Exercise 1 (10 Points). Let $K_n$ denote the continuant, which is the $(1,1)$-entry of the product

$\left( \begin{array}{ccc} x_1 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{ccc} x_2 & 1 \\ 1 & 0 \end{array} \right) \cdots \left( \begin{array}{ccc} x_n & 1 \\ 1 & 0 \end{array} \right)$

What is the coefficient of the monomial $x_{i_1} \cdot x_{i_2} \cdots x_{i_\ell}$ in $K_n$?

Solution 1. For this, we can look at the ABP computing $K_n$. By looking at this ABP, we note that if a monomial $x_{i_1} \cdot x_{i_2} \cdots x_{i_\ell}$ appears in $K_n$ and a variable $x_k$ is not present in $x_{i_1} \cdot x_{i_2} \cdots x_{i_\ell}$ then at least one of $x_k$ or $x_{k-1}$ must also be missing from $x_{i_1} \cdot x_{i_2} \cdots x_{i_\ell}$. Moreover, this is a sufficient condition also for a monomial $x_{i_1} \cdot x_{i_2} \cdots x_{i_\ell}$ to appear in $K_n$. Thus $x_{i_1} \cdot x_{i_2} \cdots x_{i_\ell}$ appears in $K_n$ iff $x_{i_1} \cdot x_{i_2} \cdots x_{i_\ell}$ can be obtained by $x_1 \cdot x_2 \cdots x_n$ by removing disjoint pairs of consecutive variables. Also, it is easy to observe that all the monomials in $K_n$ have coefficient one. Thus the coefficient of the monomial $x_{i_1} \cdot x_{i_2} \cdots x_{i_\ell}$ in $K_n$ is 1 iff $x_{i_1} \cdot x_{i_2} \cdots x_{i_\ell}$ can be obtained from $x_1 \cdot x_2 \cdots x_n$ by removing disjoint pairs of consecutive variables. Otherwise this coefficient is zero.

Exercise 2 (10 Points). Prove that $K_n(x_1, x_2, \ldots, x_n) = K_n(x_n, x_{n-1}, \ldots, x_1)$.

Solution 2 (Typeset by all students in the lecture). This directly follows since the $(1,1)$ entry does not change under transposing a matrix and $(AB)^t = B^t A^t$. Thus we have

$K_n(x_1, x_2, \ldots, x_n) = \left( \begin{array}{ccc} x_1 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{ccc} x_2 & 1 \\ 1 & 0 \end{array} \right) \cdots \left( \begin{array}{ccc} x_n & 1 \\ 1 & 0 \end{array} \right)_{(1,1)}$

$= \left( \begin{array}{ccc} x_1 & 1 \\ 1 & 0 \end{array} \right)^t \left( \begin{array}{ccc} x_2 & 1 \\ 1 & 0 \end{array} \right)^t \cdots \left( \begin{array}{ccc} x_n & 1 \\ 1 & 0 \end{array} \right)^t_{(1,1)}$

$= \left( \begin{array}{ccc} x_{n-1} & 1 \\ 1 & 0 \end{array} \right)^t \left( \begin{array}{ccc} x_{n-2} & 1 \\ 1 & 0 \end{array} \right)^t \cdots \left( \begin{array}{ccc} x_1 & 1 \\ 1 & 0 \end{array} \right)^t_{(1,1)}$

$= \left( \begin{array}{ccc} x_n & 1 \\ 1 & 0 \end{array} \right)^t \left( \begin{array}{ccc} x_{n-1} & 1 \\ 1 & 0 \end{array} \right)^t \cdots \left( \begin{array}{ccc} x_1 & 1 \\ 1 & 0 \end{array} \right)^t_{(1,1)} = K_n(x_n, x_{n-1}, \ldots, x_1)$
One can also prove $K_n(x_1, x_2, \ldots, x_n) = K_n(x_n, x_{n-1}, \ldots, x_1)$ by reversing every edge in the ABP computing $K_n$.

Remark by Ikenmeyer: One can also use Exercise 1: Adjacency of variables does not change when we reverse the order.

**Exercise 3 (10 Points).** Prove that

$$K_n = \det \begin{pmatrix} x_1 & 1 & 0 & \ldots & 0 \\ -1 & x_2 & 1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \ldots & 0 & -1 & x_n \end{pmatrix}.$$ 

**Solution 3.** In order to prove this we will first prove the following claim:

$$K_n(x_1, x_2, \ldots, x_n) = x_n \cdot K_{n-1}(x_1, x_2, \ldots, x_{n-1}) + K_{n-2}(x_1, x_2, \ldots, x_{n-2}).$$

This is easy to see by looking at the ABP computing $K_n$ or by a short calculation using the definition of $K_n$ as the $(1,1)$-entry of the product of matrices.

Now the actual exercise can be solved by induction. The base case for $K_2$ is trivial since

$$\det \begin{pmatrix} x_1 & 1 \\ -1 & x_2 \end{pmatrix} = x_1 x_2 + 1.$$ 

For the induction step we use Laplace to compute the determinant:

$$\det \begin{pmatrix} x_1 & 1 & 0 & \ldots & 0 \\ -1 & x_2 & 1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \ldots & 0 & -1 & x_{n+1} \end{pmatrix} = x_{n+1} \cdot \det A_n + \det A_{n-1}$$

where $A_n$ denotes the matrix that is given in the exercise. Thus $\det A_n$ satisfies the same recursion formula as $K_n$ which concludes the proof.

**Exercise 4 (10 Points).** For a homogeneous degree $m$ polynomial $h$ we define $L(h)$ to be the smallest $n$ such that $x_1^{n-m} h \in \mathbb{GL}_n \mathbb{K}_n$ (as usual, the variables in $h$ are ordered consecutively).
In the lecture we showed that a sequence \((h_m)\) is in \(\overline{\mathbb{P}_e}\) iff its sequence \(L(h_m)\) is polynomially bounded. Prove that this is still true if we replace \(K_n\) by any of the two polynomials

\[
K'_n = \det \begin{pmatrix} x_1 & 1 & 0 & \ldots & 0 \\ 1 & x_2 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & x_n \end{pmatrix}
\]

or

\[
K''_n = \text{tr} \left( \begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} x_n & 1 \\ 1 & 0 \end{pmatrix} \right)
\]

**Solution 4.** For this, we need to show that \(K'_n\) and \(K''_n\) can be used to approximate any \((h_m) \in \overline{\mathbb{P}_e}\) with \(n \leq \text{poly}(m)\), and vice-versa. As usual, we use the notation \(i = \sqrt{-1}\). Now we show that \(i^n \cdot K_n\) reduces to \(K'_n\) under suitable reductions.

Note that \(K'_n(i \cdot x_1, i \cdot x_2, \ldots, i \cdot x_n) = i^n \cdot (K_n(x_1, x_2, \ldots, x_n))\). This can be proved by induction on \(n\) and using the recurrence for \(K_n\), which was derived in the solution of exercise 3. Thus if \((h_m)\) can be approximated by \(K_{\text{poly}(m)}\) then it can also be approximated by \(K'_{\text{poly}(m)}\). Moreover, this reduction from \(K_n\) to \(K'_n\) which we showed above can be easily modified to work in other direction also. Thus \((h_m)\) can be approximated by \(K_{\text{poly}(m)}\) iff \((h_m)\) can be approximated by \(K'_{\text{poly}(m)}\). Thus \((h_m) \in \overline{\mathbb{P}_e}\) iff \((h_m)\) can be approximated by \(K'_{\text{poly}(m)}\).

Now we prove that \(K''_{\text{poly}(m)}\) can be used to approximate any \((h_m) \in \overline{\mathbb{P}_e}\). This directly follows from Proposition 3.6 from the paper “On algebraic branching programs of small width”, see Proposition 3.6 in [link to paper]. Moreover we also know that \((K''_n) \in \mathbb{P}_e\). Thus \((h_m) \in \overline{\mathbb{P}_e}\) iff \((h_m)\) can be approximated by \(K''_{\text{poly}(m)}\).