Brief Announcement: Local Approximability of Minimum Dominating Set on Planar Graphs

Miikka Hilke
Helsinki Institute for Information Technology HIIT, Department of Computer Science, University of Helsinki, Finland
miikka.hilke@helsinki.fi

Christoph Lenzen
Computer Science and Artificial Intelligence Laboratory, MIT
clenzen@csail.mit.edu

Jukka Suomela
Helsinki Institute for Information Technology HIIT, Department of Information and Computer Science, Aalto University, Finland
jukka.suomela@aalto.fi

ABSTRACT

We show that there is no deterministic local algorithm (constant-time distributed graph algorithm) that finds a $(7-\epsilon)$-approximation of a minimum dominating set on planar graphs, for any positive constant ϵ. In prior work, the best lower bound on the approximation ratio has been $5-\epsilon$; there is also an upper bound of 52.

Categories and Subject Descriptors

C.2.4 [**Computer-Communication Networks**]: Distributed Systems; F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems

Keywords

Approximation algorithm; dominating set problem; local distributed algorithm; lower bound; planar graph

1. INTRODUCTION

This work studies one of the last uncharted corners in the area of deterministic local algorithms: planar graphs.

A *local algorithm* is a distributed graph algorithm that runs in $O(1)$ communication rounds, independently of the size of the network. While the theory of *randomised* local algorithms is still in its infancy, we have nowadays a good understanding of the capabilities of *deterministic* local algorithms.

For many classical graph problems, there are exactly matching upper and lower bounds on the best possible approximation ratio that can be achieved by a deterministic local algorithm [6]. In many cases, we can apply a straightforward two-step procedure to derive tight lower bounds:

1. Prove tight bounds for anonymous networks (without unique identifiers).
2. Apply a simulation argument [2] to show that unique identifiers do not help.

However, there are some isolated examples of natural questions in which the above two-step procedure fails badly. Perhaps the most intriguing example is *dominating sets on planar graphs*:

1. We do not have tight bounds for this problem in anonymous networks.
2. Planar graphs are not closed under lifts, and therefore the simulation argument [2] cannot be applied.

In this work we are interested in the smallest α such that there is a deterministic local algorithm that finds an α-approximation of a minimum dominating set in any planar graph. The current bounds are very far from being tight:

- $5-\epsilon < \alpha \leq 636$ for anonymous networks [1, 7],
- $5-\epsilon < \alpha \leq 52$ in the LOCAL model [1, 3, 4, 8].

In this work we give the first improvement on the lower bounds in six years: we prove a lower bound $\alpha > 7-\epsilon$ for both models, for any positive constant ϵ.

2. PROOF OVERVIEW

Let $\mathcal{A}$ be a deterministic distributed algorithm with running time $T = O(1)$ in the LOCAL model. Assume that $\mathcal{A}$ finds a dominating set $D = \mathcal{A}(G)$ in any planar graph G (that is, each node that is not in D is adjacent to at least one node of D).

Pick sufficiently large $m \gg T$ and r. Let $m' = m - 2T$. We will construct a planar graph G with $n = m^2 r$ nodes as shown in Figure 1a. There are r *blocks* with $m \times m$ nodes in each block. The nodes of each block are partitioned to *internal nodes* and *boundary nodes*: there are $m' \times m'$ internal nodes, and they are surrounded by boundary areas of width T. Let B_i be the set of nodes in block i, and let $I_i \subseteq B_i$ be the set of internal nodes in block B_i. We will prove the following lemma.

LEMMA 1. *For any m and any sufficiently large r, we can assign unique identifiers in G so that $I_i \subseteq \mathcal{A}(G)$ for all $1, 2, \ldots, r-\ell$, for some $\ell = o(r)$.*

In other words, all internal nodes of blocks $1, 2, \ldots, r-\ell$ are in the dominating set $D = \mathcal{A}(G)$ produced by algorithm $\mathcal{A}$. Now if we choose large enough m and r, we can make the contributions of the boundary nodes and the contributions of the remaining $o(r)$ blocks arbitrarily small. In particular, for any positive constant ϵ', we can pick m and r such that $|D| \geq (1-\epsilon')n$.

PODC'14, July 15–18, 2014, Paris, France.
ACM 978-1-4503-2944-6/14/07.
http://dx.doi.org/10.1145/2611462.2611504.

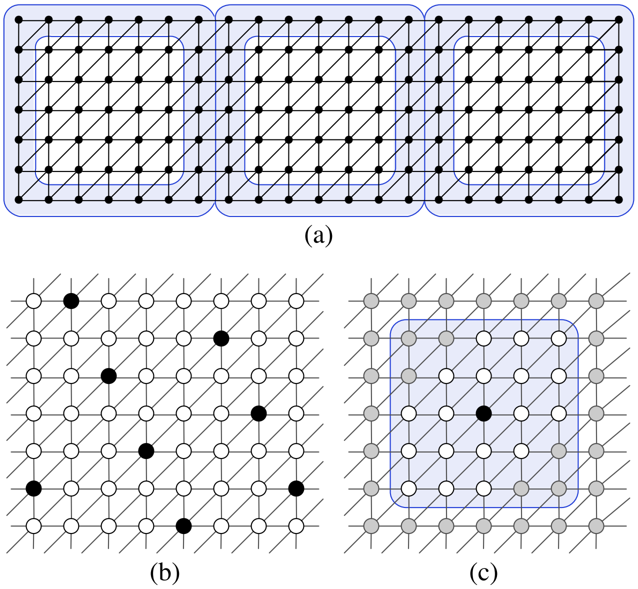

Figure 1: (a) Construction of graph G for $T = 1$, $m = 7$, and $r = 3$. There are 3 blocks. In each block there are 7×7 nodes: 5×5 internal nodes (white area), surrounded by a boundary area of width 1 (shaded). (b) A dominating set D^* of G that contains only a fraction $1/7$ of internal nodes—the figure shows only a part of a large triangular grid. (c) The local output of an internal node v (black node) only depends on its radius-T neighbourhood (white nodes, here $T = 2$). In particular, if we know the unique identifiers in the $k \times k$ region R_v around v (shaded area), we know the local output of node v.

On the other hand, there is a dominating set D^* which contains only a fraction $1/7$ of the internal nodes; see Figure 1b. Therefore $|D^*| \le (1/7 + \epsilon')n$, and the claim follows: for any positive constant ϵ we can show that algorithm $\mathcal{A}$ cannot find a factor $7 - \epsilon$ approximation of a minimum dominating set on planar graphs.

3. PROOF OF LEMMA 1

The proof uses the strategy of repeated applications of Ramsey's theorem; cf. Czygrinow et al. [1, Lemma 4]. We will use the notation $\mathcal{A}(G, v) \in \{0, 1\}$ to refer the *local output* of node v when we apply algorithm $\mathcal{A}$ to graph G; we have $\mathcal{A}(G, v) = 1$ if node v is in the dominating set computed by algorithm $\mathcal{A}$. By definition, $\mathcal{A}(G, v)$ only depends on the radius-T neighbourhood of v in G.

Let $k = 2T + 1$, $K = k^2$, and $M = m^2$. Consider any internal node $v \in I_i$ of any block B_i. The structure of graph G in the radius-T neighbourhood does not depend on the choice of v. The local output of node v only depends on the unique identifiers in the local neighbourhood. The local neighbourhood is contained within a rectangular $k \times k$ region $R_v \subseteq B_i$; see Figure 1c.

Let $V = \{1, 2, \ldots, n\}$ be the set of unique identifiers. Consider any K-subset of identifiers $X \subseteq V$, $|X| = K$. We will associate a *colour* $c(X) \in \{0, 1\}$ with each such set, as follows:

1. Pick an internal node v.
2. Assign the identifiers from X to region R_v in an increasing order by rows: the smallest k identifiers to the bottom row from left to right, etc. Assign the identifiers from $V \setminus X$ to the remaining nodes arbitrarily.
3. Apply algorithm $\mathcal{A}$, and set $c(X) = \mathcal{A}(G, v)$.

Now we have defined a colouring of all K-subsets of V; by restriction, we also have a colouring of all K-subsets of any $V' \subseteq V$. We say that $Y \subseteq V$ is *monochromatic* if $c(X_1) = c(X_2)$ for any K-subsets X_1 and X_2 of Y. By Ramsey's theorem [5] there exists an integer $N = N(K, M)$ such that the following holds: if V' is any N-subset of V, then there always exists a monochromatic subset $Y \subseteq V'$ of size M.

Now we will pick r and ℓ so that $\ell M > N$ and $\ell = o(r)$. Let $V_1 = V$. For each $i = 1, 2, \ldots, r - \ell$, we define the identifiers of block i as follows.

1. As $|V_i| \ge N$, we can find a monochromatic subset $Y_i \subseteq V_i$ of size M.
2. Assign the identifiers from Y_i to block B_i in an increasing order by rows: the smallest m identifiers to the bottom row from left to right, etc.
3. Set $V_{i+1} = V_i \setminus Y_i$.

Finally, assign the remaining ℓM identifiers from $V_{r-\ell+1}$ to blocks $r - \ell + 1, \ldots, r$ arbitrarily.

To complete the proof, consider a block i, where $1 \le i \le r - \ell$. Let $v \in I_i$ be an internal node of the block. Consider the $k \times k$ region R_v around v, and let X_v be the set of unique identifiers assigned to region R_v. Observe that the identifiers of X_v are assigned in an increasing order by rows. It follows that $\mathcal{A}(G, v) = c(X_v)$, i.e., the local output of the internal node v is simply the colour of subset X_v. Furthermore, $X_v \subseteq Y_i$ and Y_i was monochromatic. Hence all internal nodes of block i produce the same output. The common output cannot be 0; otherwise there would be nodes that are not dominated. Hence $I_i \subseteq \mathcal{A}(G)$.

4. ACKNOWLEDGEMENTS

Many thanks to Wojciech Wawrzyniak for discussions, and to anonymous reviewers for their helpful feedback. This work was supported in part by the Deutsche Forschungsgemeinschaft (DFG, reference number Le 3107/1-1), by the Academy of Finland, Grant 252018, and by the Research Funds of the University of Helsinki.

5. REFERENCES

[1] Andrzej Czygrinow, Michał Hańćkowiak, and Wojciech Wawrzyniak. Fast distributed approximations in planar graphs. In *Proc. 22nd International Symposium on Distributed Computing (DISC 2008)*, volume 5218 of *Lecture Notes in Computer Science*, pages 78–92. Springer, 2008. doi:10.1007/978-3-540-87779-0_6.

[2] Mika Göös, Juho Hirvonen, and Jukka Suomela. Lower bounds for local approximation. *Journal of the ACM*, 60(5):39:1–23, 2013. doi:10.1145/2528405. arXiv:1201.6675.

[3] Christoph Lenzen. *Synchronization and Symmetry Breaking in Distributed Systems.* PhD thesis, ETH Zurich, January 2011.

[4] Christoph Lenzen, Yvonne Anne Pignolet, and Roger Wattenhofer. Distributed minimum dominating set

approximations in restricted families of graphs. *Distributed Computing*, 26(2):119–137, 2013. doi:10.1007/s00446-013-0186-z.

[5] Frank P. Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society*, 30:264–286, 1930. doi:10.1112/plms/s2-30.1.264.

[6] Jukka Suomela. Survey of local algorithms. *ACM Computing Surveys*, 45(2):24:1–40, 2013. doi:10.1145/2431211.2431223. http://www.cs.helsinki.fi/local-survey/.

[7] Wojciech Wawrzyniak. Brief announcement: a local approximation algorithm for MDS problem in anonymous planar networks. In *Proc. 32nd Annual ACM Symposium on Principles of Distributed Computing (PODC 2013)*, pages 406–408. ACM Press, 2013. doi:10.1145/2484239.2484281.

[8] Wojciech Wawrzyniak. A strengthened analysis of a local algorithm for the minimum dominating set problem in planar graphs. *Information Processing Letters*, 114(3):94–98, 2014. doi:10.1016/j.ipl.2013.11.008.