamma\) denotes the vector space of tableaux of shape \(\lambda\) and content \(\gamma\).

\(\bigoplus_{\gamma \in \mathbb{N}^m, |\gamma| = d} \{\lambda\}^\gamma\) is an \(S_m\)-representation. As seen in the proof for the power sum, we group together with respect to the content:
\[
\{\lambda, \lambda', \lambda''\}^{D_m} \otimes \Phi_m = \bigoplus_{\varrho} (\bigoplus_{\gamma \in \Phi_m \otimes \varrho} \{\lambda\}^\gamma \otimes \{\lambda'\}^\gamma \otimes \{\lambda''\}^\gamma)^{\Phi_m}
\]

Completely analogously to the proof for the power sum, we can take \(\text{stab}_\varrho\)-invariants instead of \(\Phi_m\)-invariants:
\[
\dim (\bigoplus_{\gamma \in \Phi_m \otimes \varrho} \{\lambda\}^\gamma \otimes \{\lambda'\}^\gamma \otimes \{\lambda''\}^\gamma)^{\Phi_m} = \dim (\{\lambda\}^\varrho \otimes \{\lambda'\}^\varrho \otimes \{\lambda''\}^\varrho)^{\text{stab}_\varrho}
\]
22.4.4 Claim.

\[
\{\lambda\}^\omega_{\text{stab}_\omega\text{-repr}} \cong \bigoplus_{\beta^{i_1} \cdots \beta^{i_m} \vdash \lambda} \sum_{\nu^{i_1} \cdots \nu^{i_m}} c^\lambda_{\nu^{i_1}, \cdots, \nu^{i_m}} \left( \prod_{i=1}^m a_{\nu^i}(\beta^i, i) \right)[\beta^1] \otimes \cdots \otimes [\beta^m],
\]

where \(\kappa\) is the frequency notation of \(\nu\).

**Proof.** Recall that \(\{\lambda\}_{m \times k} = \bigoplus_{\mu \vdash \lambda} a_{\lambda}(\mu, k)[\mu]\).

We first prove (\(\ast\)): \(\bigotimes^i \text{Sym}^m V = (\bigotimes^i \text{Sym}^m V)^{G_e} = \bigoplus_{\nu^{i_1} \cdots \nu^{i_m}} [\nu^{i_1}] \otimes \cdots \otimes [\nu^{i_m}] = \bigoplus_{\nu^{i_1}, \cdots, \nu^{i_m}} a_{\nu^{i_1}, \cdots, \nu^{i_m}}(\nu) \otimes [\nu]\), where for the last equality we use the generalized Gay’s theorem.

Now we can calculate:

\[
\bigoplus_{\lambda \vdash d} \{\lambda\} \otimes \{\nu\}^\omega = \bigoplus_{\lambda \vdash d} \{\lambda\} \otimes [\lambda]^{G_e} = (\bigotimes^d \text{Sym}^m V)^{G_e} = \text{Sym}^{e_1} V \otimes \cdots \otimes \text{Sym}^{e_m} V
\]

\[
\text{Sym}^{e_1} V \otimes \cdots \otimes \text{Sym}^{e_m} V
\]

\[
\bigoplus_{\nu^{i_1} \cdots \nu^{i_m}} a_{\nu^{i_1}}(\beta^{i_1}, 1) \{\nu^{i_1}\} \otimes [\beta^{i_1}] \otimes \cdots \otimes \bigoplus_{\nu^{i_1} \cdots \nu^{i_m}} a_{\nu^{i_1}}(\beta^{i_1}, d) \{\nu^{i_1}\} \otimes [\beta^{i_1}]
\]

\[
= \bigoplus_{\nu^{i_1} \cdots \nu^{i_m}} \prod_{i=1}^m a_{\nu^i}(\beta^i, i) \{\nu^i\} \otimes [\beta^i] \otimes \cdots \otimes [\beta^m].
\]

Taking HWVs of weight \(\lambda\) on both sides we obtain

\[
\{\lambda\}^\omega = \bigoplus_{\nu, \beta} \sum_{\nu^{i_1} \cdots \nu^{i_m}} \prod_{i=1}^m a_{\nu^i}(\beta^i, i) [\beta^1] \otimes \cdots \otimes [\beta^m].
\]

Since the dimension of the \(\mathfrak{S}_\kappa\)-invariant space of \([\beta^1] \otimes [\beta^2] \otimes [\beta^m]\) is given by the Kronecker coefficient \(k(\beta^1, \beta^2, \beta^m)\), we obtain:

\[
\text{dim}(\{\lambda\}^\omega \otimes \{\lambda^i\}^\omega \otimes \{\lambda^m\}^\omega)_{\text{stab}_\omega} = \sum_{\beta^1, \beta^2, \beta^m} j_{\beta, \omega, \rho}(\lambda) j_{\beta^1, \omega, \rho}(\lambda^i) j_{\beta^2, \omega, \rho}(\lambda^m) \prod_{i=1}^m k(\beta^i, \beta^m),
\]

where the sum for \(\beta\) is over all lists of partitions such that \(\beta^i \vdash \kappa_i\) and analogously for \(\beta^1\) and \(\beta^2\).

The following second proof for (22.0.1) is taken from the lecture notes [BI]:

22.4.5 Corollary. As at the beginning of Chapter 22, let \(\lambda = \lambda^i = \lambda^m\) be the hook partition with \(3k + 1\) boxes and \(2k + 1\) rows. Then \(\text{mult}_{\lambda, \lambda^i, \lambda^m}(\mathfrak{G}_3^{3k}(3k)) = 0\).
Proof. We use the formula in Theorem 22.4.3. Since it has no signs, we can assume (for the sake of contradiction) that the formula yields a is positive result and derive conditions on the partitions that are involved in positive summands.

We use a few standard facts about Littlewood-Richardson coefficients, plethysm coefficients, and Kronecker coefficients, each marked with a †.

First observation: \( \nu^1 = \beta^1 \), because of the plethysm \( a_{\nu^1}(\beta^1,1) = \text{mult}_{\nu^1}(S^{\beta^1}(Sym^1 V)) \).

A multi-LR-coefficients can only be positive if all small partitions are contained in the large partition, i.e., the small Young diagrams are subsets of the large Young diagram (†). In our case, all large partitions are hooks, so all \( \nu^i \) are hooks. Thus also \( \beta^1, \beta^2, \beta^{m1} \) are hooks.

Let \( d \) be the number of boxes. For a hook \( \nu^1 \) define the inner leg length as \( \ell(\nu^1) = 1 \). For hook triples with inner leg lengths \( a_1, a_2, a_3 \), Kronecker positivity requires (†, see e.g. [Ros01, Pf. of Thm. 3(4.)]):

\[
2d - a_1 - a_2 - a_3 - 2 \geq 0.
\]

Thus not all three \( a_1, a_2, a_3 \) can be large. Indeed, let \( a = \min\{a_1, a_2, a_3\} \), then \( 2d - 3a - 2 \geq 0 \) and thus \( a \leq \frac{2d - 2}{3} \). In particular this holds for \( k(\nu^1, \nu^1, \nu^{m1}) = k(\beta^1, \beta^2, \beta^{m1}) > 0 \). W.l.o.g. \( \nu^1 \) is the shortest of \( \nu^1, \nu^1, \nu^{m1} \). Then

\[
\ell(\nu^1) - 1 \leq \frac{2|\nu^1| - 2}{3} < \frac{2}{3} |\nu^1| - \frac{2}{3}
\]

and thus

\[
\ell(\nu^1) \leq \frac{2}{3} |\nu^1| + \frac{1}{3}.
\]

All partitions appearing in \( \bigotimes^a Sym^b V \) have at most \( a \) rows, as the basis of HWVs is given by semistandard tableaux with content \( (b, b, \ldots, b) \). Therefore the positive plethysm coefficients in the formula imply

\[
\ell(\nu^i) \leq |\beta^1| = \kappa_i = \nu^i_i
\]

Adding up the lengths we obtain

\[
\ell(\nu^1) + \cdots + \ell(\nu^f) \leq \frac{2}{3} |\nu^1| + \frac{1}{3} + \frac{1}{2} (|\nu^2| + \cdots + |\nu^f|) = 3k + 1 - |\nu^1|
\]

\[
= \frac{2}{3} |\nu^1| + \frac{1}{3} + \frac{3}{2} k + \frac{1}{2} - \frac{1}{2} |\nu^1| = \frac{3}{2} k + \frac{1}{6} |\nu^1| + \frac{5}{6}
\]

We now use that for a positive multi-LRC the lengths of the small partitions add up to at least the length of the large partition (†):

\[
\ell(\nu^1) + \cdots + \ell(\nu^f) \geq \ell(\lambda) = 2k + 1.
\]

Therefore

\[
\frac{3}{2} k + \frac{5}{6} |\nu^1| \geq 2k + 1 \iff -\frac{1}{2} k - \frac{1}{6} \geq 0 \iff |\nu^1| \geq 3k + 1.
\]

Since \( |\nu^1| = \kappa_1 \), this means that \( g^1 = (1^{3k+1}) \), but the sum is only over \( g^1 \) with at most \( 3k \) rows.

Occurrence obstructions for matrix multiplication
The tensor setting is analogous polynomial setting, but the group \( \text{GL}_{n^2} \) is replaced by \( \text{GL}_n \times \text{GL}_n \times \text{GL}_n \).

One can explicitly construct occurrence obstructions that show border rank lower bounds on the matrix multiplication tensor. Obstruction designs help visualizing the arguments.
Appendix
Appendix A

Some basic algebraic vocabulary

This appendix contains some basic notions from algebra.

A monoid \((G, \cdot, e)\) is set with a binary operation \(\cdot : G \times G \to G\) and a so-called neutral element \(e \in G\) such that the following conditions hold:

1. Associativity: For all \(a, b, c \in G\) we have \(a \cdot (b \cdot c) = (a \cdot b) \cdot c\).
2. Existence of identity: For all \(a \in G\) we have \(e \cdot a = a = a \cdot e\).

We omit the multiplication dot if there is no possibility of confusion. For example, the set \(\mathbb{C}^{n \times n}\) of complex \(n \times n\) matrices with binary operation the matrix multiplication is a monoid. A subset of a monoid \(G\) which contains the identity element and satisfies associativity is called a submonoid of \(G\).

For example, the set of upper triangular complex \(n \times n\) matrices with operation the matrix multiplication is a monoid. A monoid homomorphism from a monoid \((G, \cdot, e)\) to a monoid \((G', \cdot', e')\) is a map \(\varphi : G \to G'\) which satisfies \(\varphi(g \cdot h) = \varphi(g) \cdot' \varphi(h)\) for all \((g, h) \in G \times G\) and \(\varphi(e) = e'\).

A group is a monoid in which for each element \(a \in G\) we have an element \(a^{-1} \in G\) such that \(a^{-1} \cdot a = e = a \cdot a^{-1}\). The element \(a^{-1}\) is called the inverse element of \(a\). For example, the set of invertible (i.e., nonzero determinant) complex \(n \times n\) matrices with operation the matrix multiplication is a group, the so-called general linear group \(\text{GL}_n\), where \(a^{-1}\) is the inverse matrix. Another example is the symmetric group on \(n\) letters \(\mathfrak{S}_n\), which consists of all bijective maps from the set \(\{1, 2, \ldots, n\}\) to \(\{1, 2, \ldots, n\}\) and the operation is the composition of maps, where \(a^{-1}\) is the inverse permutation. A monoid homomorphism between groups is called a group homomorphism. A group \(G\) is called abelian if it is commutative as a monoid. For example, the set \((\mathbb{C}, +, 0)\) of complex numbers with addition is an abelian group. Moreover, the set \((\mathbb{C} \setminus \{0\}, \cdot, 1)\) of complex numbers (without zero) with multiplication is an abelian group. Also the set \((\mathbb{C}^{n \times m}, +, 0)\) of \(n \times m\) matrices with addition is an abelian group.

A field \((\mathbb{F}, +, \cdot, 0, 1)\) is a set with two binary operations and two specific elements that satisfies:

1. \((\mathbb{F}, +, 0)\) and \((\mathbb{F} \setminus \{0\}, \cdot, 1)\) are abelian groups, and
2. distributivity holds, i.e., for all \(a, b, c \in \mathbb{F}\) we have \(a \cdot (b + c) = (a \cdot b) + (a \cdot c)\).

For example, the complex numbers \(\mathbb{C}\) form a field.

If we do not require the existence of multiplicative inverse elements, then our algebraic structure is called a ring, more precisely: A ring \((R, +, \cdot, 0, 1)\) is a set with two binary operations and two neutral elements that satisfies:

1. \((R, +, 0)\) is an abelian group,
2. \((R, \cdot, 1)\) is a monoid,
3. distributivity holds, i.e., for all \(a, b, c \in R\) we have \(a \cdot (b + c) = (a \cdot b) + (a \cdot c)\).

If the monoid \((R, \cdot, 1)\) is commutative, then we call the ring \(R\) a commutative ring. For example, the set \(\mathbb{C}[X_1, X_2, \ldots, X_n]\) of polynomials in \(n\) variables is a commutative ring. The set \(\mathbb{C}^{n \times n}\) of \(n \times n\) matrices with matrix multiplication and addition is a noncommutative ring for \(n > 1\). A
ring homomorphism between two rings $R_1$ and $R_2$ is a map from $R_1$ to $R_2$ which at the same time is a group homomorphism for the additive structure and a monoid morphism for the multiplicative structure. A bijective ring homomorphism is called a ring isomorphism.

An ideal $I$ of a commutative ring $(R, +, \cdot, 0, 1)$ is a subset $I \subseteq R$ that forms an abelian group $(I, +)$ with the rings addition and moreover is closed under ring multiplication, i.e., for all $a \in R$, $b \in I$ we have $a \cdot b \in I$. For example, the set of polynomials $f$ in $\mathbb{C}[X_1, X_2, \ldots, X_n]$ that are divisible by $X_2$ form an ideal of $\mathbb{C}[X_1, X_2, \ldots, X_n]$. If $\mathbb{F}$ is a field, $\alpha \in \mathbb{F}$, and $a$ is an element in a vector space over $\mathbb{F}$, then we write $\alpha \cdot a$ for the scalar multiplication. A vector space over the complex numbers is also called a $\mathbb{C}$-vector space. A ring $(\mathbb{A}, +, \cdot, 0, 1)$ that is also $\mathbb{C}$-vector space (with the same addition) is called a $\mathbb{C}$-algebra, if $\mathbb{A}$ satisfies $(\alpha \cdot 1) \cdot a = \alpha \cdot a$ for all $\alpha \in \mathbb{C}$ and all $a \in \mathbb{A}$. If the $\mathbb{C}$-algebra $\mathbb{A}$ is a commutative ring, then we call $\mathbb{A}$ a commutative $\mathbb{C}$-algebra. For example the set $\mathbb{C}[X_1, X_2, \ldots, X_n]$ of polynomials in $n$ variables is a commutative $\mathbb{C}$-algebra.

A vector space $V$ is a direct sum of linear subspaces $V_i \subseteq V$, written $V = \bigoplus_i V_i$, if the union $\bigcup_i V_i$ spans $V$ and each intersection $V_i \cap \sum_{j \neq i} V_j$ is the zero space. If $V = \bigoplus_i V_i$, then each element in $V$ has a unique representation as a sum of elements from the $V_i$.

A $\mathbb{C}$-algebra $\mathbb{A}$ is called graded, if the vector space $\mathbb{A}$ is a direct sum $\mathbb{A} = \bigoplus_{d \in \mathbb{N} \cup \{0\}} \mathbb{A}_d$ such that the ring multiplication satisfies $a \cdot a' \in \mathbb{A}_{d+d'}$ for all $a \in \mathbb{A}_d$ and $a' \in \mathbb{A}_{d'}$. For example the algebra $\mathbb{C}[X_1, X_2, \ldots, X_n]$ is graded as follows: The linear subspace $\mathbb{C}[X_1, X_2, \ldots, X_n]_d$ is spanned by the monomials of degree $d$, where the degree is the sum of exponents, e.g., the monomial $X_1^2X_2^3$ has degree 5. We call $\mathbb{C}[X_1, X_2, \ldots, X_n]_d$ the homogeneous degree $d$ component of $\mathbb{C}[X_1, X_2, \ldots, X_n]$ and elements of $\mathbb{C}[X_1, X_2, \ldots, X_n]_d$ are said to be homogeneous of degree $d$. For example $X_1^2X_2^3 - X_1X_2^3$ is not homogeneous. The degree of a nonhomogeneous polynomial is defined to be the maximal degree of its monomials. We define $\mathbb{C}[X_1, X_2, \ldots, X_n]_{\leq d}$ to be the vector space of (not necessarily homogeneous) polynomials of degree at most $d$. A homomorphism of $\mathbb{C}$-algebras is a linear map that is a ring homomorphism. An isomorphism of $\mathbb{C}$-algebras is a linear map that is a ring isomorphism. An isomorphism of graded $\mathbb{C}$-algebras $f: \mathbb{A} \to \mathbb{B}$ is defined to be an isomorphism of graded $\mathbb{C}$-algebras such that the restriction of $f$ to each homogeneous degree $i$ part $\mathbb{A}_i$ is a vector space isomorphism to $\mathbb{B}_i$. 

APPENDIX A. SOME BASIC ALGEBRAIC VOCABULARY 144
Bibliography


