MULTI-COLOR DISCREPANCIES II

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Abstract. We use a refined recursive coloring approach to show improved upper bounds for the \(c\)-color discrepancy in the six standard deviation situation of \(\mathcal{O}(\sqrt{\frac{c}{n}} \log c)\) and for the arithmetic progressions of \(\mathcal{O}(c^{-0.16} n^{0.25})\). We also investigate the weighted discrepancy in two colors, that is, the problem of partitioning the vertex set in such a way that all hyperedges are split in a given ratio \((q, 1 - q)\) by this partition. In the six standard deviation situation, this \((q, 1 - q)\)-discrepancy is at most \(\mathcal{O}(\sqrt{qn \log \frac{1}{q}})\).

1. General approach

For technical reasons as well as independent interest we investigate a slightly stronger discrepancy notion in this section. In simple words, we will require that the set of all vertices is perfectly balanced. For the traditional 2-color discrepancy as well as for \(c\)-color discrepancy this shall mean that any two color classes deviate in size by at most one. Equivalently, we ask for a coloring \(\chi : X \to [c]\) such that \(|X \cap \chi^{-1}(i)| - \frac{1}{c}|X| < 1\) holds for each color \(i \in [c]\). When talking about weighted discrepancy with respect to a weight \(p \in [0, 1]^c\), we require \(|X \cap \chi^{-1}(i)| - p_i |X| < 1\) to hold for each color \(i \in [c]\). A coloring of this kind will be called a fair coloring. The corresponding discrepancy notion (where the minimum is taken over all fair colorings) will be called fair discrepancy.

One remark that eases work with the fractional parts: Let us call a weight \(p \in [0, 1]^c\) integral (with respect to \(\mathcal{H} = (X, E)\)) if all \(p_i, i \in [c]\) are multiples of \(\frac{1}{|X|}\). From the definition it is clear that a fair coloring \(\chi\) with respect to an integral weight \(p\) fulfills \(|\chi^{-1}(i)| = p_i |X|\) for all colors \(i \in [c]\). On the other hand, suppose that we know that for a given hypergraph and for all integral weights \(p\) there is a fair coloring

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*supported by the graduate school 'Effiziente Algorithmen und Multiskalenmethoden', Deutsche Forschungsgemeinschaft.*
with respect to \( p \) that has discrepancy at most \( k \). Then there are fair colorings having discrepancy at most \( k + 1 \) for any weight: For an arbitrary weight \( p \) there is an integral weight \( p' \) such that \( |p_i - p'_i| < \frac{1}{\kappa \sqrt{n}} \) holds for all \( i \in [c] \). Therefore, a fair coloring with respect to \( p' \) is also fair with respect to \( p \), and its discrepancy with respect to \( p \) is larger (if at all) than the one with respect to \( p' \) by less than one. For these reasons we may restrict ourselves to the more convenient case that all weights are integral.

To state the following theorem in its strongest form and also for its proof we need the following constants: For each \( p \in [0, 1] \) define \( v_\alpha(p) \) by

\[
\max\left\{ \sum_{i=1}^{k} \prod_{j=1}^{i} q_j^\alpha \prod_{j=i+1}^{k} q_j^k \in \mathbb{N}, q_1, \ldots, q_{k-1} \in \left[0, \frac{2}{3}\right], q_k \in [0, 1], \prod_{j=1}^{k} q_j = p. \right\}
\]

For \( \alpha \in ]0, 1[ \) set \( C(\alpha) := 1 + \sum_{i=0}^{\infty} \left( \frac{2}{3} \right)^{(1-\alpha)i} \). Then we have

**Lemma 1.** Let \( \alpha \in ]0, 1[ \). Then

(i) Let \( 0 < p < q \leq \frac{2}{3} \). Then \( q^\alpha v_\alpha(\frac{q}{q}) + q^\alpha \frac{q}{q} \leq v_\alpha(p) \).

(ii) For all \( p \in [0, 1] \), \( v_\alpha(p) \leq C(\alpha)p^\alpha \).

**Proof.** Let \( k \in \mathbb{N}, q_1, \ldots, q_{k-1} \in \left[0, \frac{2}{3}\right], q_k \in [0, 1] \) such that \( \prod_{j=1}^{k} q_j = \frac{q}{q} \) and \( v_\alpha(\frac{q}{q}) = \sum_{i=0}^{k} \prod_{j=1}^{i} q_j^\alpha \prod_{j=i+1}^{k} q_j^k \). With \( q_0 := q \) we have

\[
q^\alpha v_\alpha(\frac{q}{q}) + q^\alpha \frac{q}{q} = q^\alpha \sum_{i=0}^{k} \prod_{j=1}^{i} q_j^\alpha \prod_{j=i+1}^{k} q_j^k + q_0^\alpha \prod_{j=1}^{k} q_j^k = \sum_{i=0}^{k} \prod_{j=0}^{i} q_j^\alpha \prod_{j=i+1}^{k} q_j \leq v_\alpha(p),
\]

since \( \prod_{j=0}^{k} q_j = q \prod_{j=1}^{k} q_j = p \). This is (i).

Let \( k \in \mathbb{N}, q_1, \ldots, q_{k-1} \in \left[0, \frac{2}{3}\right], q_k \in [0, 1] \) such that \( \prod_{j=1}^{k} q_j = p \) and \( v_\alpha(p) = \sum_{i=1}^{k} \prod_{j=1}^{i} q_j^\alpha \prod_{j=i+1}^{k} q_j^k \). For \( i \in [k] \) set \( x_i := \prod_{j=1}^{i} q_j^\alpha \prod_{j=i+1}^{k} q_j^k \). Then \( x_k = p^\alpha \) and \( x_{k-1} \leq x_k \). For \( i \in [k-2] \) we have

\[
\frac{x_{k-1-i}}{x_{k-1-i+1}} = \frac{q_{k-1-i}^\alpha}{q_{k-1-i}^k} = q_{k-1-i}^{1-\alpha} \leq \left( \frac{2}{3} \right)^{1-\alpha},
\]
and hence $x_{k-1,i} \leq \left(\frac{2}{3}\right)^{(1-\alpha)i} x_k$. Thus

$$v_{\alpha}(p) = \sum_{i=1}^{k} x_i \leq p^\alpha \left(1 + \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{(1-\alpha)i}\right) = C(\alpha)p^\alpha.$$ 

\[\square\]

In the following we deal with the situation that the weighted discrepancy of an induced subhypergraph $\mathcal{H}_{|X_0|$ is $O(|X_0|^\alpha)$ for some constant $\alpha \in [0, 1]$. Compared to the recursive approach investigated in [DS00] we assume a better bound for smaller subhypergraphs. With this assumption we get an improve upper bound of $O\left(\left(\frac{2}{3}\right)^\alpha\right)$ for the $c$-color discrepancy (instead of just $O(n^\alpha)$ with the previous assumption of uniformly bounded weighted discrepancies).

**Theorem 1.** Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph. Let $p_0, \alpha \in [0, 1]$ and $C > 0$. Assume that for all $X_0 \subseteq X$ such that $|X_0| \geq p_0|X|$ and all $q \in [0, 1]$ such that $(q, 1-q)$ is integral with respect to $\mathcal{H}_{|X_0|$ the induced hypergraph $\mathcal{H}_{|X_0|$ has fair $(q, 1-q)$-discrepancy at most $C|X_0|^\alpha$, i.e. there is a $2$-coloring such that $||X_0 \cap \chi^{-1}(1)| - q|X_0|| < 1$ and $||E \cap X_0 \cap \chi^{-1}(1)| - q|E \cap X_0|| \leq C|X_0|^\alpha$ holds for all $E \subseteq \mathcal{E}$.

Then for each integral weight $p \in [0, 1]^c$ there is a fair $c$-coloring of $\mathcal{H}$ with respect to $p$ such that the discrepancy with respect to $p$ is at most $Cv_{\alpha}(p_i)n^\alpha \leq CC(\alpha)(p_i)n^\alpha$ in all colors $i \in [c]$ such that $p_i \geq p_0$. In particular, the $c$-color discrepancy of $\mathcal{H}$ is bounded by

$$\text{disc}(\mathcal{H}, c) \leq CC(\alpha)\left(\frac{2}{c}\right)^\alpha + 1$$

for all $c \leq \frac{1}{p_0}$.

**Proof.** We start with

**Claim 1:** For each integral (with respect to $\mathcal{H}$) weight $(2^{-k}, 1 - 2^{-k}), 2^{-k} \geq p_0$ there is a fair $2$-coloring with respect to this weight that has discrepancy at most $\sum_{i=0}^{k-1} 2^{-k+1+i}2^{-ai}Cn^\alpha$.

We proceed by induction. For $k = 1$, there is nothing to show. Let $k > 1$. Let $\chi_0 : X \to [2]$ be a fair $(0.5, 0.5)$ coloring having discrepancy at most $Cn^\alpha$. Set $X_1 := \chi_0^{-1}(1)$. Let $\chi_1 : X_1 \to [2]$ be a fair $(2^{-k+1}, 1 - 2^{-k+1})$ coloring (note that $(2^{-k+1}, 1 - 2^{-k+1})$ is integral for $\mathcal{H}_{|X_1|$). By induction we may assume that $\chi_1$ has discrepancy at most $\sum_{i=0}^{k-2} 2^{-k+2+i}2^{-ai}C(\frac{2}{2})^\alpha$. Define a coloring $\chi : X \to [2]$ by $\chi(x) = 1$ if and only if $\chi_0(x) = 1$ and $\chi_1(x) = 1$. Then $\chi$ is a fair $(2^{-k}, 1 - 2^{-k})$
coloring. For an edge \( E \in \mathcal{E} \) we compute its \((2^{-k}, 1 - 2^{-k})\)-discrepancy in color 1:

\[
\|E \cap \chi^{-1}(1) - 2^{-k} |E|\|
= \|E \cap \chi_0^{-1}(1) \cap \chi_1^{-1}(1) - 2^{-k} |E|\|
\leq \|E \cap \chi_0^{-1}(1) \cap \chi_1^{-1}(1) - 2^{-k+1} |E \cap \chi_0^{-1}(1)|\|
+ 2^{-k+1} |E \cap \chi_0^{-1}(1)| - 2^{-k} |E|\|
\leq \|(E \cap X_1) \cap \chi_1^{-1}(1) - 2^{-k+1} |E \cap X_1|\| + 2^{-k+1} \|E \cap \chi_0^{-1}(1) - 0.5 |E|\|
\leq \sum_{i=0}^{k-2} 2^{-k+2+i} 2^{-\alpha i} C \left( \frac{n}{2} \right)^\alpha + 2^{-k+1} C n^\alpha
= \sum_{i=0}^{k-1} 2^{-k+1+i} 2^{-\alpha i} C n^\alpha.
\]

This proves Claim 1. From our assumptions on \( \mathcal{H} \) it is clear that the assertion of Claim 1 also holds for any induced subgraph \( \mathcal{H}_{X_0} \) of \( \mathcal{H} \) as long as \( 2^{-k} |X_0| \leq p_0 |X| \). We will use this fact to prove

Claim 2: For each integral (with respect to \( \mathcal{H} \)) weight \((q, 1 - q), q \geq p_0\) there is a fair \(2\)-coloring with respect to this weight that has discrepancy at most \(\frac{2}{2^{1-\alpha} - 1} C (qn)^\alpha\).

Let \(q' \leq 1\) be maximal subject to the condition that \(\frac{q'}{q}\) is a power of \(2\). Let \(q' = 2^k q\). Let \(\chi_0 : X \to [2]\) be a fair \((q', 1 - q')\) coloring having discrepancy at most \(C n^\alpha\). Let \(\chi_1 : \chi_0^{-1}(1) \to [2]\) be a fair \((2^k q, 1 - 2^k q)\) coloring. From Claim 1 we may assume that \(\chi_1\) has discrepancy at most \(\sum_{i=0}^{k-1} 2^{-k+1+i} 2^{-\alpha i} C (q'n)^\alpha\). Define a coloring \(\chi : X \to [2]\) by \(\chi(x) = 1\) if and only if \(\chi_0(x) = 1\) and \(\chi_1(x) = 1\). Then \(\chi\) is a fair \((q, 1 - q)\)
coloring. For an edge \( E \in \mathcal{E} \) we compute its discrepancy in color 1:
\[
|E \cap \chi^{-1}(1)| - q|E| \\
= |E \cap \chi_0^{-1}(1) \cap \chi_1^{-1}(1)| - q|E| \\
\leq |E \cap \chi_0^{-1}(1) \cap \chi_1^{-1}(1)| - \frac{q'}{q}|E \cap \chi_0^{-1}(1)| + \frac{q'}{q}|E \cap \chi_0^{-1}(1)| - q|E| \\
\leq \sum_{i=0}^{k-1} 2^{-k+1+i-2\alpha i} C(q'n)^{\alpha} + 2^{-k} Cn^{\alpha} \\
= \left(2^{-k+1}q^{\alpha}(\frac{q'}{q})^{\frac{2(1-\alpha)}{21-\alpha}} - 1 + 2^{-k}\right) Cn^{\alpha} \\
< 2q^{\alpha}\left(\frac{2-\alpha}{21-\alpha} - 1\right) Cn^{\alpha} = \frac{2}{21-\alpha} C(qn)^{\alpha}.
\]
This proves Claim 2.

Let \( p \in [0, 1]^c \) be an integral weight. Again we proceed by induction. Choose a partition \( \{C_1, C_2\} \) of the set of colors \([c]\) such that \( \|p_{|C_1}\|_1, \|p_{|C_2}\|_1 \leq \frac{2}{3} c \) or \( C_1 \) contains a single color with weight at least \( \frac{1}{3} \). In particular, \( \|p_{|C_2}\|_1 \leq \frac{2}{3} c \) holds in both cases. Set \( (q_1, q_2) := (\|p_{|C_1}\|_1, \|p_{|C_2}\|_1) \). Choose a fair coloring \( \chi_0 : X \to [2] \) with respect to the weight \( (q_1, q_2) \). From Claim 2 we may assume that \( \chi_0 \) has discrepancy at most \( C(q'n)^{\alpha} \) in color \( i = 1, 2 \) if \( q_i \geq p_0 \) (of course the discrepancy is the same in both colors, but we do not need this). Set \( X_i := \chi^{-1}(i) \) for \( i = 1, 2 \). If \( |C_i| > 1 \), then by induction, there is a fair coloring \( \chi_i : X_i \to C_i \) with respect to the weight \( \frac{1}{n} p_{|C_i} \) having discrepancy at most \( Cv_\alpha(\frac{p_i}{q_i})(qn)^{\alpha} \) in each color \( j \in C_i, p_j \geq p_0 \). If \( C_i = \{ j \} \) for some \( j \in [c] \), set \( \chi_i : X_i \to \{ j \} \). Set \( \chi = \chi_1 \cup \chi_2 \). We compute the discrepancy of an edge \( E \in \mathcal{E} \) with respect to \( \chi \) and \( p \) in color \( j \in C_i \) in the case \( |C_i| > 1 \) and \( p_j \geq p_0 \):
\[
|E \cap \chi^{-1}(j)| - p_j|E| \\
= |E \cap \chi_0^{-1}(i) \cap \chi_1^{-1}(j)| - p_j|E| \\
\leq |E \cap \chi_0^{-1}(i) \cap \chi_1^{-1}(j)| - \frac{p_j}{q_i}|E \cap \chi_0^{-1}(i)| + \frac{p_j}{q_i}|E \cap \chi_0^{-1}(i)| - p_j|E| \\
\leq |(E \cap X_i) \cap \chi_1^{-1}(j)| - \frac{p_j}{q_i}|E \cap X_i| + \frac{p_j}{q_i}|E \cap \chi_0^{-1}(i)| - q_i|E| \\
\leq Cv_\alpha(\frac{p_i}{q_i})(qn)^{\alpha} + \frac{p_j}{q_i}C(qn)^{\alpha} \\
= Cv_\alpha(p_j)n^{\alpha}
\]
by Lemma 1 (i). On the other hand, if \( C_i \) contains a single color
\[ j, \text{ then } p_j = q_i \text{ and } \|E \cap \chi^{-1}(j)\| - p_j|E| = \|E \cap \chi^{-1}(i)\| - p_j|E| \leq C(p_m)n^\alpha \leq C_{\alpha}(p_j)n^\alpha. \]

The last assertion follow from taking \( p = \frac{1}{e}1_e \) and the remark about integral and arbitrary weights. \( \square \)

2. Six Standard Deviations

The famous “Six Standard Deviation” result due to Spencer [Spe85] is

**Theorem 2.** There is a constant \( K \) such that for all hypergraphs \( \mathcal{H} = (X, \mathcal{E}) \) having \( n \) vertices and \( m \) edges

\[ \text{disc}(\mathcal{H}) \leq K \sqrt{n \ln \left( \frac{2m}{n} \right)} \]

holds.

The interesting case is of course the one where \( m = \mathcal{O}(n) \) and thus \( \text{disc}(\mathcal{H}) = \mathcal{O}(\sqrt{n}) \). The title “Six Standard Deviations Suffice” of this paper comes from the fact that for \( n = m \) large enough, \( \text{disc}(\mathcal{H}) \leq 6\sqrt{n} \) holds. Using the relation between hereditary and linear discrepancy and an easy recoloring argument, we derive from Spencer’s theorem

**Lemma 2.** For any \( X_0 \subseteq X \) and integral (with respect to \( \mathcal{H}_{|X_0} \)) weight \((q, 1 - q)\) there is a fair \((q, 1 - q)\)-coloring of \( \mathcal{H}_{|X_0} \) that has discrepancy at most \( 2K \sqrt{|X_0| \ln \left( \frac{2m}{|X_0|} \right)} \).

**Proof.** Let \( X_0 \subseteq X \). Then any induced subgraph of \( \mathcal{H}_{|X_0} \) has discrepancy at most \( K \sqrt{|X_0| \ln \left( \frac{2m}{|X_0|} \right)} \), simply because Spencer’s bound is monotone in the number of vertices. From the fact that \( \text{ldisc}(\mathcal{H}) \leq 2\text{herdisc}(\mathcal{H}) \) holds for any hypergraph (cf. [LSV86]), we have \( \text{wd}(\mathcal{H}_{|X_0}, (q, 1 - q)) \leq 2\text{ldisc}(\mathcal{H}_{|X_0}) \leq \text{herdisc}(\mathcal{H}_{|X_0}) \leq K \sqrt{|X_0| \ln \left( \frac{2m}{|X_0|} \right)} \).

It remains to show the existence of a fair coloring. This is an easy, but nice trick. Let \( \overline{\mathcal{H}} \) denote the hypergraph arising from \( \mathcal{H} \) by adding the set \( X \) as an additional edge (unless of course \( X \in \mathcal{E} \) already holds). Then \( \overline{\mathcal{H}}_{|X_0} \) has at most \( m + 1 \) edges, and from the previous paragraph we know \( \text{wd}(\overline{\mathcal{H}}_{|X_0}, (q, 1 - q)) \leq K \sqrt{|X_0| \ln \left( \frac{2m}{|X_0|} \right)} \).

Let \( \chi \) be a coloring realizing this bound. Set \( x := q|X_0| - |\chi^{-1}(1)| \). Since \( X_0 \) is an edge in \( \overline{\mathcal{H}}_{|X_0} \), \( x \leq K \sqrt{|X_0| \ln \left( \frac{2m}{|X_0|} \right)} \). Let \( \overline{\chi} \) denote a coloring
arising from $\chi$ by changing the color of $|x|$ points in such a way that $X_0$ is perfectly balanced with respect to $\chi$, that is, $q |X_0| = |\chi^{-1}(1)|$. Now $\chi$ is a fair coloring with respect to the weight $(q, 1 - q)$. For an edge $E \cap X_0$, $E \in \mathcal{E}$ we compute

$$ |q|E \cap X_0| - |\chi^{-1}(1)||$$

$$ \leq |q|E \cap X_0| - |\chi^{-1}(1)|| + ||\chi^{-1}(1)| - |\chi^{-1}(1)||$$

$$ \leq 2K \sqrt{|X_0| \ln \left(\frac{2m+2}{p_0}\right)}.$$

\hfill \Box

Lemma 2 and Theorem 1 yield

**Theorem 3.** Let $\mathcal{H} = (X, \mathcal{E})$ denote a hypergraph and $p \in [0, 1]^c$ an integral weight. Set $p_0 := \min_{i \in [c]} p_i$. Then there is a fair coloring with respect to $p$ having discrepancy at most $13K \sqrt{p_0 n \ln \left(\frac{2m+2}{p_0}\right)}$ in color $i$.

In particular, in the case $|X| = |\mathcal{E}| = n$ we have

$$ \text{disc}(\mathcal{H}, c) \leq O\left(\sqrt{\frac{e}{c} \ln c}\right).$$

**Proof.** By Lemma 2 we may apply Theorem 1 with $\alpha = \frac{1}{2}$, $C = 2K \sqrt{n \ln \left(\frac{2m+2}{p_0}\right)}$ and $p_0$. This yields a fair coloring with respect to $p$ having discrepancy at most $CC(\frac{1}{2}) \sqrt{p_0 n}$ in color $i \in [c]$. The claim follows from $C(\frac{1}{2}) \leq 6.44949$.

This is quite close to the optimum. In [DS00] we showed a lower bound of $\Omega\left(\sqrt{\frac{e}{c}}\right)$ in the case $|X| = |\mathcal{E}| = n$.

Let us remark that we did not try to optimize the implicit constant in our multi-color version of the six standard deviation theorem. Our approach yields $\text{disc}(\mathcal{H}, c) \leq 78 \sqrt{\frac{e}{c} \ln c}$ if $\frac{c}{e}$ is sufficiently large.

In the case of (equi-weighted) $c$-color discrepancy, a better way of recursively partitioning the set of colors exists: If $c$ is not a power of 2 and $c = \sum_{i=0}^{d} a_i 2^i$ is its binary expansion, then partition $[c]$ into set of size $2^{\min\{i \mid a_i \neq 0\}}$ and $c - 2^{\min\{i \mid a_i \neq 0\}}$. If $c$ is a power of 2, split $[c]$ into equal-sized sets. This partitioning method was used in [DS00]. It should replace $C(\frac{1}{2}) \approx 6.4$ by some smaller number.

We also believe that the proof of Spencer’s result can be modified in such a way that the additional constraint of fair coloring does not inflict
an extra factor of 2 (as does our recoloring trick), but something much smaller, it all at.

Another corollary worth mentioning can already be derived from combining Claim 2 of the proof of Theorem 1 and Lemma 2:

**Corollary 3.** Let $\mathcal{H} = (X, E)$ denote a hypergraph such that $|X| = |E| =: n$ and $(q, 1 - q)$ an integral 2-color weight. Assume $q \leq \frac{1}{2}$. Then the weighted discrepancy $\text{disc}(\mathcal{H}, (q, 1 - q))$ is at most $10K \sqrt{qn \ln \left(\frac{2}{q}\right)}$.

### 3. Arithmetic progressions

**Theorem 4.** For a suitable constant $C'$ the following holds: Let $\mathcal{H} = ([n], E)$ denote the hypergraph of arithmetic progressions in $[n]$. Let $p \in [0, 1]^e$ be a weight. Then there is a fair coloring with respect to $p$ having discrepancy at most $C'p_1^{0.16}n^{0.25}$ in each color $i$ such that $p_i \geq n^{0.25}$.

**Proof.** From Lemma 5.3 of [MS96] we learn that an induced subgraph $\mathcal{H}_0 = \mathcal{H}[X_0]$ of $\mathcal{H}$ on $|X_0| = \rho n \geq n^{0.25}$ vertices has discrepancy at most $C_1\rho^{0.16}n^{0.25}$. We first show that $\text{herdisc}(\mathcal{H}_0) \leq 2C_1\rho^{0.16}n^{0.25}$.

Let $\mathcal{H}_1 = (X_1, E_1)$ be an induced subhypergraph of $\mathcal{H}_0$. If $|X_1| \geq n^{0.25}$ we are done by the Lemma of Matousek and Spencer. Let us therefore assume $|X_1| < n^{0.25}$. We show that $(\mathcal{H}_1)[\frac{n}{2}]$ and $(\mathcal{H}_1)[\frac{n}{2}]$ have discrepancy at most $C_1\rho^{0.16}n^{0.25}$ and conclude $\text{disc}(\mathcal{H}_1) \leq 2C_1\rho^{0.16}n^{0.25}$. Consider the hypergraph $\mathcal{H}_2 := \mathcal{H}(X_1[n/2], [n - n^{0.25} + |X_1|/2, n/2 + 1, \ldots, n])$. This hypergraph has exactly $n^{0.25} \leq \rho n$ vertices and thus discrepancy at most $C_1\rho^{0.16}n^{0.25}$. As every edge of $(\mathcal{H}_1)[\frac{n}{2}]$ is also an edge of $\mathcal{H}_2$, we conclude $\text{disc}(\mathcal{H}_1)[\frac{n}{2}] \leq C_1\rho^{0.16}n^{0.25}$. A similar reasoning shows $\text{disc}(\mathcal{H}_1)[\frac{n}{2}] \leq C_1\rho^{0.16}n^{0.25}$.

Thus we proved $\text{herdisc}(\mathcal{H}_0) \leq 2C_1\rho^{0.16}n^{0.25}$. The relation between the linear and hereditary discrepancy yields that all weighted discrepancies of $\mathcal{H}_0$ are bounded by $2C_1\rho^{0.16}n^{0.25}$. As $[n]$ is an arithmetic progression, we may apply the recoloring trick from the previous section and conclude that twice this discrepancy may be achieved by a fair coloring respecting the underlying weight.

We are now in a position to apply Theorem 1 with $C = 4C_1n^{0.09}$, $\alpha = 0.16$ and $p_0 = n^{0.25}$, which proves our claim. \hfill $\square$
Again we did not try to optimize the constants. As a corollary we improve the upper bound for the $c$-color discrepancy of the arithmetic progressions:

**Theorem 5.** For the hypergraph $\mathcal{H}$ of the arithmetic progressions in $[n]$, we can bound the discrepancy in $c \leq n^{0.25}$ colors by

$$\text{disc}(\mathcal{H}, c) = \mathcal{O}(e^{-0.16} n^{0.25}).$$

Let us comment a few words on this result. First of all, we note that the discrepancy decreases with increasing number of colors. This seems natural, but does not hold for all hypergraphs. Together with the lower bound of $\Omega(e^{-0.5} n^{0.25})$ we feel that we have shed some light on the discrepancy behavior of the arithmetic progressions in different numbers of colors.

On the other hand, our lower and upper bounds deviate by a larger extent than in the six standard deviations case where we just had a factor of $\mathcal{O}(\sqrt{\log c})$. One way of improvement is to tighten the constants due to Matousek and Spencer. They remark that the exponent of $\rho^{0.16}$ could be improved by more careful calculations. For them it does not matter, however, as the 0.16 just influences some constants that are not explicitly stated anyway. For us, of course, it would directly improve our upper bound.

This approach is limited to $\rho^{0.25}$. This can easily be seen from the fact that the hypergraph of arithmetic progressions on $[n]$ contains the hypergraph of arithmetic progressions on $[n_0]$ as an induced subhypergraph for all $n_0 \leq n$. This observation as well as vague feeling of the kind that a very regular hypergraph as the one of arithmetic progression should display a regular discrepancy behavior suggests that an investigation of the lower bound could give some improvement as well.

We would not be surprised if the correct value for the discrepancy in $c$ colors is $(cn)^{0.25}(\log c)^C$ for some constant $C$.

**References**


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