As in the lecture, to each obstruction design $\mathcal{H}_n$ we have the corresponding highest weight vector function $f_{\mathcal{H}_n}$.

**Exercise 1** (10 Points). Consider two obstruction designs $\mathcal{H}_1$ and $\mathcal{H}_2$. Take their union (the usual union of hyper-graphs by just drawing them next to each other) and call the resulting obstruction design $\mathcal{H}_3$. Prove that

$$f_{\mathcal{H}_3} = f_{\mathcal{H}_1} \cdot f_{\mathcal{H}_2}$$

as functions.

**Solution 1** (Typeset by all students in the lecture). We have the following definition for $f_{\mathcal{H}}$:

$$f_{\mathcal{H}}(w) = \sum_{J: \mathcal{H} \to \tau} \text{eval}_\mathcal{H}(J) = \sum_{J: \mathcal{H} \to \tau} \prod_{k=1}^{3} \text{eval}_{E(k)}(J^{(k)})$$

$$= \sum_{J: \mathcal{H} \to \tau} \prod_{k=1}^{3} \prod_{e \in E(k)} \det J^{(k)}|_e$$

Now we get:

$$f_{\mathcal{H}_1} \cdot f_{\mathcal{H}_2} = (\sum_{J_1: \mathcal{H}_1 \to \tau} \prod_{k=1}^{3} \prod_{e_1 \in E_1^{(k)}} \det J_1^{(k)}|_{e_1}) \cdot (\sum_{J_2: \mathcal{H}_2 \to \tau} \prod_{l=1}^{3} \prod_{e_2 \in E_2^{(l)}} \det J_2^{(l)}|_{e_2})$$

Now since $\mathcal{H}_1$ and $\mathcal{H}_2$ are disjoint sets we can combine both functions to a new function $J: \mathcal{H}_3 \to \tau$ where $\mathcal{H}_3 = \mathcal{H}_1 \cup \mathcal{H}_2$. Here $J_3$ basically covers all possible pairs of $J_1$ and $J_2$.

Now we can rewrite the product above as follows:

$$\sum_{J_3: \mathcal{H}_3 \to \tau} (\prod_{k=1}^{3} \prod_{e_1 \in E_1^{(k)}} \det J_1^{(k)}|_{e_1}) \cdot (\prod_{l=1}^{3} \prod_{e_2 \in E_2^{(l)}} \det J_2^{(l)}|_{e_2})$$
Now each summand has basically 6 factors each (where each of these factors is again a product of determinants, but we do not care about that right now). In the next step we just divide these 6 factors into 3 groups which consists of pairs. Doing that we obtain:

$$\sum_{J_3: \mathcal{H}_3 \rightarrow \tau} \prod_{k=1}^{3} \left( \prod_{e_1 \in E_1^{(k)}} \det J_1^{(k)}|e_1| \cdot \prod_{e_2 \in E_2^{(k)}} \det J_2^{(k)}|e_2| \right)$$

Again we just the fact that $\mathcal{H}_1$ and $\mathcal{H}_2$ were disjoint obstruction designs to combine $E_1^{(k)}$ and $E_2^{(k)}$ into $E_3^{(k)}$ which then gives us:

$$\sum_{J_3: \mathcal{H}_3 \rightarrow \tau} \prod_{k=1}^{3} \prod_{e \in E_3^{(k)}} \det J_3^{(k)}|e| = f_{\mathcal{H}_3}$$

**Exercise 2** (15 Points). Consider the following sequence of obstruction designs $\mathcal{H}_n$.

$$n = 1 \quad n = 2 \quad n = 3 \quad \ldots$$

Prove that $f_{\mathcal{H}_n}$ is the zero function if $n > 1$ is odd.

**Solution 2** (Typeset by all students in the lecture). We can directly translate the obstruction design into the following tensor of young tableaux (in this case the example of $n = 3$):

$$\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\otimes & \otimes & \otimes
\end{array}$$

Note that the values in the columns of the second tableaux are exactly the rows in the third one and vice-versa.

If we now symmetrize over the subgroup of $\mathfrak{S}_{n^2}$ given by swapping two fixed rows in the second tableaux we get 0, because the first tableaux is always invariant, the second one will get a negative sign since we swap an odd number of elements inside the columns and the third one stays invariant since just swapping back the columns. Thus symmetrizing over $\mathfrak{S}_{n^2}$ will also yield the $\mathfrak{S}_{n^2}$ tensor so $f_{\mathcal{H}_n}$ is also the zero function if $n > 1$ is odd.

**Exercise 3** (15 Points). For odd $n$, find an obstruction design $\mathcal{H}_n'$ of the same type as $\mathcal{H}_n$ in the previous exercise, but with $f_{\mathcal{H}_n'} \neq 0$.

**Solution 3.** We can achieve this by making the dashed and the continuous hyper-edges the same. If we now evaluate we get:

$$f_{\mathcal{H}_n'}(w) = \sum_{J: \mathcal{H} \rightarrow \tau} \prod_{e \in E_1^{(1)}} \det J_1^{(1)}|e| \cdot \prod_{e \in E_2^{(2)}} \det J_2^{(2)}|e| \cdot \prod_{e \in E_3^{(3)}} \det J_3^{(3)}|e|$$
If we evaluate \( f_{H_n}(w) \) on \( w = \sum_{i=1}^{n} e_i \otimes e_i \otimes g e_i \) for some \( g \in \text{GL}_n(\mathbb{R}) \), then \( \prod_{e \in E^{(2)}} \det J^{(2)}|e \) and \( \prod_{e \in E^{(3)}} \det J^{(3)}|e \) always evaluate to the same real number and thus \( \prod_{e \in E^{(2)}} \det J^{(2)}|e \cdot \prod_{e \in E^{(3)}} \det J^{(3)}|e \) is always non-negative. If we assume that the first row of \( g \) is positive, \( \prod_{e \in E^{(1)}} \det J^{(1)}|e \) is always positive.

It is also easy to observe that there are terms \( \prod_{e \in E^{(1)}} \det J^{(1)}|e \cdot \prod_{e \in E^{(2)}} \det J^{(2)}|e \cdot \prod_{e \in E^{(3)}} \det J^{(3)}|e \) in above sum which evaluate to non-zero. Thus \( f_{H_n}(w) \neq 0 \).