Exercise 1 (15 Points). Let $A = M \otimes N^*$, $B = N \otimes L^*$, and $C = L \otimes M^*$ with $\dim M = m$, $\dim N = n$, and $\dim L = \ell$, that is, $A$ can be viewed as the vector space of $m \times n$-matrices, etc. Let $(m, n, \ell) \in A \otimes B \otimes C$ be the tensor of the multiplication of $m \times n$-matrices with $n \times \ell$-matrices. Prove that $(m, n, \ell) \cong \text{Id}_M \otimes \text{Id}_N \otimes \text{Id}_L$.

Solution 1 (Typeset by all students in the lecture). Our first observation is the following fact

$$A \otimes B \otimes C \cong (M \otimes N^*) \otimes (N \otimes L^*) \otimes (L \otimes M^*) \cong (M \otimes M^*) \otimes (N \otimes N^*) \otimes (L \otimes L^*) \quad (0.1)$$

We see $X \otimes X^*$ as the space of $(x \times x)$ matrices where $x$ is the dimension of $X$. Furthermore we can see $(m, n, \ell)$ as a trilinear map that lives in $A^* \times B^* \times C^* \rightarrow \mathbb{F}$.

Now since $T := (m, n, \ell)$ denotes the matrix multiplication tensor we can conclude that for an element $(x_{i,i'}, y_{j,j'}, z_{h,h'}) \in A^* \times B^* \times C^*$ we have:

$$T(x_{i,i'}, y_{j,j'}, z_{h,h'}) = \begin{cases} 1 & \text{if } i' = j, j' = h \text{ and } i = h' \\ 0 & \text{else} \end{cases} \quad (0.2)$$

Using the canonical isomorphism that we get from our first observation and the conditions we get from (0.2) we can see $T$ as the trilinear map

$$T : (M^* \otimes M) \times (N^* \otimes N) \times (L^* \otimes L) \rightarrow \mathbb{F}, (x_{h',i}, y_{i',j}, z_{j',h}) \mapsto \begin{cases} 1 & \text{if } h' = i, i' = j \text{ and } j' = h \\ 0 & \text{else} \end{cases}$$

Thus only elements of the form $(x_{i,i}, y_{j,j}, z_{h,h})$ get mapped to 1 and everything else becomes 0.

Now that is exactly the map that corresponds to the tensor

$$\text{Id}_M \otimes \text{Id}_M \otimes \text{Id}_L$$
Exercise 2 \(15\) Points. Let \(V\) be an irreducible \(\text{GL}_n\)-representation. Prove that \(V\) is also irreducible as an \(\text{SL}_n\)-representation, where \(\text{SL}_n\) is the group of matrices with determinant equal to \(1\).

Solution 2 (Typeset by all students in the lecture). Let \(V\) be an irreducible \(\text{GL}_n\)-representation. Now assume that \(V\) is not irreducible as an \(\text{SL}_n\)-representation. That means that there exists a non-trivial linear subspace \(W \subseteq V\) such that \(W\) is closed under the action of \(\text{SL}_n\):

\[
\forall s \in \text{SL}_n : \forall w \in W : sw \in W \quad (0.3)
\]

Since \(V\) is irreducible as a \(\text{GL}_n\)-representation there is a \(g \in \text{GL}_n\) with \(\det(g) \neq 1\) (due to (3)) and \(w^* \in W\) such that:

\[
gw^* \notin W \quad (0.4)
\]

Now define:

\[
s := \frac{1}{\sqrt{|\det(g)|}} g
\]

Note that \(\det(s) = 1\) such that \(s \in \text{SL}_n\).

Using \((0.3)\) we further get:

\[
sw^* \in W
\]

But since \(W\) was a linear subgroup of \(V\) we also have that

\[
\sqrt{|\det(g)|} \cdot sw^* = \sqrt{|\det(g)|} \cdot \frac{1}{\sqrt{|\det(g)|}} gw^* = gw^* \in W
\]

which contradicts \((0.4)\).

Thus \(W\) is irreducible as an \(\text{SL}_n\)-representation.

Exercise 3 \((10\) Points\). Let \(W\) be a vector space of dimension \(2\) and let \(\ell, \ell_1, \ell_2, \ldots, \ell_{n-1} \in W\) be pairwise independent, i.e., \(\dim(\langle a, b \rangle) = 2\) for all \(a, b \in \{\ell, \ell_1, \ell_2, \ldots, \ell_{n-1}\}\) with \(a \neq b\). Prove that there is a \(g \in S^{n-1}W^*\) that vanishes on \(\ell_1, \ell_2, \ldots, \ell_{n-1}\), but not on \(\ell\).

Solution 3. Since \(\dim(\langle \ell, \ell_1 \rangle) = 2\), we can assume that \(\{\ell, \ell_1\}\) is a basis of \(W\). Thus there exist scalars \(\lambda_2, \mu_2, \lambda_3, \mu_3, \ldots, \lambda_{n-1}, \mu_{n-1}\) such that \(\ell_j = \lambda_j \ell + \mu_j \ell_1\) for \(j \in \{2, 3, \ldots, n-1\}\). Note that \(\lambda_2, \mu_2, \lambda_3, \mu_3, \ldots, \lambda_{n-1}, \mu_{n-1}\) are all non-zero because otherwise some \(\ell_j\) (for \(j > 2\)) would not be independent of \(\ell\) or \(\ell_1\). Now let \(x, y \in W^*\) be such that \(x(\ell) = 1, x(\ell_1) = 0, y(\ell) = 0, y(\ell_1) = 1\).

Thus \(\{x, y\}\) is a basis of \(W^*\). Now consider the following elements \(f_1, f_2, \ldots, f_{n-1}\) of \(W^*\), where \(f_1 = x\) and \(f_j = \mu_j x - \lambda_j y\) for \(j \in \{2, 3, \ldots, n-1\}\). It is easy to observe that \(f_j(\ell_j) = 0\) for \(j \in \{1, 2, \ldots, n-1\}\). Consider now \(g = f_1 f_2 \cdots f_{n-1}\). Note that \(f\) is a bi-variate homogeneous polynomial in \(x, y\) of degree \(n-1\). Thus \(g \in S^{n-1}W^*\), and by the choice of \(f_j\)’s we know that \(g\) vanishes on \(\ell_1, \ell_2, \ldots, \ell_{n-1}\). Also \(g(\ell) = 1 \cdot \mu_2 \cdot \mu_3 \cdots \mu_{n-1}\), since all \(\mu_j\)’s are non-zero we get that \(g(\ell) \neq 0\). Thus \(g\) vanishes on \(\ell_1, \ell_2, \ldots, \ell_{n-1}\), but not on \(\ell\).