New Low Discrepancy Random Colorings*†
— Short Extended Abstract —

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Abstract

We treat the problem of 2–coloring a hypergraph in such a way that every hyperedge has about the same number of points in each color (discrepancy problem for hypergraphs). The only efficient approach for the general case involves a random coloring obtained by independently flipping a coin for each vertex to decide its color. We analyze a different random coloring. Our method yields better bounds for the general case, as it reduces the number and size of the hyperedges to be considered.

It also allows to prescribe that some subsets of the vertex set have to be colored perfectly. This is used successfully to exploit additional structural knowledge on the hypergraph.

1 Introduction

Combinatorial discrepancy is a measure of how well a hypergraph can be partitioned into two classes such that both classes contain roughly the same number of vertices in each hyperedge. Let \( \mathcal{H} = (X, \mathcal{E}) \) denote a finite hypergraph, i.e. \( X \) is a finite set and \( \mathcal{E} \) is a family of subsets of \( X \). Set \( n := |V| \) and \( m := |\mathcal{E}| \) for convenience. Let \( \chi : X \to \{-1, +1\} \) be a 2-coloring of \( X \).

For \( E \in \mathcal{E} \) define \( \chi(E) = \sum_{x \in E} \chi(x) \). Then the discrepancy of \( \mathcal{H} \) w.r.t. \( \chi \) is defined by

\[
\text{dis}_\chi(\mathcal{H}) = \max_{E \in \mathcal{E}} |\chi(E)|.
\]

Discrepancy is an \( NP \)-hard problem. It is even \( NP \)-hard to decide whether a zero discrepancy coloring exists or not. Efficient algorithms finding an optimal coloring therefore are not to be expected. Indeed, very little is known about the algorithmic aspect of discrepancy. For some restrictions of the problem nice solutions exist, e.g. for hypergraphs having degree at most \( t \) Beck and Fiala gave a polynomial time algorithm leading to a coloring having discrepancy less than \( 2t \).

For the general case, the best efficient result known to us is derived from “coin flipping”. For each vertex independently choose a color with equal probability \( \frac{1}{2} \). From the Chernoff inequality, we know \( P(\lambda < \text{dis}_\chi(\mathcal{H}) \leq \sqrt{2n \ln(4m)}) \leq 2e^{-\lambda^2 / 2} \) for any \( E \subseteq V \). In particular, we compute \( P(\text{dis}_\chi(\mathcal{H}) \leq \sqrt{2n \ln(4m)}) \leq \frac{1}{2} \). This naturally leads to a randomized algorithm. All this as well as a derandomization can be found in [Spe87].

This paper sketches a new idea that (a) yields an improved bound of \( \text{dis}_\chi(\mathcal{H}) \leq \sqrt{n \ln(4m)} \) with probability \( \frac{1}{2} \), (b) shows that with no extra cost we may assume some sets to be perfectly balanced, in particular, \( \chi \) to be an equicoloring, (c) allows to use structural information about the hypergraph in generating a random coloring.

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2 The Basic Idea

Suppose that we have the following: A partition \( \mathcal{P} = \{P_1, \ldots, P_r\} \) of the vertex set and colorings \( \chi_i : P_i \to \{-1, +1\} \) such that \( |\chi_i(E \cap P_i)| \leq 1 \) holds for all edges \( E \in \mathcal{E} \). We will shortly see that this is less artificial than it may seem at first. For each \( i \in [r] := \{1, \ldots, r\} \) "flip a coin", i.e. independently and uniformly choose a sign \( \varepsilon_i \in \{-1, +1\} \). Let \( \chi : X \to \{-1, +1\} \) denote the union of the \( \varepsilon_i \chi_i \). Set \( I := \{i \in [r] |\chi_i(E \cap P_i) \neq 0\} \), hence \( \chi_i(E \cap P_i) \in \{-1, +1\} \) for all \( i \in I \). The Chernoff bound (exactly in the same way as above) now yields \( P(|\chi(E)| > \lambda) < e^{-\frac{\lambda^2}{2 \tau}} \), that is, we replaced the cardinality of \( E \) by the possibly smaller number of \( P_i \) such that \( \chi_i(E \cap P_i) \neq 0 \).

Set \( E\mathcal{P} := \bigcup_{\chi_i(E \cap P_i) \neq 0} (E \cap P_i) \) for all \( E \in \mathcal{E} \). Depending on the partition \( \mathcal{P} \) and the colorings \( \chi_i \), the mapping \( E \mapsto E\mathcal{P} \) is not injective, that means we need to consider fewer edges.

3 Arbitrary Hypergraphs

The method described above can be applied to any hypergraph. Let us assume that \( n \) is even. Let \( \mathcal{P} \) be any matching on \( V \), that is, \( \mathcal{P} \) consist of \( \frac{n}{2} \) disjoint sets each holding 2 vertices. For each \( P \in \mathcal{P} \), \( \{v_1, v_2\} \in \mathcal{P} \) independently 'flip a coin' to decide a color for \( v_1 \) and the opposite color for \( v_2 \). Let \( \chi \) denote the resulting (random) coloring. If a hyperedge \( E \in \mathcal{E} \) contains a matching edge \( P \in \mathcal{P} \), we have \( \chi(E) = \chi(E \setminus P) \), since \( P \) contains one point in each color. Therefore \( |E\mathcal{P}| \leq \frac{n}{2} \), and we immediately derive the bound \( \text{disc}(\mathcal{H}) \leq \sqrt{n \ln(4m)} \) with probability \( \frac{1}{2} \).

Note that this works for any matching! In particular by choosing a suitable matching we can enforce several subsets of \( V \) to be perfectly balanced. All we need is to choose a matching which also induces a matching on these subsets. For example any disjoint family of subsets of \( V \) can be ensured a perfect coloring in addition to the discrepancy guarantee for the hyperedges.

Another aspect is that we might be able to choose a matching such that many matching edges are contained in hyperedges of \( \mathcal{H} \). This then reduces the relevant size \( |E\mathcal{P}| \) of the hyperedges and further reduces the discrepancy.

4 Exploiting the Structure

A major difficulty using the probabilistic method in discrepancy theory so far was to use structural information in the design of the random experiment. Here we show a way:

Let us consider the hypergraph of \( d \)-dimensional boxes in \([n]^d\), that is \( \mathcal{H} = ([n]^d, \{ S_1 \times \ldots \times S_d | S_i \subseteq [n]\}) \). This is (simply?) the \( d \)-fold cartesian product of the complete hypergraph \([n], 2^{[n]}\) on \( n \) points. The usual probabilistic argument yields a bound of \( \text{disc}(\mathcal{H}) = O(n \frac{\sqrt{d}}{\sqrt{n}} \sqrt{d}) \).

Set \( \mathcal{P} := \{ \prod_{i \in [d]} (2x_i - 1, 2x_i) | x_1, \ldots, x_d \in \mathbb{Z}\} \), that is, we split the \( n^d \)-cube into \( 2^d \)-cubes in a rather canonical way. The coloring corresponding to each such small cube shall be such that adjacent corners always receive the opposite color. More formally, a vertex is colored +1 if and only if an even number of its coordinates is even. Let \( \chi \) again be the random coloring obtained from independently taking these colorings or their inverse colorings. Simple facts about cartesian products now show that any box \( E \) containing more than one vertex in a small cube \( P \) already fulfills \( \chi(E \cap P) = 0 \). Hence \( \text{disc}(\mathcal{H}) = O(\frac{2^{d+1}}{\sqrt{d}}) \).

Furthermore, just \( 2^{-d} n^{d/2} \leq 1.74^{nd} \) different edges (instead of \( 2^{nd} \) ones) have to be considered. This yields another 11% improvement.

References