Determinantal complexity

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Chapter 1

Determinantal complexity

The question whether $\text{VP}_{ws} = \text{VNP}$ can be rephrased as the question whether $\text{det}$ is $p$-projection of $\text{per}$. Related questions have been studied. One of them is the so-called determinantal complexity.

1.0.1 Definition. The determinantal complexity $D(f)$ of a polynomial $f \in F[X_1, \ldots, X_n]$ is the smallest $s$ such that there are affine linear forms $\alpha_{i,j} \in F[X_1, \ldots, X_n]$, $1 \leq i, j \leq s$, such that we can write $f = \text{det}_s(\alpha_{i,j})$.

1.0.2 Lemma. $(f_n) \in \text{VP}_{ws}$ iff $\text{dc}(f_n)$ is $p$-bounded.

Proof. If $(f_n) \in \text{VP}_{ws}$, then it is a $p$-projection of $\text{det}$. Therefore, its determinantal complexity is $p$-bounded. For the other direction, note that the determinant has weakly skew circuits of polynomial size. We can compute the affine linear forms by weakly skew circuits of polynomial size. Therefore, $(f_n)$ has weakly polynomial circuits of polynomial size. \qed

1.1 Mignon-Ressayre bound

In this section, we prove the best lower bound for the determinantal complexity, due to Mignon and Ressayre [MR04].

1.1.1 Observation. $\frac{\partial^2}{\partial X_{i,j} \partial X_{k,\ell}} \text{per}(X)$ is the permanent of the $(n-1) \times (n-1)$ matrix obtained from $X$ by deleting the $i$th row and the $j$th column. The same is true for the determinant.

For a matrix $A \in F^{n \times n}$, let $H_{\text{per}}(A)$ denote the $n^2 \times n^2$-matrix with the entry in row $(i, j)$ and column $(k, \ell)$ being equal to

$$\frac{\partial^2}{\partial X_{i,j} \partial X_{k,\ell}} \text{per}_n(A).$$

That is, we take the permanent polynomial, differentiate it twice and then we plug in the values from the matrix $C$.

For the proof, we need to construct a matrix $A$ such that $\text{per}(A) = 0$ but $H_{\text{per}}(A)$ has full rank $n^2$. Let $A$ be the matrix

$$A = \begin{pmatrix}
1 - n & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 1
\end{pmatrix},$$

that is, $a_{1,1} = 1 - n$ and all other entries are 1.

1.1.2 Lemma. $\text{per}(A) = 0$. 

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**Proof.** Like for the determinant, we can do a Laplace expansion of the permanent. It is even easier, since there are no signs to keep track off. (If you prefer to think in terms of cycle covers, a Laplace expansion along the \(i\)th row just groups the cycle covers depending on which node is visited right after node \(i\).

So we do a Laplace expansion along the first row. Since all other rows are filled only with 1’s, all the submatrices that we get are the same. We get it once multiplied by \(1 - n\) and \(n - 1\) times multiplied by 1. So the sum is 0. □

**1.1.3 Lemma.** \(H_{\text{per}}(A)\) has rank \(n^2\).

**Proof.** When \(i = k\) or \(j = \ell\), then

\[
\frac{\partial^2}{\partial X_{i,j} \partial X_{k,\ell}} \text{per}_n(X) = 0.
\]

This is due to the fact that every monomial contains only one variable from each row or column. (This property is called set-multilinear).

If \(i \neq k\) and \(j \neq \ell\), then

\[
\frac{\partial^2}{\partial X_{i,j} \partial X_{k,\ell}} \text{per}_n(X)
\]

is the permanent of the submatrix obtained by deleting rows \(i\) and \(k\) and columns \(j\) and \(\ell\) from \(X\). If \(1 \notin \{i, j, k, \ell\}\), then

\[
\frac{\partial^2}{\partial X_{i,j} \partial X_{k,\ell}} \text{per}_n(A) = (n - 2)!,
\]

since the matrix that we obtain from \(A\) after deleting the rows and columns is the all-ones-matrix of size \((n - 2) \times (n - 2)\), the permanent of which is \((n - 2)!\). If \(1 \notin \{i, j, k, \ell\}\), then

\[
\frac{\partial^2}{\partial X_{i,j} \partial X_{k,\ell}} \text{per}_n(A) = -2(n - 3)!,
\]

since the matrix that we obtain from \(A\) in this case has \(n - 1\) in position \((1, 1)\) and 1’s elsewhere. Using Laplace expansion, one can easily see that the permanent of this matrix is \(-2(n - 3)!\).

Therefore, we have that

\[
H_{\text{per}}(A) = (n - 3)!
\]

where

\[
B = \begin{pmatrix}
0 & B & B & \ldots & B \\
B & 0 & C & \ldots & C \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
B & C & \ldots & C & 0
\end{pmatrix}
\]

and

\[
C = \begin{pmatrix}
0 & n - 2 & n - 2 & \ldots & n - 2 \\
n - 2 & 0 & -2 & \ldots & -2 \\
n - 2 & -2 & 0 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
n - 2 & -2 & \ldots & -2 & 0
\end{pmatrix}.
\]
The matrix $B$ has full rank: If we subtract the $(n-1)$th row from the $n$th, then the $(n-2)$th row from the $(n-1)$th and so on until we subtract the first row from the second, we get the matrix

$$
(n-2) \begin{pmatrix}
0 & 1 & 1 & \ldots & 1 \\
1 & -1 & 0 & \ldots & 0 \\
0 & 1 & -1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & -1 
\end{pmatrix}
$$

From the structure of the rows 2 to $n$ it follows that every nontrivial vector in the kernel of the matrix has to be a nonzero multiple of the all-ones-vector. The scalar product of this vector with the first row is however nonzero. Therefore, $B$ has full rank. Doing the same transformation with $C$, but stopping one row earlier, we get the matrix

$$
\begin{pmatrix}
0 & n-2 & n-2 & \ldots & n-2 \\
n-2 & 0 & -2 & \ldots & -2 \\
0 & -2 & 2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & -2 & 2 
\end{pmatrix}
$$

From the structure of the third to $n$th row, we get that the entries at positions 2 to $n$ of every nontrivial vector in the kernel have to be the same. From the first row, it follows that these entries have to be 0. And finally, the second row tells us that also the first entry has to be zero then. Therefore, $C$ is also invertible.

We have

$$
\begin{pmatrix}
CB^{-1} & 0 & \ldots & 0 \\
0 & I & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & I 
\end{pmatrix} H_{\text{per}}(A) \begin{pmatrix}
B^{-1}C & 0 & \ldots & 0 \\
0 & I & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & I 
\end{pmatrix} = (n-3)! \begin{pmatrix}
0 & C & C & \ldots & C \\
C & 0 & C & \ldots & C \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
C & C & \ldots & 0 & C \\
0 & 1 & \ldots & 1 
\end{pmatrix} \otimes C.
$$

Since the Kronecker product of two full rank matrices has itself full rank, $H_{\text{per}}(A)$ has full rank.

**1.1.4 Theorem.** $\text{dc}(\text{per}_n) \geq n^2/2$

**Proof.** Let $s$ be the determinantal complexity of $\text{per}_n$. We know that there are affine linear forms $\alpha_{i,j}(X), 1 \leq i,j \leq s$ such that

$$
\text{per}_n(X) = \det_s(\alpha_{i,j}(X)).
$$

Let $A = (a_{i,j})$ be the matrix from the previous lemma. Write $\alpha_{i,j}(X) = \lambda_{i,j}(X-A) + y_{i,j}$, where $\lambda_{i,j}$ is a homogeneous linear form and $y_{i,j}$ is a constant, i.e., we perform a translation on the coordinates. Thus,

$$
\text{per}_n(X) = \det_s(\lambda_{i,j}(X-A) + y_{i,j}).
$$

Since $\text{per}_n(A) = 0$, we have $\det_s(y_{i,j}) = 0$, so $Y := (y_{i,j})$ is not of full rank. Let $S$ and $T$ be invertible matrices such that

$$
\text{SYT} = \begin{pmatrix}
0 & 0 \\
0 & I_t
\end{pmatrix}
$$
for some $t < s$. Multiplying the matrix on the righthand side of (1.1.5) by
\[
\begin{pmatrix}
1 / \det S & 0 \\
0 & I_{s-1}
\end{pmatrix}
S \quad \text{and} \quad T \begin{pmatrix}
1 / \det T & 0 \\
0 & I_{s-1}
\end{pmatrix}
\]
from the left and the right, respectively, does not change its determinant. As
\[
\begin{pmatrix}
1 / \det S & 0 \\
0 & I_{s-1}
\end{pmatrix}
\text{SYT} \begin{pmatrix}
1 / \det T & 0 \\
0 & I_{s-1}
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & I_t
\end{pmatrix}
\]
(recall that $t < s$), we can assume w.l.o.g. that $(y_{i,j})$ is of this form. Define $H_{\det}$ in the same way as $H_{\per}$. Now we differentiate both sides of (1.1.5). We get
\[
H_{\per}(X) = LH_{\det}(\alpha_{i,j}(X - A) + y_{i,j})L^T
\]
for some matrix $L$ with entries from $F$ by the chain rule (see below). Therefore,
\[
\text{rk} \ H_{\per}(A) = \text{rk} \ H_{\det}(Y)L^T
\]
and
\[
\text{rk}(H_{\det}(Y)) \leq 2s.
\]
By the previous Lemma, we know that $\text{rank} \ H_{\per}(A) = n^2$. Therefore, we are done when we show that $\text{rank}(H_{\det}(Y)) \leq 2s$.

Let us first consider the case when $t = s - 1$. An entry of $H_{\det}(y_{i,j})$ is of the form
\[
\frac{\partial^2}{\partial X_{e,f} \partial X_{k,\ell}} \det_s(y_{i,j}).
\]
This entry can only be nonzero, if differentiating removes the first row and the first column and the $h$th row and $h$th column for any other $h$. This means that
1. $(e, f) = (1, 1)$ and $(k, \ell) = (h, h)$,
2. $(e, f) = (1, 1)$ and $(k, \ell) = (h, 1)$,
3. $(e, f) = (h, 1)$ and $(k, \ell) = (1, h)$,
4. $(e, f) = (h, h)$ and $(k, \ell) = (1, 1)$.

In the first case, we get one row with $s - 1$ ones in it. In the fourth case, we get one column with $s - 1$ ones in it. In the second case, we get $s - 1$ different rows, each having a single one and zeros elsewhere. The third case is similar. Altogether, we get that the rank is at most $2 + 2(s - 1) = 2s$.

When $t = s - 2$, then every entry of $H_{\det}(Y)$ will be zero.

1.1.6 Observation. Let $f$ be a polynomial in $Y_1, \ldots, Y_m$ variables and $\ell_1, \ldots, \ell_m$ be affine linear forms in $X_1, \ldots, X_n$. Then by the chain rule
\[
\frac{\partial^2}{\partial X_i \partial X_j} f(\ell_1, \ldots, \ell_n) = \sum_{s=1}^m \sum_{t=1}^m \frac{\partial^2}{\partial Y_s \partial Y_t} f(\ell_1, \ldots, \ell_n) \frac{\partial}{\partial X_i} \ell_s \frac{\partial}{\partial X_j} \ell_t
\]
Note that $\frac{\partial}{\partial X_i} \ell_s$ and $\frac{\partial}{\partial X_j} \ell_t$ are just constants. Let $L = \left( \frac{\partial}{\partial X_i} \ell_s \right)_{1 \leq s \leq m, 1 \leq i \leq n}$. Then
\[
\left( \frac{\partial^2}{\partial X_i \partial X_j} f(\ell_1, \ldots, \ell_n) \right) = L \left( \frac{\partial^2}{\partial Y_s \partial Y_t} f(\ell_1, \ldots, \ell_n) \right) L^T
\]
Bibliography