LINEAR DISCREPANCY OF TOTALLY UNIMODULAR MATRICES

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Abstract. We show that the linear discrepancy of a totally unimodular $m \times n$ matrix $A$ is at most

$$\text{lindisc}(A) \leq \min\{1 - \frac{1}{m+1}, 1 - \frac{1}{n+1}\}.$$ 

This bound is sharp. In particular, this result proves Spencer's conjecture $\text{lindisc}(A) \leq (1 - \frac{1}{m+1})\text{herdis}(A)$ in the special case of totally unimodular matrices. As a side product we derive some bounds for the problem of efficiently rounding a given $p \in [0,1]^n$ to a 0,1 vector $z$ such that $\|A(p - z)\|\infty$ is small, and a characterization of those totally unimodular matrices which have linear discrepancy $1 - \frac{1}{n+1}$.

1. Introduction and Results

Let $A \in \mathbb{R}^{m \times n}$ be any real matrix and $p \in [0,1]^n$. The linear discrepancy of $A$ with respect to $p$ is defined by

$$\text{lindisc}(A, p) := \min_{z \in [0,1]^n} \|A(p - z)\|\infty,$$

the linear discrepancy of $A$ is

$$\text{lindisc}(A) := \max_{p \in [0,1]^n} \text{lindisc}(A, p).$$

The linear discrepancy can be seen as a generalized notion of discrepancy as well as a measure of how well a solution $p$ of a linear system of equations $Ax = b$ can be rounded to an integer one (also called lattice approximation problem).

Define the discrepancy of $A$ by

$$\text{disc}(A) := \min_{\chi \in \{-1,1\}^n} \|A\chi\|\infty.$$

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Discrepancy is a measure of how well the columns of $A$ can be partitioned into two classes such that for each row the two sums of its entries in each class are nearly equal. Immediately we see $\text{disc}(A) = 2\text{lindisc}(A, \frac{1}{2}1_n) \leq 2\text{lindisc}(A).$ Therefore we may view the linear discrepancy as a generalization of the discrepancy, where we have weights assigned to the columns describing the ratio in which (in average) we would like this column to belong to each of the two partition classes.

The following elementary remark fixes the connection between linear discrepancy and the problem of approximate integer solutions of linear systems.

**Remark.** Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ such that the linear system $Ax = b$ has a solution $x$. Let $p \in [0, 1]^n$ such that $p - x \in \mathbb{Z}^n$. Then there is a $z \in \mathbb{Z}^n$ such that $\|x - z\|_\infty \leq 1$ and $\|Az - b\|_\infty \leq \text{lindisc}(A, p)$.

An $m \times n$ matrix $A$ is called **totally unimodular** if each square submatrix has determinant $-1, 0$ or $1$. In particular, $A \in \{-1, 0, 1\}^{m \times n}$. The discrepancy problem for totally unimodular matrices is well-understood. Their discrepancy is at most one. By definition, submatrices of totally unimodular matrices are totally unimodular, hence the discrepancy of all submatrices is at most one as well. The beautiful theorem of Ghouila-Houri [GH62] states that also the converse holds:

**Theorem** (Ghouila-Houri, 1962). A matrix is totally unimodular if and only if each submatrix has discrepancy at most one.

Defining the **hereditary discrepancy** $\text{herdisc}(\cdot)$ to be the maximal discrepancy among all submatrices, we have that totally unimodular matrices are exactly the ones having hereditary discrepancy at most one. Contrary to the discrepancy and the hereditary discrepancy, for the linear discrepancy of totally unimodular matrices a sharp upper bound was missing so far. This paper solves the problem.

**Previous Results.** Using the result due to Lovász, Spencer and Vesztergombi [LSV86] that

$$\text{lindisc}(A) \leq \text{herdisc}(A)$$


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1Sometimes the linear discrepancy is defined by $\max_{p \in [-1, 1]^n} \min_{\chi \in \{-1, 1\}^n} \|A(p - \chi)\|_\infty$. This makes the linear discrepancy a direct generalization of the discrepancy, but puts less emphasis on the connection to the rounding problem. This notion is larger than the one we use by a factor of 2.
holds for any matrix $A$, immediately we have $\operatorname{lindisc}(A) \leq 1$ for a totally unimodular matrix $A$. Spencer conjectures that even $\operatorname{lindisc}(A) \leq (1 - \frac{1}{n+1}) \operatorname{herdisc}(A)$ holds for any $A$. This would yield $\operatorname{lindisc}(A) \leq 1 - \frac{1}{n+1}$, but this over 15 years old conjecture seems far from being proven. It is backed up by the fact that Spencer provides an example of a matrix $A$ such that $\operatorname{lindisc}(A) \leq (1 - \frac{1}{n+1}) \operatorname{herdisc}(A)$. As this $A$ is totally unimodular, it also shows that $\operatorname{lindisc}(A) \leq (1 - \frac{1}{n+1})$ is the best possible general upper bound for the linear discrepancy of totally unimodular matrices.

For a special class of totally unimodular matrices, Peng and Yan [PY99] used a combinatorial approach. A matrix $A$ is called strongly unimodular, if it is totally unimodular and if each matrix obtained from $A$ by replacing a single non-zero entry by zero is also totally unimodular. Peng and Yan show that for a strongly unimodular 0,1 matrix $A$,

$$\operatorname{lindisc}(A) \leq 1 - 3^{-\frac{n+1}{2}}$$

holds. They use a decomposition lemma due to Crama, Loebl and Poljak [CLP92], which states that such a matrix is the union of incidence matrices of digraphs, if all rows contain an even number of ones. In the same paper they show an upper bound of $1 - \frac{1}{n+1}$ for strongly unimodular 0,1 matrices which have at most two non-zeros in every row.

**Our Contribution.** In this paper we do not follow the approach via the hereditary discrepancy, nor do we use any structure theory for totally unimodular matrices. Instead, we consider a suitable linear program and apply the theorem of Hoffman and Kruskal. This easily yields

$$\operatorname{lindisc}(A) \leq 1 - \frac{1}{m}.$$ 

Using some basic linear algebra, we refine this result to

**Theorem 1.** For any $m \times n$ totally unimodular matrix $A$ we have

$$\operatorname{lindisc}(A) \leq \min \{1 - \frac{1}{n+1}, 1 - \frac{1}{m} \}.$$ 

This result is sharp, as the example of Spencer shows. As side products, this approach yields some bounds for $\operatorname{lindisc}(A,p)$ for fixed $p$, as well as a characterization of all totally unimodular matrices such that $\operatorname{lindisc}(A) = 1 - \frac{1}{n+1}$:

**Theorem 2.** Let $A$ be a $m \times n$ totally unimodular matrix. Then $\operatorname{lindisc}(A) = 1 - \frac{1}{n+1}$ holds exactly if there is a collection of $n+1$ rows of
A such that each \( n \) thereof are linearly independent. If \( \text{linindisc}(A, p) = \frac{1}{n} \) for some \( p \in [0, 1]^n \), then \( p_i \in \{ \frac{1}{n+1}, \ldots, \frac{n}{n+1} \} \) for all \( i \in [n] \).

2. Definitions and Notation

For a real number \( r \in \mathbb{R} \) write \( \lfloor r \rfloor := \max \{ z \in \mathbb{Z} | z \leq r \} \) for the largest integer not greater than \( r \), and \( \lceil r \rceil := \min \{ z \in \mathbb{Z} | z \geq r \} \) for the smallest integer not being less than \( r \). Set \( \{ r \} := r - \lfloor r \rfloor \), the fractional part of \( r \).

Let \( b \in \mathbb{R}^n \). We assume the above notation lifted to vectors in the natural way, e. g. \( \lceil b \rceil := (\lceil b_i \rceil)_{i \in [m]} \). Part of our strategy will be to round those components of \( b \) to the nearest integer which are already very close to an integer. For \( d \in [0, \frac{1}{2}] \) we define \( I^-(b, d) := \{ i \in [m] | [b_i] < d \} \), the set of indices such that \( b_i \) is less than \( d \) above the nearest integer (and hence a candidate for being rounded down), and \( I^+(b, d) := \{ i \in [m] | 1 - \{ b_i \} < d \} \), the set of indices such that \( b_i \) is less than \( d \) below the nearest integer. Set \( I(b, d) := I^-(b, d) \cup I^+(b, d) \). Let \( r(b, d) \in \mathbb{R}^n \) denote the vector resulting from rounding the components with index in \( I(b, d) \) to the nearest integer, i. e. for all \( i \in [m] \) we have

\[
r(b, d) = \begin{cases} 
\lceil b_i \rceil & \text{if } i \in I^-(b, d) \\
\lfloor b_i \rfloor & \text{if } i \in I^+(b, d) \\
b_i & \text{else}
\end{cases}
\]

The total error of this rounding is described by

\[
e(b, d) := \| r(b, d) - b \|_1 = \sum_{i \in I^-(b, d)} (b_i - \lfloor b_i \rfloor) + \sum_{i \in I^+(b, d)} (\lceil b_i \rceil - b_i).
\]

Let \( g(b) \) denote the maximum value of \( d \in [0, \frac{1}{2}] \) such that \( e(b, d) < 1 \) (the maximum exists, since \( d \mapsto e(b, d) \) is left-continuous). For a matrix \( A \in \mathbb{R}^{m \times n} \) set \( g(A) := \max_{p \in [0, 1]^n} g(Ap) \).

**Lemma 1.** Let \( b \in \mathbb{R}^n \) and \( d \in [0, \frac{1}{2}] \). Then

\( (i) \) \( e(b, d) < |I(b, d)|d \leq md \),
\( (ii) \) \( g(b) \geq \frac{1}{m} \).

In particular \( g(A) \geq \frac{1}{m} \) holds for any \( m \times n \) matrix \( A \).
Proof. We have
\[ e(b, d) = \sum_{i \in I^-(b, d)} (b_i - \lfloor b_i \rfloor) + \sum_{i \in I^+(b, d)} (\lfloor b_i \rfloor - b_i) \]
\[ < \sum_{i \in I^-(b, d)} d + \sum_{i \in I^+(b, d)} d \]
\[ = |I(b, d)| d \leq md. \]
In particular \( e(b, \frac{1}{m}) < 1. \) Thus \( g(b) \geq \frac{1}{m} \) by definition.

These bounds are sharp. The vector \((d - \varepsilon)1_m, \varepsilon > 0\) shows that (i) does not allow any further improvement, and \( b = \frac{1}{m}1_m \) is an example for \( g(b) = \frac{1}{m}. \)

3. The Linear Program

The next lemma analyses the linear discrepancy problem for fixed \( A \) and \( p \in [0, 1]^n. \) It shows that the rounding problem for totally unimodular matrices can be solved efficiently with error at most \( 1 - g(Ap). \) The proof is self-contained apart from the well-known theorem of Hoffman and Kruskal [HK56], which states that the set of feasible solutions of a linear program is an integral polyhedron, if the constraint matrix is totally unimodular and the right-side vector is integral. Hence in this case there are optimal solutions which are integral.

**Lemma 2.** Let \( A \in \mathbb{R}^{m \times n} \) be a totally unimodular matrix and \( p \in [0, 1]^n. \) Then
\[ \operatorname{lin} \operatorname{disc}(A, p) \leq 1 - g(Ap). \]

A \( z \in \{0, 1\}^n \) such that \( \|A(p - z)\|_\infty \leq 1 - g(Ap) \) holds, can be found efficiently by solving a linear optimization problem in \( \mathbb{R}^n \) having \( 2(m + n) \) inequalities.

**Proof.** Set \( b := Ap. \) Let \( P := \{x \in [0, 1]^n | [b] \leq Ax \leq \lfloor b \rfloor\}. \) As \( A \) is totally unimodular, \( P \) is an integral polyhedron (this is [HK56]). Define \( f : P \to \mathbb{R} \) by
\[ f(x) = \sum_{i \in I^-(b, g(b))} ((Ax)_i - [b_i]) + \sum_{i \in I^+(b, g(b))} ([b_i] - (Ax)_i). \]
for all \( x \in P \). Thus \( f(x) \) is the total error inflicted by rounding \( Ax \) in that way that was used to get \( r(b,g(b)) \) from \( b \). By definition, \( f \) is non-negative. We first show that for all \( x \in P \cap \mathbb{Z}^n \) we have

\[
(2) \quad f(x) < 1 \iff \forall i \in [m] \left\{ \begin{array}{ll}
\{ b_i \} < g(b) & \Rightarrow (Ax)_i = \lfloor b_i \rfloor \\
\{ b_i \} > 1 - g(b) & \Rightarrow (Ax)_i = \lfloor b_i \rfloor - 1
\end{array} \right.
\]

Suppose that \( f(x) < 1 \) for \( x \in P \cap \mathbb{Z}^n \). As \( A \) and \( x \) are integral, \( f(x) \leq 0 \), and since \( f \) is non-negative, \( f(x) = 0 \). Since all parts of the sum in (1) are non-negative, they are all zero. Hence \((Ax)_i = r(b,g(b))_i\) for all \( i \in I(b,g(b))\). This is the right-hand-side of (2). On the other hand, if the right-hand-side of (2) is fulfilled, we have \( f(x) = 0 \) by (1). Thus (2) holds.

Consider the linear optimization problem

\[
\min_{x \in P} f(x).
\]

\( p \) is a feasible solution and \( f(p) = e(Ap,g(b)) = e(b,g(b)) < 1 \). Hence there is an optimal solution \( x^* \) such that \( f(x^*) < 1 \). As \( P \) is integral, we may assume \( x^* \in \mathbb{Z}^n \).

Let us compute \( \|A(p - x^*)\|_\infty = \|b - Ax^*\|_\infty \). Let \( i \in [m] \). If \( i \in I^-(b,g(b)) \), then \((Ax^*)_i = \lfloor b_i \rfloor \) by (2). Hence \( |b_i - (Ax^*)_i| < g(b) \).

Similarly for \( i \in I^+(b,g(b)) \). Thus we may assume \( i \in [m] \setminus I(b,g(b)) \), i.e. \( b_i \in (\lfloor b_i \rfloor + g(b), \lfloor b_i \rfloor - g(b)) \). As \((Ax^*)_i \in \{\lfloor b_i \rfloor, \lfloor b_i \rfloor \} \) due to \( x^* \in P \), we conclude \( |b_i - (Ax^*)_i| \leq 1 - g(b) \). \( \square \)

Lemma 2 is sharp in the worst-case, as this example due to Spencer shows: Set \( m := n + 1 \). Let \( A \in \{0,1\}^{m \times n} \) denote the \( m \times n \) matrix with

\[
a_{ij} = \begin{cases} 
1 & \text{if } i = j \text{ or } i = n + 1 \\
0 & \text{else}
\end{cases}
\]

Set \( p = \frac{1}{m}1_n \). It is easy to see that any \( z \in \{0,1\}^n \) fulfills \( \|A(p - z)\|_\infty \geq 1 - \frac{1}{m} \): If any \( z_j, j \in [n] \) equals 1, then \((A(p - z))_j = p_j - z_j = \frac{1}{m} - 1 \). Otherwise we have \((A(p - z))_{n+1} = n\frac{1}{m} = 1 - \frac{1}{m} \).

Lemma 1 and 2 yield

**Corollary 3.** Let \( A \in \mathbb{R}^{m \times n} \) be a totally unimodular matrix. Then

\[
l\text{indisc}(A) \leq 1 - g(A).
\]

In particular \( l\text{indisc}(A) \leq 1 - \frac{1}{m} \).
4. The Refinement

In this section we refine the result of the previous one and finally prove Theorem 1. Before doing so let us remark that a weaker bound in terms of $n$ follows from a purely combinatorial argument. If $A \in \mathbb{R}^{m \times n}$ is totally unimodular and any two rows of $A$ are linearly independent, then

\begin{equation}
    m \leq \binom{n}{2} + \binom{n}{1}
\end{equation}

holds. We would show (3) using the connection between the VC-dimension and the primal shatter function of hypergraphs, but maybe more direct ways are possible as well. Unfortunately, (3) is sharp. To prove Theorem 1, we therefore need a different approach.

Proof of Theorem 1. Let $p \in [0,1]^n$ and $b := Ap$. Denote the rows of $A$ by $a_1, \ldots, a_m$. Set $I := I(b, \frac{1}{n+1}) = \{i \in [m] \mid \|b_i - \text{rd}(b_i, \frac{1}{2})\| < \frac{1}{n+1}\}$. Let us call these rows 'critical' for the moment, because they are the ones where a rounding error of more than $1 - \frac{1}{n+1}$ can occur when using the approach of the previous section.

We proceed by showing that it is enough to consider at most $n$ critical rows. Let $I_0 \subseteq I$ be chosen such that $\{a_i \mid i \in I_0\}$ is a basis for the vector space generated by all critical rows. In particular, $|I_0| \leq n$. Let $A_0$ denote the matrix obtained from $A$ by deleting all critical rows except $a_i, i \in I_0$. Then $A_0p = b|_{\{i \in [m] \mid I_0\}} =: b_0$. From Lemma 1 and $I(b_0, \frac{1}{n+1}) = I_0$ we conclude $e(b_0, \frac{1}{n+1}) < \frac{n}{n+1}$. Hence $g(A_0p) \geq \frac{1}{n+1}$. By Lemma 2 there is a $z \in \{0,1\}^n$ such that $\|A_0(p - z)\|_\infty < 1 - \frac{1}{n+1}$. In particular, for all $i \in I_0$ we have

\begin{equation}
    |a_i \cdot (p - z)| < \frac{1}{n+1},
\end{equation}

where $\cdot$ denote the usual inner product on $\mathbb{R}^n$.

We end the proof by showing that this $z$ also fulfills $\|A(p - z)\|_\infty \leq 1 - \frac{1}{n+1}$. Let $j \in I \setminus I_0$. As $I_0$ is a basis for the vector space generated by all critical rows, there are $\lambda_i, i \in I_0$ such that $a_j = \sum_{i \in I_0} \lambda_i a_i$. Since $A$ is totally unimodular, Cramer's rule implies $\lambda_i \in \{-1, 0, 1\}$ for all $i \in I_0$. Now

\begin{equation}
    |a_j \cdot (p - z)| \leq \sum_{i \in I_0} |\lambda_i a_i \cdot (p - z)| \leq n \frac{1}{n+1}
\end{equation}

by (4).\qed
5. A Characterization

The proof above yields some more information, which we now use for a characterization of totally unimodular matrices that have \( \text{disc}(A) = 1 - \frac{1}{n+1} \).

**Proof of Theorem 2.** Let \( \text{disc}(A) = 1 - \frac{1}{n+1} \). Choose \( p \in [0, 1]^n \) such that \( \text{disc}(A, p) = 1 - \frac{1}{n+1} \). Set \( b := Ap \) and \( I := \{ i \in [n] | |b_i - \text{rd}(b, \frac{1}{2})| \leq \frac{1}{n+1} \} \). Note that \( I = I(b, d) \) for some \( d > \frac{1}{n+1} \). For any \( J \subseteq [n] \) define \( V_J \) to be the vector space generated by the rows \( a_j, j \in J \). Let \( I_0 \) be a minimal subset of \( I \) such that \( V_{I_0} = V_I \). In particular, the rows \( a_i, i \in I_0 \) form a basis of \( V_I \). If there is an \( i \in I_0 \) such that \( \{ b_i \} \not\subseteq \{ \frac{1}{n+1}, 1 - \frac{1}{n+1} \} \), or if \( |I_0| < n \), then by mimicking the proof above we get a \( z \in \{0, 1\}^n \) such that \( \|A(p - z)\|_{\infty} \leq \max\{1 - d, \sum_{i \in I_0} |b_i - \text{rd}(b, \frac{1}{2})| \} < 1 - \frac{1}{n+1} \). We conclude \( |I_0| = n \) and \( \{ b_i \} \subseteq \{ \frac{1}{n+1}, 1 - \frac{1}{n+1} \} \) for all \( i \in I_0 \), and hence also for all \( i \in I \). From Lemma 1 we get \( |I| \geq n + 1 \) (otherwise \( g(b) \geq d \), and Lemma 2 yields a contradiction).

From the fact that \( A \) is totally unimodular, we know that each \( a_i, i \in I \setminus I_0 \), can be expressed in the form \( a_i = \sum_{j \in I_0} \lambda_j a_j \) with some \( \lambda_j \in \{-1, 0, 1\}, j \in I_0 \). Let us assume that for each \( i \in I \setminus I_0 \) there is such an expression \( a_i = \sum_{j \in I_0} \lambda_j a_j \) such that at least one of the \( \lambda_j, j \in I_0 \) is zero. Then by mimicking the proof of Theorem 1 above (using this \( I_0 \)), we find a \( z \in \{0, 1\}^n \) such that \( |a_i \cdot (p - z)| = \frac{1}{n+1} \) for all \( i \in I \setminus I_0 \) and \( |a_i \cdot (p - z)| \leq \frac{n+1}{n+1} \) for all \( i \in I \setminus I_0 \). This is again a contradiction to our choice of \( p \). Hence there is an \( i \in I \setminus I_0 \) such that \( a_i = \sum_{j \in I_0} \lambda_j a_j \) with some (by the way unique) \( \lambda_j \in \{-1, 1\}, j \in I_0 \). In particular, any \( n \) of the rows with index in \( I_0 \cup \{ i \} \) are linearly independent.

Let \( A' \) and \( b' \) denote the restrictions of \( A \) and \( b \) on the rows with index in \( I_0 \). Then \( A' \) is non-singular, and thus \( p \) is already determined by \( A'p = b' \). As \( (n+1)b' \in \{1, n\}^n \) was shown in the first paragraph, we have \( (n+1)p \in \mathbb{Z}^n \) (the inverse of a totally unimodular matrix is totally unimodular, and thus integral). Clearly, none of the \( p_i, i \in [n] \) is 0 or 1 — otherwise we may just put \( z_i = p_i \) reducing the dimension of the problem by one. Hence all \( p_i, i \in [n] \) are in \( \{ \frac{1}{n+1}, \ldots, \frac{n}{n+1} \} \) as claimed.

Now let \( A \) be such that there are \( n + 1 \) rows each \( n \) thereof being linearly independent. Without loss of generality we may assume these to be the rows \( a_1, \ldots, a_{n+1} \). As above there are \( \lambda_1, \ldots, \lambda_n \in \{-1, 1\} \)
such that $a_{n+1} = \sum_{i \in [n]} \lambda_i a_i$. Define $b' \in \mathbb{R}^n$ by
\[
   b'_i := \begin{cases} 
      \frac{1}{n+1} & \text{if } \lambda_i = 1 \\
      1 - \frac{1}{n+1} & \text{else}
   \end{cases}
\]
for all $i \in [n]$. Let $A_0$ denote the matrix consisting of the rows $a_1, \ldots, a_n$ only. As $A_0$ has full rank, the system $A_0 x = b'$ has a unique solution $x$. Since $A_0$ is totally unimodular and $(n+1)b' \in \mathbb{Z}^n$, $(n+1)x$ is also integral. Set $p = \{x\}$ and $b = A p$. Then $\{b'_i\} = \{b_i\}$ for $i \in [n]$. We claim that any $z \in \{0,1\}^n$ fulfills $\|A(p - z)\|_\infty \geq 1 - \frac{1}{n+1}$. Let us assume $|a_i \cdot (p - z)| < 1 - \frac{1}{n+1}$ for all $i \in [n]$ (otherwise we are done). Then $\lambda_i a_i \cdot (p - z) = \frac{1}{n+1}$ holds for all $i \in [n]$ by definition of $b'$. Thus $a_{n+1} \cdot (p - x) = \sum_{i \in [n]} \lambda_i a_i \cdot (p - x) = n \frac{1}{n+1}$. This proves the claim. \( \square \)

It is a trivial consequence of the definition of the linear discrepancy that if a matrix $B$ consists of some rows of the matrix $A$, then $\text{linDisc}(B) \leq \text{linDisc}(A)$. In the light of Theorem 2 it makes sense to call a totally unimodular $m \times n$ matrix critical, if $m = n+1$ and $\text{linDisc}(A) = 1 - \frac{1}{n+1}$. Theorem 2 then states that a totally unimodular $m \times n$ matrix has linear discrepancy $1 - \frac{1}{n+1}$ if and only if it contains a critical one. The reasoning above also shows that for critical matrices $A$, there are just two different $p$ such that $\text{linDisc}(A,p) = 1 - \frac{1}{n+1}$ holds, namely the one constructed, call it $p^{(1)}$, and $p^{(2)} := 1 - p^{(1)}$.

\textbf{References}


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