Exercise 1 (10 points).
Let $G$ be a finite group and let $V$ and $W$ be two $G$-representations (in particular, $V$ and $W$ are finite dimensional). Then the tensor product $V \otimes W$ of vector spaces is a $G$-representation via
\[
g(v \otimes w) := gv \otimes gw
\]
and linear continuation. Prove that the character $\chi_{V \otimes W}$ satisfies $\chi_{V \otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$.

Exercise 2 (10 points).
The tensor product of Specht modules $[(2, 1)] \otimes [(2, 1)]$ is a 4-dimensional $S_3$-representation. Compute its character and decompose it as a linear combination of characters of irreducible $S_3$-representations.

Exercise 3 (10 points).
The tensor power of Specht modules $W := [(2, 1)]^\otimes n$ is a $2^n$-dimensional $S_3$-representation via
\[
g(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_n
\]
and linear continuation. Determine the multiplicities of the irreducible $S_3$-representations in $W$.

Exercise 4 (10 points).
Let $H \leq G$ be a subgroup of a (not necessarily finite) group. For each $g \in G$ we define the coset as its orbit under the right multiplication:
\[
gH := \{gh \mid h \in H\}.
\]
Prove that distinct cosets have empty intersection. Also prove that all cosets have the same cardinality.