In the following lectures we want to study the representation theoretic multiplicities in the coordinate rings of orbits. These give upper bounds for the multiplicities in the coordinate rings of orbit closures. The main tool is the algebraic Peter-Weyl theorem.

1 The algebraic Peter-Weyl theorem

Let \( \mathbb{A} \) be a finite dimensional complex vector space with a polynomial action of \( G = \text{GL}_k \). One main example is \( \mathbb{A} = \text{Sym}^n \mathbb{C}^k \). (For tensors \( G = \text{GL}_k^3 \), \( \mathbb{A} = \otimes^3 \mathbb{C}^k \)).

Let \( Z \subseteq \mathbb{A} \) be locally closed. One main example is a group orbit \( Z = Gv \).

Recall (last semester, Chapter 15) that for a locally closed set \( Z \subseteq \mathbb{A} \) we defined the coordinate ring \( \mathbb{C}[Z] \) as the ring of regular functions (i.e., locally defined by rational functions) on \( Z \). Therefore \( \mathbb{C}[Z] \subseteq \mathbb{C}[\mathbb{A}] \), because \( \mathbb{C}[\mathbb{Z}] \) is the ring of polynomial functions on \( Z \), and polynomial functions are regular.

For a point \( v \in \mathbb{A} \) let \( \text{stab}_G(v) \subseteq G \) denote its stabilizer (or its symmetry group):
\[
\text{stab}_G(v) := \{ g \in G \mid gv = v \}.
\]

The algebraic Peter-Weyl theorem implies that
\[
\text{mult}_\lambda(\mathbb{C}[Gv]) = \dim(\{\lambda\}^{\text{stab}_G(v)}).
\]

Thus
\[
\text{mult}_\lambda(\mathbb{C}[Gv]) \leq \dim(\{\lambda\}^{\text{stab}_G(v)}).
\]

In our situations more is known by now about the relationship between both rings: One is a so-called localization of the other\(^1\).

\(^1\)Bürgisser and Ikenmeyer, *Fundamental invariants of orbit closures*, Journal of Algebra Volume 477, 1 May 2017, Pages 390–434


2 Characterization by the stabilizer

2.1 Theorem. Every connected reductive algebraic subgroup $H \subseteq G$ is characterized (up to group isomorphism) by its dimension data, which is the map $\lambda \mapsto \dim(\lambda)^H$.

2.2 Definition. A point $v \in \mathbb{A}$ is characterized by its stabilizer (or alternatively characterized by its symmetries) if

$$\forall w \in \mathbb{A} : (\text{stab}_G(v) \subseteq \text{stab}_G(w) \Rightarrow w \in Cv).$$

We will see that many points $v$ of interest are characterized by their stabilizer. Since $\text{mult}_\lambda(\mathbb{C}[Gv])$ determines $\text{stab}_G(v)$ up to isomorphism, if in our specific situations a slightly stronger version of Theorem 2.1 holds, then this means that $\text{mult}_\lambda(\mathbb{C}[Gv])$ determines $\text{stab}_G(v)$ and thus $v$.

If something comparable holds for orbit closures (in specific situations), then this would mean that every lower bound can be proved using multiplicity obstructions.

CAVEAT: There are situations in which $\overline{Gu} \nsubseteq \overline{Gv}$ cannot be proved using multiplicity obstructions! But in all known cases $v$ and $w$ are not characterized by their stabilizer. For example, let $G$ be the trivial group and let $v \neq w$. Then both $\overline{Gu}$ and $\overline{Gv}$ are just a single point each. But $\mathbb{C}[\overline{Gu}] = \mathbb{C} = \mathbb{C}[\overline{Gv}]$.

3 Main Examples

We now determine several stabilizers (or black-box them), prove that the points are characterized by the stabilizer and determine the multiplicities in $\mathbb{C}[Gv]$. On the way we will discuss the basics of the character theory of the symmetric group.

3.1 Product of homogeneous linear forms

3.1 Proposition. Let $g \in \text{GL}_k$. If $g(x_1 \cdots x_k) = x_1 \cdots x_k$, then $g$ is the product of a permutation matrix and a diagonal matrix with determinant 1 (the so-called $\text{SL}_k$-torus $T_{\text{SL}_k}$). Notation: $T_{\text{SL}_k} \rtimes \mathfrak{S}_k$ (this is called a semidirect product).

Proof. Clearly every product of a permutation matrix and a diagonal matrix with determinant 1 fixes $x_1 \cdots x_k$.

Let $g(x_1 \cdots x_k) = \ell_1 \cdots \ell_k$. Let $\ell_i = \alpha_{i,1}x_1 + \cdots + \alpha_{i,k}x_k$.

For each $x_{k'}$: Set all other variables to 1. Then $\prod_i \ell_i(1, \ldots, 1, x_{k'}, 1, \ldots, 1) = x_{k'}$. Thus exactly one $\ell_i(1, \ldots, 1, x_{k'}, 1, \ldots, 1)$ is not just a constant, but an affine linear form. This affine linear form is homogeneous, because the polynomial $x_{k'}$ is homogeneous. We conclude that among

\cite{LarsenPink90}

the \( \ell_i \) at most one can have a nonzero coefficient for the variable \( x_{k'} \). On the other hand each \( \ell_i \) must have at least one nonzero coefficient, so each \( \ell_i = c_i x_{\pi(i)} \) for some permutation \( \pi \) and nonzero constants \( c_i \). Clearly the constants satisfy \( \prod c_i = 1 \).

3.2 Proposition. The polynomial \( x_1 \cdots x_k \) is characterized by its stabilizer.

Proof. The action of \( T_{\mathfrak{sl}_k} \) preserves the monomial structure (in other words, the support of the coefficient vector of a polynomial is invariant under the action of \( T_{\mathfrak{sl}_k} \)). Thus if a polynomial \( w \) is stabilized by \( T_{\mathfrak{sl}_k} \), each monomial of \( w \) is stabilized independently. A monomial is stabilized by \( T_{\mathfrak{sl}_k} \) iff each variable appears the same number of times. In degree \( k \) there is only one monomial that has this property: \( x_1 \cdots x_k \).

3.3 Proposition. For \( \lambda \vdash kd \), \( \text{mult}_{\lambda}(\mathbb{C}[\text{GL}_k x_1 \cdots x_k])_d = a_\lambda(k, d) \).

Proof. We use the algebraic Peter-Weyl theorem. \( \text{mult}_{\lambda}(\mathbb{C}[\text{GL}_k x_1 \cdots x_k])_d = \dim \{ \lambda \} T_{\mathfrak{sl}_k} \rtimes \mathfrak{S}_k = \dim(\{ \lambda \} T_{\mathfrak{sl}_k})^{\mathfrak{S}_k} \). Recall the vector space of tableaux with the Grassmann-Plücker relations. A basis of \( \{ \lambda \} \) is given by semistandard tableaux with entries \( 1, \ldots, k \). Each basis vector gets rescaled by the action of \( T_{\mathfrak{sl}_k} \). The \( T_{\mathfrak{sl}_k} \)-invariants are the tableaux for which each number appears equally often. Since \( \lambda \vdash kd \), each number appears exactly \( d \) times. Taking the \( \mathfrak{S}_k \)-invariants of this space of tableaux, its dimension is precisely the plethysm coefficient \( a_\lambda(k, d) \).

CAVEAT: Let \( b_\lambda(d, k) := \text{mult}_{\lambda}(\mathbb{C}[\text{GL}_k(x_1 \cdots x_k)])_d \). We just saw \( b_\lambda(d, k) \leq a_\lambda(k, d) \). It might be confusing that we also know \( b_\lambda(d, k) \leq a_\lambda(d, k) \), because \( \text{GL}_k(x_1 \cdots x_k) \) is a subvariety of \( \text{Sym}^k \mathbb{C}^k \).

3.2 Power sum

3.4 Proposition. Let \( m \geq 3 \). Let \( G = \text{GL}_k \) and let \( v = x_1^m + \cdots + x_k^m \). Then \( \text{stab}_G(v) \) is generated by the permutation matrices and the diagonal matrices with \( m \)th roots of unity on the main diagonal. Notation: \( \mathbb{Z}_m^k \rtimes \mathfrak{S}_k \)

Proof. Clearly the listed matrices stabilize \( v \). The rest of the proof uses partial derivatives. We postpone it for a few minutes.

CAVEAT:

\[
\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (x^2 + y^2) = \frac{1}{2} ((x + y)^2 + (x - y)^2) = x^2 + y^2.
\]

Let us therefore assume \( m \geq 3 \).

3.5 Proposition. The power sum \( x_1^m + \cdots + x_k^m \) is characterized by its stabilizer.
Proof. The action of \( \mathbb{Z}_m^k \) preserves the monomial structure. Thus if a polynomial \( w \) is stabilized by \( \mathbb{Z}_m^k \), each monomial is stabilized. A monomial is stabilized by \( \mathbb{Z}_m^k \) iff each variable appears a multiple of \( m \) times. In \( \text{Sym}^m \mathbb{C}^k \) there are only \( k \) such monomials: \( x_i^m, 1 \leq i \leq k \). Invariance under \( S_k \) ensures that they all have the same coefficient. Thus \( w \) is a multiple of \( x_1^m + \cdots + x_k^m \). □

The multiplicities \( \mathbb{C}[\text{GL}_k(x_1^m + \cdots + x_k^m)] \) can be determined, but it is a bit tricky and we postpone it for a few lectures to Section 3.6. But the simplest case goes as follows.

3.6 Proposition. For \( \lambda \vdash km \), mult\( \lambda \mathbb{C}[\text{GL}_k(x_1^m + \cdots + x_k^m)] \geq a_{\lambda}(k,m) \).

Proof. We use the algebraic Peter-Weyl theorem. mult\( \lambda \mathbb{C}[\text{GL}_k(x_1^m + \cdots + x_k^m)]_d = \dim \{ \lambda \}^{\mathbb{Z}_m^k \times S_k} = \dim \{ \lambda \}^{\mathbb{Z}_m^k} S_k \). A basis of \( \{ \lambda \} \) is given by semistandard tableaux. Each basis vector gets rescaled by the action of \( \mathbb{Z}_m^k \). The \( \mathbb{Z}_m^k \)-invariants are the tableaux for which each number appears a multiple of \( m \) times often. In particular we obtain the tableaux in which each number appears exactly \( m \) times. Taking the \( S_k \)-invariants of this space of tableaux, its dimension is precisely the plethysm coefficient \( a_{\lambda}(k,m) \). □

Remark: One can show that mult\( \lambda \mathbb{C}[\text{GL}_k(x_1^m + \cdots + x_k^m)] = a_{\lambda}(k,m) \) for \( \lambda \vdash km \).

Rest of the proof of Proposition 3.4. This is taken from 3.

Let \( \bar{x} := (x_1, \ldots, x_k) \). Let \( v := x_1^m + \cdots + x_k^m \). Define the Hessian \( H_v(\bar{x}) \) as the \( k \times k \) matrix whose \((i,j)\)-entry is

\[
\frac{\partial^2}{\partial x_i \partial x_j} v(\bar{x}).
\]

The matrix \( H_v(\bar{x}) \) is diagonal with entry \((i,i)\) being \( m(m-1)x_i^{m-2} \). Thus

\[
\det H_v(\bar{x}) = \prod_{i=1}^{k} m(m-1)x_i^{m-2}.
\]

3.7 Claim. \( H_{g^{-1}v}(\bar{x}) = g^T \cdot H_v(g\bar{x}) \cdot g \). In particular \( \det(H_{g^{-1}v}(\bar{x})) = \det(g)^2 \det(H_v(g\bar{x})) \).

Proof of claim. Let \( F := g^{-1}v \), i.e., \( F(\bar{x}) = v(g\bar{x}) \). We use the chain rule, \( 1 \leq i \leq k \):

\[
\frac{\partial F}{\partial x_i}(\bar{x}) = \sum_{q=1}^{k} g_{q,i} \cdot \frac{\partial v}{\partial x_q}(g\bar{x}).
\]

---

We use the chain rule again:
\[
\frac{\partial^2 F}{\partial x_i \cdot \partial x_j}(\vec{x}) = \sum_{q=1}^{k} g_{q,i} \left( \sum_{p=1}^{k} g_{p,j} \cdot \frac{\partial^2 v}{\partial x_q \partial x_p}(g \vec{x}) \right)
\]
\[
= \sum_{1 \leq p, q \leq k} g_{q,i} \cdot \frac{\partial^2 v}{\partial x_q \partial x_p}(g \vec{x}) \cdot g_{p,j}.
\]

In matrix form: \( H_F(\vec{x}) = g^T \cdot H_v(g \vec{x}) \cdot g \).

Now let \( v = g^{-1} \cdot v \). Then also the Hessians coincide:

\[
H_v(\vec{x}) = H_{g^{-1} \cdot v}(\vec{x}) = g^T \cdot H_v(g \vec{x}) \cdot g.
\]

In particular their determinants coincide:

\[
\det(H_v(\vec{x})) = \det(g)^2 \det(H_v(g \vec{x})).
\]

\[
\prod_{i=1}^{k} m(m-1)x_i^{m-2} = \underbrace{\det(g)^2 \prod_{i=1}^{k} \left( \sum_{j=1}^{k} g_{i,j}x_j \right)^{m-2}}_{\text{constant}}.
\]

Now we use the uniqueness of factorization: Each \( \sum_{i=1}^{k} g_{i,j}x_j \) is a scalar multiple of some \( x_i \).

Thus \( g \) has at most 1 nonzero entry in each row and column, since \( g \in \text{GL}_k \), \( g \) has at exactly 1 nonzero entry in each row and column. Clearly any permutation fixes \( v \), so we can assume that \( g \) is diagonal. The diagonal matrices that fix \( v \) are precisely those whose diagonal entries are \( m \)th roots of unity.

3.3 Determinant

We now discuss the determinant and the permanent. We prove that both are characterized by their stabilizer. One part of the proof requires some basic character theory, which we discuss in Section 3.5.

The stabilizer

Let \( X = (x_{i,j}) \) be an \( n \times n \) variable matrix.

\[
\det(gXh) = \det(g) \det(h) \det(X) \quad \text{Thus } \det(X) = \det(gXh) \text{ with } \det(g) \cdot \det(h) = 1.
\]

Moreover, \( \det(X) = \det(gX^t h) \) with \( \det(g) \cdot \det(h) = 1. \)

These are the only symmetries, as was first shown by Frobenius in 1897\(^4\). Hence \( \text{stab}_{\text{GL}_n}(\text{det}_n) = (\text{GL}_n \times \text{GL}_n)/(\mathbb{C}^* \times \mathbb{Z}_2). \)

\(^4\)Frobenius, Über die Darstellung der endlichen Gruppen durch lineare Substitutionen. Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin, pages 994–1015, 1897., §7, Satz I
Characterization by the stabilizer

We want to see that $\det_n$ is characterized by its stabilizer. We need the following preliminary lemma.

3.8 Lemma. From the exercises we know that every irreducible $\text{GL}_n$-representation $\{\lambda\}$ is irreducible as an $\text{SL}_n$-representation. We have that $\{\lambda\}$ is the trivial $\text{SL}_n$-representation iff $\lambda = n \times d$ for some $d$.

Proof. In order for $\{\lambda\}$ to be the trivial $\text{SL}_n$-representation, we need $\{\lambda\}$ to be 1-dimensional. Recall that $\dim\{\lambda\}$ equals the number of semistandard tableaux of shape $\lambda$ with entries $1, \ldots, n$. Thus the only 1-dimensional $\text{GL}_n$-representations $\{\lambda\}$ satisfy $\lambda = n \times d$. Indeed, these correspond to the representation $gv = \det(g)^d.v$. In particular these are trivial with respect to the $\text{SL}_n$-action.

3.9 Theorem. $\det_n$ is characterized by its stabilizer.

Proof. $\otimes^n(\mathbb{C}^n \otimes \mathbb{C}^n)$ decomposes w.r.t. the $\text{GL}_n \times \text{GL}_n$-action (Schur-Weyl duality) as

$$\otimes^n(\mathbb{C}^n \otimes \mathbb{C}^n) = \bigoplus_{\lambda, \mu \vdash n} \{\lambda\} \otimes \{\mu\} \otimes [\lambda] \otimes [\mu].$$

Thus $\text{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n)$ decomposes as

$$\text{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n) = \bigoplus_{\lambda, \mu \vdash n} \{\lambda\} \otimes \{\mu\} \otimes ([\lambda] \otimes [\mu])^{S_n}.$$

Now we use $([\lambda] \otimes [\mu])^{S_n} = \begin{cases} \mathbb{C} & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$ (which we prove below using character theory):

$$\text{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n) = \bigoplus_{\lambda \vdash n} \{\lambda\} \otimes \{\lambda\}.$$

Taking $\text{SL}_n \times \text{SL}_n$-invariants (Lemma 3.8):

$$(\text{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n))^{\text{SL}_n \times \text{SL}_n} = \{1^n\} \otimes \{1^n\},$$

which is 1-dimensional.

3.4 Permanent

The stabilizer

Let $X = (x_{i,j})$ be an $n \times n$ variable matrix.

$\text{per}(gXh) = \text{per}(X)$ if $g$ and $h$ are permutation matrices.
Moreover, \( \per(gXh) = \per(X) \) if \( g \) and \( h \) are diagonal matrices with \( \det(gh) = 1 \).
Moreover, \( \per(X) = \per(X^t) \).
These are the only symmetries\(^5\). Hence \( \text{stab}_{\text{GL}_n^2}(\per_n) = (Q_n \times Q_n)/\mathbb{C}^\times \times \mathbb{Z}_2 \), where \( Q_n = T_n \times \mathfrak{S}_n \).

**Characterization by the stabilizer**

Analogously to \( \det_n \) we see that \( \per_n \) is characterized by its stabilizer.

**3.10 Theorem.** \( \per_n \) is characterized by its stabilizer.

**Proof.** \( \otimes^n(\mathbb{C}^n \otimes \mathbb{C}^n) \) decomposes w.r.t. the \( \text{GL}_n \times \text{GL}_n \)-action (Schur-Weyl duality) as

\[
\otimes^n(\mathbb{C}^n \otimes \mathbb{C}^n) = \bigoplus_{\lambda, \mu \vdash n} \{\lambda\} \otimes \{\mu\} \otimes [\lambda] \otimes [\mu].
\]

Thus \( \text{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n) \) decomposes as

\[
\text{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n) = \bigoplus_{\lambda, \mu \vdash n} \{\lambda\} \otimes \{\mu\} \otimes ([\lambda] \otimes [\mu])^\mathfrak{S}_n.
\]

Now we use \( ([\lambda] \otimes [\mu])^\mathfrak{S}_n = \begin{cases} \mathbb{C} & \text{iff } \lambda = \mu \\ 0 & \text{otherwise} \end{cases} \) (which we prove below using character theory):

\[
\text{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n) = \bigoplus_{\lambda \vdash n} \{\lambda\} \otimes \{\lambda\}.
\]

For \( \lambda \vdash n \) Gay’s theorem states that \( \{\lambda\}^{T_n} = [\lambda] \), but this can easily be generalized: Let \( T_{\text{SL}_n} := T_n \cap \text{SL}_n \). Then for \( \lambda \vdash n \) we have \( \{\lambda\}^{T_{\text{SL}_n}} = [\lambda] \).

Thus taking \( T_{\text{SL}_n} \times T_{\text{SL}_n} \)-invariants we obtain:

\[
(\text{Sym}^n(\mathbb{C}^n \otimes \mathbb{C}^n))^{T_{\text{SL}_n} \times T_{\text{SL}_n}} = \bigoplus_{\lambda \vdash n} [\lambda] \otimes [\lambda].
\]

Taking \( \mathfrak{S}_n \times \mathfrak{S}_n \)-invariants yields

\[
\bigoplus_{\lambda \vdash n} [\lambda]^{\mathfrak{S}_n} \otimes [\lambda]^{\mathfrak{S}_n} = [n] \otimes [n],
\]

which is 1-dimensional. \( \Box \)

3.5 Dual representations and character theory

In this section we discuss some basic character theory to prove the following statement, which is a missing part in our arguments in Sections 3.3 and 3.4.

3.11 Proposition. \( \dim([\lambda] \otimes [\mu])^S_n = 1 \) iff \( \lambda = \mu \) (0 otherwise).

Remark: Using what we have learned last semester, with obstruction designs and explicit highest weight vectors it is easy to see that \( \dim([\lambda] \otimes [\lambda])^S_n \geq 1 \). For the corresponding hypergraph let the layer 1 hyperedges agree with the layer 2 hyperedges. This hypergraph decomposes into disjoint hypergraphs. The semigroup property implies that we only need to verify \( ([1^n] \otimes [1^n])^S_n > 0 \). This can be verified directly by studying the \( S_n \)-action on the column tableau pair.

The complete proof of Prop. 3.11 can be easily seen using classical ideas from representation theory that we want to introduce now.

3.12 Definition. Let \( G \) be a group, \( V \) be a finite dimensional vector space, and let \( \varrho : G \to \text{GL}(V) \) be a representation. Let \( V^* \) be the dual space to \( V \), i.e., the space of homogeneous linear forms on \( V \). Then \( V^* \) is a representation via \((gf)(x) := f(g^{-1}x)\), which is called the dual representation or the contragredient representation.

3.13 Lemma. Let \( V \) be a \( G \)-representation. \( V \) is irreducible iff \( V^* \) is irreducible.

Proof. Let \( V^* \) be irreducible. Let \( W \subseteq V \) by a \( G \)-subrepresentation, in particular a linear subspace. Then the vanishing ideal \( I(W)_1 \) in degree 1 is called the annihilator \( W^\perp \), which is a \( G \)-subrepresentation of \( V^* \):

\[
W^\perp := \{ f \in V^* : f(W) = \{0\} \},
\]

Since \( V^* \) is irreducible, either \( W^\perp = 0 \) or \( W^\perp = V^* \). If \( W^\perp = V^* \), then all linear polynomials vanish on \( W \). Since \( W \) is a linear subspace, \( W = 0 \). If \( W^\perp = 0 \), then no linear polynomial vanishes on \( W \). Since \( W \) is a linear subspace, \( V = W \). In both cases \( W \) is a trivial \( G \)-subrepresentation of \( V \). Thus \( V \) is irreducible.

We finish the argument by showing that \( V^{**} = V \) are isomorphic \( G \)-representations.

\[
V^{**} = \{ \Phi : V^* \to \mathbb{C} \mid \Phi \text{ linear} \}
\]

The canonical isomorphism \( \Xi : V \to V^{**} \) is known from linear algebra as follows:

\[
(\Xi(v))(\varphi) := \varphi(v), \quad v \in V, \varphi \in V^*.
\]

But \( \Xi \) is \( G \)-equivariant:

\[
(\Xi(gv))(\varphi) \overset{(\dagger)}{=} \varphi(gv) = (g^{-1}\varphi)(g) \overset{(\dagger)}{=} (\Xi(v))(g^{-1}\varphi) = (\varphi)(g(\Xi(v))),
\]

i.e., \( \Xi(gv) = g(\Xi(v)) \).
If $V$ is irreducible of type $\lambda$, then we denote by $\lambda^*$ the type of $V^*$.

The following two propositions prove Proposition 3.11.

**3.14 Proposition (A).** Let $G$ be a linearly reductive group and $W$ be a $G$-representation and let $\{\lambda\}$ denote the irreducible $G$-representation of type $\lambda$. Then

$$\operatorname{mult}_\lambda(W) = \dim((\{\lambda^*\} \otimes W)^G).$$

In particular $\dim((\{\lambda^*\} \otimes \{\lambda\})^G) = 1$ and for $\lambda \neq \mu$ we have $\dim((\{\mu^*\} \otimes \{\lambda\})^G) = 0$.

We prove this in Section (A).

**3.15 Proposition (B).** $[\lambda]$ and $[\lambda]^*$ are isomorphic Specht modules.

We prove this in Section (B).

CAVEAT: Proposition 3.15 holds for the symmetric group, but it is false for example for $\GL_n$, $T_n$, or the cyclic group of order $> 2$.

**Proof of Proposition 3.11.**

$$\dim([\lambda] \otimes [\mu])^S_n \overset{(B)}{=} \dim([\lambda^*] \otimes [\mu])^S_n \overset{(A)}{=} \operatorname{mult}_{[\lambda]}([\mu]) = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$

(A) “Contraction” of representations: Proof of Proposition 3.14

**3.16 Lemma.** Given two $G$-representations $V$ and $W$, $\operatorname{Hom}(V, W) = V^* \otimes W$ is a $G \times G$-representation via

$$(g', g) \varphi(v) := g'(\varphi(g^{-1}v)).$$

**Proof.** We prove $((g'h', gh))\varphi = (g', g)((h', h)\varphi)$.

$$(g'h', gh)\varphi(v) = (g'h')\varphi((gh)^{-1}v) = g'(h'h^{-1}g^{-1}v)) = g((h', h)\varphi)(g^{-1}v) = ((g', g)\Psi)(v)$$

3.17 Lemma. Embed $G \hookrightarrow G \times G$, $g \mapsto (g, g)$. In this way $\operatorname{Hom}(V, W)$ is a $G$-representation. Its invariant space is the space of equivariant linear maps: $\operatorname{Hom}(V, W)^G = \operatorname{Hom}_G(V, W)$.

**Proof.** Recall that $\varphi \in \operatorname{Hom}_G(V, W)$ iff $g(\varphi(v)) = \varphi(gv)$ for all $g \in G$.

If $\varphi \in \operatorname{Hom}(V, W)$ is $G$-invariant, then $\varphi(v) = g(\varphi(g^{-1}v))$ and thus

$$g^{-1}(\varphi(v)) = (g^{-1}g)(\varphi(g^{-1}v)) = \varphi(g^{-1}v),$$

hence $\varphi$ is $G$-equivariant. The proof works analogously in the other direction.
Proposition 3.14 is now readily proved:

Schur’s lemma says that
\[ \dim(\text{Hom}_{G}(\{\lambda\},W)) = \operatorname{mult}_{\lambda}(W). \]
We just saw that
\[ \text{Hom}_{G}(\{\lambda\},W) = (\text{Hom}(\{\lambda\},W))^{G} = (\{\lambda\}^{\ast} \otimes W)^{G}. \]
Therefore
\[ \operatorname{mult}_{\lambda}(W) = \dim(\{\lambda\}^{\ast} \otimes W)^{G}. \]

(B) Some character theory of finite groups: The Specht modules are self-dual (Prop. 3.15)

We will use character theory to prove that the Specht modules \([\lambda]\) and \([\lambda]^{\ast}\) are isomorphic. Character theory is also currently the most efficient tool to compute Kronecker coefficients.

3.18 Definition. Let \(\varrho : G \to \text{GL}(V)\) be a representation. Then the map
\[ \chi_{\varrho} : G \to \mathbb{C}, \quad \chi_{\varrho}(g) = \text{tr}(\varrho(g)) \]
is called the character of \(\varrho\).

3.19 Observation. The character is a function that is constant on conjugacy classes. We call these functions class functions.

Proof. \(\chi_{\varrho}(h^{-1}gh) = \text{tr}(\varrho(h^{-1}gh)) = \text{tr}(\varrho(h^{-1})\varrho(g)\varrho(h)) = \text{tr}(\varrho(h)) = \chi_{\varrho}(g)\).

3.20 Proposition. Isomorphic representations have coinciding characters. (We will prove the other direction later)

Proof. Let \((V,\varrho_{V})\) and \((W,\varrho_{W})\) be isomorphic representations with isomorphism \(\gamma : V \to W\). As a product of matrices this means \(\gamma\varrho_{V}(g)v = \varrho_{W}(g)\gamma v\) for all \(v \in V\), thus \(\gamma\varrho_{V}(g) = \varrho_{W}(g)\gamma\).
In other words
\[ \gamma\varrho_{V}(g)\gamma^{-1} = \varrho_{W}(g). \]
Thus the traces of \(\varrho_{V}(g)\) and \(\varrho_{W}(g)\) coincide.

3.21 Example. We calculate the characters of \(S_{3}\).

\[
\begin{array}{cccc}
\text{id} & (12) & (13) \\
(3) & 1 & 1 & 1 \\
(13) & 1 & -1 & 1 \\
(2,1) & ? & ? & ? \\
\end{array}
\]
Let \( a := \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \), \( b := \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} \),

\[
\text{id} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix},
\]

thus

\[
\varrho_{(2,1)}(\text{id}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Therefore \( \chi_{(2,1)}(\text{id}) = \text{tr}(\varrho_{(2,1)}(\text{id})) = 2 \).

\[
(12) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a - b \\ -b \end{pmatrix},
\]

thus

\[
\varrho_{(2,1)}((12)) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.
\]

Therefore \( \chi_{(2,1)}((12)) = 0 \).

\[
(123) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ a - b \end{pmatrix},
\]

thus

\[
\varrho_{(2,1)}((123)) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.
\]

Therefore \( \chi_{(2,1)}((123)) = -1 \).

This is called a character table.

From this example we see that in the case of the symmetric group the characters are integers.

3.22 Lemma. \( \varrho^* (g) = \varrho(g)^{−t} \) (i.e., the representation matrix of the dual is the transpose inverse matrix).

Proof. Instead of \( \varrho(g) : V \to V \) we start out more generally and let \( D : V \to W \) be a linear map, not necessarily an isomorphism. The transpose \( D^t : W^* \to V^* \) is defined via

\[
f \mapsto f \circ D
\]

or, in other words

\[
(D^t(f))(x) = f(D(x)) \text{ for all } x \in V.
\]

If \( D \) is invertible, we can analyze \( D^{-1} \) instead of \( D \):

\[
(D^{-t}(f))(x) = f(D^{-1}(x)).
\]

If \( D = \varrho(g) \), then the right-hand side becomes \( f(g^{-1}x) \), which means that the left-hand side is equal to \( (gf)(x) \). \( \square \)
3.23 Theorem. Let \( g \in G \) have finite order (e.g., if \( G \) is finite). The character of the dual representation is the complex conjugate: \( \chi^*(g) = \overline{\chi(g)} \).

Proof. Since \( \varrho(g) \) is of finite order, \( \varrho(g) \) is diagonalizable. Let \( d = A^{-1} \varrho(g) A \) be diagonal. Then \( d^{-1} = A^{-1} \varrho(g)^{-1} A \) (invert all three and switch the order). Clearly \( \chi(g) := \text{tr}(\varrho(g)) = \text{tr}(d) \) and

\[
\chi^*(g) = \text{tr}(\varrho(g)^{-1}) = \text{tr}((\varrho(g)^{-1})^t) = \text{tr}(d^{-1}) = \text{tr}(d^-1).
\]

Since \( d \) has finite order, the entries on the diagonal of \( d \) are roots of unity. Thus \( d^-1 = d \). Summing up the trace gives: \( \chi^*(g) = \overline{\chi(g)} \). ☐

Remark: We know from our explicit description of the Specht modules (straightening algorithm) that the entries of the representation matrices \( \varrho(g) \), \( g \in S_n \), are all real-valued. Thus the characters of the symmetric group are real-valued. Thus \( [\lambda]^* = [\lambda] \).

3.24 Corollary. If \( g \) has finite order, then \( \chi(g^{-1}) = \overline{\chi(g)} \).

Proof.

\[
\chi(g^{-1}) = \text{tr}(\varrho(g^{-1})) = \text{tr}(\varrho(g)^{-1}) \overset{\text{Lem. 3.22}}{=} \text{tr}(\varrho(g)^{-t}) = \text{tr}(\varrho(g)^{-1}) \overset{\text{Thm. 3.23}}{=} \chi^*(g) = \overline{\chi(g)}.
\]

Characters characterize the representation

Let \( G \) be finite.

For a linear map \( \varphi : U \to V \) define a \( G \)-morphism \( \tilde{\varphi} : U \to V \) via

\[
\tilde{\varphi}(u) = \frac{1}{|G|} \sum_{g \in G} \varrho_V(g) \varphi(\varrho_U)^{-1} u.
\]

In matrix presentation:

\[
\tilde{\varphi} = \frac{1}{|G|} \sum_{g \in G} \varrho_V(g) \cdot \varphi \cdot \varrho_U(g^{-1}).
\]

3.25 Corollary. Let \( U, V \) be irreducible \( G \)-representations and \( \varphi : U \to V \) a linear map.

1. If \( U \not\simeq V \), then \( \tilde{\varphi} = 0 \).
2. If \( U = V \), then \( \tilde{\varphi} = \frac{\text{tr} \varphi}{n} \text{id}_V \).

Proof. The first claim is clear by Schur’s lemma, because \( \tilde{\varphi} \) is a \( G \)-morphism. Furthermore, also by Schur’s lemma, if \( U = V \), then \( \tilde{\varphi} = \alpha \text{id}_V \).

\[
na = \text{tr}(\alpha \text{id}_V) = \text{tr}(\tilde{\varphi}) = \frac{1}{|G|} \sum_{g \in G} \text{tr}(\varrho_V(g) \cdot \varphi \cdot \varrho_U(g^{-1})) = \text{tr}(\varphi).
\]

Thus \( \alpha = \frac{\text{tr}(\varphi)}{n} \). ☐
3.26 Corollary. Let \( U, V \) be irreducible \( G \)-representations and \( R(g) := \varrho_U(g) \), \( S(g) := \varrho_V(g) \) are the representation matrices.

1. If \( U \ncong V \), then \( \forall i, j, k, l : \frac{1}{|G|} \sum_{g \in G} S(g)_{ij} R(g^{-1})_{kl} = 0 \)

2. If \( U = V \), \( \text{dim} U = n \), then \( \forall i, j, k, l : \frac{1}{|G|} \sum_{g \in G} R(g)_{ij} R(g^{-1})_{kl} = \frac{1}{n} \delta_{il} \delta_{jk} \).

Proof. Let \( E_{jk} \) be the zero matrix with a single 1 in row \( j \), column \( k \). For matrices \( S, R \) we have

\[
(S \cdot E_{jk} \cdot R)_{il} = \sum_{a,b} S_{ia}(E_{jk})_{ab} R_{bl} = S_{ij} R_{kl}. 
\]

(†)

We want to apply Cor. 3.25 to \( \varphi = E_{jk} \):

\[
\tilde{\varphi}_{il} = \left( \frac{1}{|G|} \sum_{g \in G} S(g) E_{jk} R(g^{-1}) \right)_{il} = \frac{1}{|G|} \sum_{g \in G} (S(g) E_{jk} R(g^{-1}))_{il} = \frac{1}{|G|} \sum_{g \in G} S(g)_{ij} R(g^{-1})_{kl}.
\]

If \( U \ncong V \), then Cor. 3.25 implies \( \tilde{\varphi}_{il} = 0 \), which proves the first claim. If \( U = V \), then Cor. 3.25 implies \( \tilde{\varphi}_{il} = \frac{n}{n} \delta_{il} \delta_{il} = \frac{1}{n} \delta_{jk} \delta_{il} \), which proves the second claim. \( \Box \)

3.27 Definition. Let \( \varphi, \psi \) be functions \( G \to \mathbb{C} \). We define

\[
\langle \varphi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}.
\]

3.28 Remark. \( \langle ., . \rangle \) is an inner product on the complex vector space of functions \( G \to \mathbb{C} \). An orthonormal system is a set \( \{ \chi_1, \ldots, \chi_k : G \to \mathbb{C} \} \) such that \( \langle \chi_i, \chi_j \rangle = \delta_{ij} \). Every orthonormal system is linearly independent.

3.29 Theorem (Fundamental theorem (orthogonality relations)). Let \( U \) and \( V \) be irreducible \( G \)-representations. Then

\[
\langle \chi_U, \chi_V \rangle = \begin{cases} 1 & \text{if } U \cong V \\ 0 & \text{otherwise} \end{cases}
\]

Proof.

\[
\langle \chi_U, \chi_V \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_U(g) \overline{\chi_V(g)} \overset{\text{Cor. 3.24}}{=} \frac{1}{|G|} \sum_{g \in G} \chi_U(g) \chi_V(g^{-1})
\]

\[
= \frac{1}{|G|} \sum_{g \in G} \left( \sum_i S_{ii}(g) \right) \left( \sum_j R_{jj}(g^{-1}) \right) = \sum_{i,j} \frac{1}{|G|} \sum_{g \in G} S_{ii}(g) R_{jj}(g^{-1}).
\]

If \( U \ncong V \), this is 0 by Corollary 3.26.

If \( U = V \), then Corollary 3.26 gives

\[
\langle \chi_U, \chi_V \rangle = \sum_{i,j} \frac{1}{n} \delta_{ij} \delta_{ij} = \sum_{i=1}^{n} \frac{1}{n} = 1.
\]

\( \Box \)
3.30 Theorem. $G$-representations are isomorphic iff their characters coincide.

Proof. One direction is known from Prop. 3.20. For the other direction, let $U = \bigoplus \lambda c_\lambda \{\lambda\}$ be a decomposition into irreducible representations, and analogously for $V = \bigoplus \lambda d_\lambda \{\lambda\}$. Then $\chi_U = \sum \lambda c_\lambda \chi_\lambda$ and $\chi_V = \sum \lambda d_\lambda \chi_\lambda$ for some natural numbers $c_\lambda, d_\lambda$. Since $\{\chi_\lambda\}$ is an orthonormal system, it is linearly independent. Therefore $c_\lambda = d_\lambda$ for all $\lambda$. We conclude $U \cong V$. \qed

Proof of Prop. 3.15. Let $V = [\lambda]$ with character $\chi_V$. Since $V$ is a Specht module, $\chi_V(g) \in \mathbb{R}$. Thus $\chi_V^\ast(g)^{\text{Thm. 3.23}} = \overline{\chi_V(g)} = \chi_V(g)$. Hence $\chi_V = \chi_V^\ast$. Using Theorem 3.30 we conclude that $V \cong V^\ast$. \qed

3.6 Power sum revisited

To be continued...