Knowledge Representation for the Semantic Web
Lecture 5: Description Logics IV

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slides based on Reasoning Web 2011 tutorial “Foundations of Description Logics and OWL” by S. Rudolph

Max Planck Institute for Informatics
D5: Databases and Information Systems group

WS 2017/18
Unit Outline

Satisfaction and Entailment

Other Reasoning Problems

Algorithmic Approaches to DL Reasoning
Satisfaction and Satisfiability
Satisfaction and Satisfiability of Knowledge Bases

Satisfaction of a KB by an interpretation

An interpretation $\mathcal{I}$ satisfies (or is a model of) a knowledge base $\mathcal{K} = \langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$, if $\mathcal{I}$ satisfies every axiom of $\mathcal{K}$, i.e., $\mathcal{I} \models \alpha$ for $\alpha \in \mathcal{K}$.
Satisfaction and Satisfiability of Knowledge Bases

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KB (un)satisfiability / (in)consistency

A KB $\mathcal{K}$ is satisfiable (also: consistent), if it has some model; otherwise it is unsatisfiable (also: inconsistent or contradictory).
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**KB (un)satisfiability / (in)consistency**

A KB $\mathcal{K}$ is satisfiable (also: consistent), if it has some model; otherwise it is unsatisfiable (also: inconsistent or contradictory).

- **unsatisfiability** of a KB hints at a design bug
- unsatisfiable axioms carry no information:
  
  $\alpha$ is unsatisfiable $\iff \neg \alpha$ is tautologic (if negation is applicable), i.e., $\mathcal{I} \models \neg \alpha$ for every interpretation $\mathcal{I}$.
Example: KB Satisfiability

\[ RBox \ R \]
\[
\text{owns} \sqsubseteq \text{caresFor} \\
"\text{If somebody owns something, s/he cares for it.}" \\
\]

\[ TBox \ T \]
\[
\text{Healthy} \sqsubseteq \neg \text{Dead} \\
"\text{Healthy beings are not dead.}" \\
\text{Cat} \sqsubseteq \text{Dead} \sqcap \text{Alive} \\
"\text{Every cat is dead or alive.}" \\
\text{HappyCatOwner} \sqsubseteq \exists \text{owns} \cdot \text{Cat} \sqcap \forall \text{caresFor} \cdot \text{Healthy} \\
"\text{A happy cat owner owns a cat and all beings he cares for are healthy.}" \\
\]

\[ ABox \ A \]
\[
\text{HappyCatOwner(\text{schroedinger})} \\
"\text{Schrödinger is a happy cat owner.}" \\
\]

Is \( \mathcal{K} = \langle R, T, A \rangle \) satisfiable?
### Example: KB Satisfiability

**RBox** $\mathcal{R}$

- `owns` $\sqsubseteq$ `caresFor`

"If somebody owns something, s/he cares for it."

**TBox** $\mathcal{T}$

- `Healthy` $\sqsubseteq$ $\neg$ `Dead`
  
  "Healthy beings are not dead."

- `Cat` $\sqsubseteq$ `Dead` $\sqcap$ `Alive`
  
  "Every cat is dead or alive."

- `HappyCatOwner` $\sqsubseteq$ $\exists$ `owns.Cat` $\sqcap$ $\forall$ `caresFor.Healthy`

  "A happy cat owner owns a cat and all beings he cares for are healthy."

**ABox** $\mathcal{A}$

- `HappyCatOwner(schroedinger)`

  "Schrödinger is a happy cat owner."

Is $\mathcal{K} = \langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$ satisfiable? **Yes!**
**Example: KB Satisfiability**

<table>
<thead>
<tr>
<th>TBox $\mathcal{T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deer $\sqsubseteq$ Mammal</td>
</tr>
<tr>
<td>”Deers are mammals.”</td>
</tr>
<tr>
<td>Mammal $\sqcap$ Flies $\sqsubseteq$ Bat</td>
</tr>
<tr>
<td>”Mammals, who fly are bats.”</td>
</tr>
<tr>
<td>Bat $\sqsubseteq$ $\forall$ worksFor.${\text{batman}}$</td>
</tr>
<tr>
<td>”Bats work only for Batman”</td>
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<table>
<thead>
<tr>
<th>ABox $\mathcal{A}$</th>
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</thead>
<tbody>
<tr>
<td>Deer $\sqcap$ $\exists$ hasNose.$\text{Red}(\text{rudolph})$</td>
</tr>
<tr>
<td>”Rudolph is a deer with a red nose.”</td>
</tr>
<tr>
<td>$\forall$ worksFor$^\neg$.($\neg$Deer $\sqcap$ Flies).($\text{santa}$)</td>
</tr>
<tr>
<td>”Only non-deers or fliers work for Santa.”</td>
</tr>
<tr>
<td>worksFor($\text{rudolph}, \text{santa}$)</td>
</tr>
<tr>
<td>”Rudolph works for Santa.”</td>
</tr>
<tr>
<td>santa $\not\approx$ batman</td>
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<td>“Santa is different from Batman.”</td>
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Is $\mathcal{K}$ satisfiable?
Example: KB Satisfiability

\textit{TBox} \mathcal{T}

\begin{align*}
\text{Deer} & \sqsubseteq \text{Mammal} \\
"\text{Deers are mammals.}" \\
\text{Mammal} \sqcap \text{Flies} & \sqsubseteq \text{Bat} \\
"\text{Mammals, who fly are bats.}" \\
\text{Bat} & \sqsubseteq \forall \text{worksFor}.\{\text{batman}\} \\
"\text{Bats work only for Batman}" \\
\end{align*}

\textit{ABox} \mathcal{A}

\begin{align*}
\text{Deer} \sqcap \exists \text{hasNose}.\text{Red}(\text{rudolph}) \\
"\text{Rudolph is a deer with a red nose.}" \\
\forall \text{worksFor}^-.(\neg \text{Deer} \sqcap \text{Flies})(\text{santa}) \\
"\text{Only non-deers or fliers work for Santa.}" \\
\text{worksFor}(\text{rudolph}, \text{santa}) \\
"\text{Rudolph works for Santa.}" \\
\text{santa} \not\approx \text{batman} \\
"\text{Santa is different from Batman.}" \\
\end{align*}

\textbf{Is } \mathcal{K} \text{ satisfiable? No!}
Entailment of Axioms

Entailment checking

A knowledge base $\mathcal{K}$ entails an axiom $\alpha$ (in symbols, $\mathcal{K} \models \alpha$), if every model $\mathcal{I}$ of $\mathcal{K}$ satisfies $\alpha$.

- Informally, $\mathcal{K} \models \alpha$ elicits implicit knowledge
- If $\alpha$ occurs in $\mathcal{K}$, then trivially $\mathcal{K} \models \alpha$
- If $\mathcal{K}$ is unsatisfiable, then $\mathcal{K} \models \alpha$ for every axiom $\alpha$
### Example: Entailment

**RBox** $\mathcal{R}$

<table>
<thead>
<tr>
<th>Relation</th>
<th>Domain</th>
<th>Rationale</th>
</tr>
</thead>
<tbody>
<tr>
<td>owns</td>
<td>caresFor</td>
<td>&quot;If somebody owns something, s/he cares for it.&quot;</td>
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**TBox** $\mathcal{T}$

<table>
<thead>
<tr>
<th>Concept</th>
<th>Subsumes</th>
<th>Rationale</th>
</tr>
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<tbody>
<tr>
<td>Healthy</td>
<td>¬Dead</td>
<td>&quot;Healthy beings are not dead.&quot;</td>
</tr>
<tr>
<td>Cat</td>
<td>Dead $\sqcup$ Alive</td>
<td>&quot;Every cat is dead or alive.&quot;</td>
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<td>HappyCatOwner</td>
<td>$\exists$ owns.Cat $\sqcap$ $\forall$ caresFor.Healthy</td>
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**ABox** $\mathcal{A}$

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<tr>
<th>Concept</th>
<th>Value</th>
<th>Rationale</th>
</tr>
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<tbody>
<tr>
<td>HappyCatOwner</td>
<td>schroedinger</td>
<td>&quot;Schrödinger is a happy cat owner.&quot;</td>
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### Example: Entailment

**RBox \( \mathcal{R} \)**

<table>
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<th>Relation</th>
<th>Expression</th>
<th>Meaning</th>
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<tr>
<td>\texttt{owns} &amp; \sqsubseteq &amp; \texttt{caresFor}</td>
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**TBox \( \mathcal{T} \)**

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<tr>
<td>\texttt{Healthy} &amp; \sqsubseteq &amp; \neg \texttt{Dead}</td>
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<td>&quot;Healthy beings are not dead.&quot;</td>
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<td></td>
</tr>
<tr>
<td>\texttt{Cat} &amp; \sqsubseteq &amp; \texttt{Dead} \sqcup \texttt{Alive}</td>
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<td></td>
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<td>\texttt{HappyCatOwner} &amp; \sqsubseteq &amp; \exists \texttt{owns} . \texttt{Cat} \sqcap \forall \texttt{caresFor} . \texttt{Healthy}</td>
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**ABox \( \mathcal{A} \)**

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- \( \mathcal{K} \models \exists \text{caresFor}. (\texttt{Cat} \sqcap \texttt{Alive})(\texttt{schroedinger})? \)
Example: Entailment

RBox \( \mathcal{R} \)
- \( \text{owns} \sqsubseteq \text{caresFor} \)
  "If somebody owns something, s/he cares for it."

TBox \( \mathcal{T} \)
- \( \text{Healthy} \sqsubseteq \neg \text{Dead} \)
  "Healthy beings are not dead."
- \( \text{Cat} \sqsubseteq \text{Dead} \sqcup \text{Alive} \)
  "Every cat is dead or alive."
- \( \text{HappyCatOwner} \sqsubseteq \exists \text{owns} . \text{Cat} \cap \forall \text{caresFor} . \text{Healthy} \)
  "A happy cat owner owns a cat and all beings he cares for are healthy."

ABox \( \mathcal{A} \)
- \( \text{HappyCatOwner}(\text{schroedinger}) \)
  "Schrödinger is a happy cat owner."

- \( \mathcal{K} \models \exists \text{caresFor} . (\text{Cat} \cap \text{Alive})(\text{schroedinger}) \)
- \( \mathcal{K} \models \forall \text{owns} . \neg \text{Cat} \sqsubseteq \neg \text{HappyCatOwner} ? \)
Example: Entailment

\( RBox \ \mathcal{R} \)

\begin{align*}
\text{owns} & \sqsubseteq \text{caresFor} \\
& \quad \text{"If somebody owns something, s/he cares for it."}
\end{align*}

\( TBox \ \mathcal{T} \)

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\begin{align*}
\text{HappyCatOwner}(\text{schroedinger}) \\
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- \( \mathcal{K} \models \exists \text{caresFor}. (\text{Cat} \sqcap \text{Alive})(\text{schroedinger}) \)
- \( \mathcal{K} \models \forall \text{owns}. \neg \text{Cat} \sqsubseteq \neg \text{HappyCatOwner} \)
- \( \mathcal{K} \models \text{Cat} \sqsubseteq \text{Healthy} \)?
### Example: Entailment

**RBox** $\mathcal{R}$

- `owns` ⊑ `caresFor`
  
  "If somebody owns something, s/he cares for it."

**TBox** $\mathcal{T}$

<table>
<thead>
<tr>
<th>Class</th>
<th>⊑</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>Healthy</code></td>
<td><code>¬Dead</code></td>
<td>&quot;Healthy beings are not dead.&quot;</td>
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<td><code>Cat</code></td>
<td><code>Dead</code> ⊔ <code>Alive</code></td>
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<tr>
<td><code>HappyCatOwner</code></td>
<td><code>∃owns.Cat</code> ⊓ <code>∀caresFor.Healthy</code></td>
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</tr>
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**ABox** $\mathcal{A}$

- `HappyCatOwner(schroedinger)`
  
  "Schrödinger is a happy cat owner."

- $\mathcal{K} \models \exists caresFor.(Cat \sqcap Alive)(schroedinger)$
- $\mathcal{K} \models \forall owns.¬Cat \sqsubseteq ¬HappyCatOwner$
- $\mathcal{K} \not\models Cat \sqsubseteq Healthy$
**Decidability of DLs**

**DLs are decidable**, i.e., there exists an algorithm that

**Given:** a **KB** and an axiom **α**,

**Output:** “yes” iff **KB** $\models \alpha$ and no otherwise.

- Likewise, there is a similar algorithm that decides whether an input **KB** is satisfiable
- Just ask **KB** $\models \top \sqsubseteq \bot$: if the answer is “yes”, then **KB** is unsatisfiable, otherwise it is satisfiable.
Standard Reasoning Problems
Standard DL Reasoning Problems

- **KB Satisfiability:** verify whether the KB is satisfiable

- **Entailment:** verify whether the KB entails a certain axiom
e.g., \( \mathcal{K} \models \text{CatOwner}(\text{Schroedinger}) \)

- **Concept Satisfiability:** verify whether a given concept is
  (un)satisfiable, e.g., \( \mathcal{K} \models \text{Dead} \sqcap \text{Alive} \sqsubseteq \bot \)

- **Coherence:** verify whether none of the concepts in the KB is
  unsatisfiable

- **Classification:** compute the subsumption hierarchy of all atomic
  concepts, e.g. \( \mathcal{K} \models \text{Healthy} \sqsubseteq \neg \text{Dead}, \text{etc.} \)

- **Instance Retrieval:** retrieve all the individuals known to be
  instances of a certain concept, e.g., find all \( a \), s.t. \( \exists \text{caresFor}(a) \)
Deciding KB Satisfiability

- deciding KB satisfiability is a basic inference task (the “mother” of all standard reasoning tasks)
- directly needed in the process of KB engineering
  - detect severe modeling errors
- other reasoning tasks can be reduced to checking KB (un)satisfiability (and vice versa)

**Theorem 1: Reducing reasoning problems to KB satisfiability**

Let $\mathcal{K}$ be a KB and $a$ an individual name not in $\mathcal{K}$. Then

2. $C$ is satisfiable w.r.t. $\mathcal{K}$ iff $\mathcal{K} \cup C(a)$ is satisfiable;
3. $\mathcal{K}$ is coherent iff, for each concept name $C$, $\mathcal{K} \cup C(a)$ is satisfiable;
4. $\mathcal{K} \models A \sqsubseteq B$ iff $\mathcal{K} \cup (A \sqcap \neg B)(a)$ is unsatisfiable;
5. $\mathcal{K} \models B(b)$ iff $\mathcal{K} \cup \neg B(b)$ is unsatisfiable.
Entailment Checking

- used in the KB modeling process to check, whether the specified knowledge has the intended consequences
- used for querying the KB if certain propositions are necessarily true

Reduction of entailment problem \( \mathcal{K} \models \alpha \) to checking KB inconsistency (see Th.1) follows the idea of proof by contradiction:

- negate the axiom \( \alpha \)
- add the negated axiom \( \neg \alpha \) to \( \mathcal{K} \)
- check for inconsistency of the resulting KB \( \mathcal{K} \cup \{\neg \alpha\} \)

If an axiom cannot be negated directly, its negation can be emulated (\( \{\neg \alpha\} \leadsto A_\alpha \)).
Entailment Checking, cont’d

Axiom sets $A_\alpha$ such that $\mathcal{K} \models \alpha$ exactly if $\mathcal{K} \cup A_\alpha$ is unsatisfiable:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$A_\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1 \circ \ldots \circ r_n \sqsubseteq r$</td>
<td>${\neg r(c_0, c_n), r_1(c_0, c_n), \ldots, r_n(c_{n-1}, c_n)}$</td>
</tr>
<tr>
<td>$\text{Dis}(r, r')$</td>
<td>${r(c_1, c_2), r'(c_1, c_2)}$</td>
</tr>
<tr>
<td>$C \sqsubseteq D$</td>
<td>${(C \sqcap \neg D)(c)}$ or $\top \sqsubseteq \exists u(C \sqcap \neg D)$</td>
</tr>
<tr>
<td>$C(a)$</td>
<td>${\neg C(a)}$</td>
</tr>
<tr>
<td>$\neg C(a)$</td>
<td>${C(a)}$</td>
</tr>
<tr>
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</tr>
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</tr>
<tr>
<td>$a \approx b$</td>
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</tr>
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- Individual names $c$ with possible subscripts are supposed to be fresh$^1$.
- For GCI (third line), the first variant is normally employed; the second is logical equivalent instead of just emulating.

---

$^1$Fresh individuals are those not appearing in the given KB $\mathcal{K}$. 


## Concept Satisfiability

A concept expression $C$ is called **satisfiable** with respect to a knowledge base $\mathcal{K}$, if there exists a model $\mathcal{I}$ of $\mathcal{K}$ such that $C^\mathcal{I} \neq \emptyset$.

- Unsatisfiable atomic concepts normally indicate KB modeling errors.
- Concept satisfiability can be reduced to KB consistency (Th. 1) and non-entailment resp.:

\[ C \text{ is satisfiable wrt. } \mathcal{K} \iff \mathcal{K} \cup \{C(a)\} \text{ is consistent, where } a \text{ is a fresh individual name} \]

\[ C \text{ is satisfiable wrt. } \mathcal{K} \iff \mathcal{K} \not\models C \sqsubseteq \bot \]
Concept Satisfiability, cont’d

Entailment of general concept inclusions $C \sqsubseteq D$ and equivalences $C \equiv D$ can be reduced to both concept (un)satisfiability and KB (un)satisfiability.

- $C \sqsubseteq D \iff C \sqcap \neg D$ is unsatisfiable
  $\iff$ KB $\{C(a), \neg D(a)\}$ is unsatisfiable

- $C \equiv D \iff (C \sqcap \neg D) \sqcup (D \sqcap \neg C)$ is unsatisfiable
  $\iff$ both $C \sqcap \neg D$ and $D \sqcap \neg C$ are unsatisfiable
  $\iff$ both KBs $\{C(a), \neg D(a)\}$ and $\{D(a), \neg C(a)\}$ are unsatisfiable
Classification

**KB Classification**

Classification of a knowledge base $\mathcal{K}$ is to determine for any two concept names $A, B$, whether $\mathcal{K} \models A \sqsubseteq B$ holds.

- This is useful at KB design time for checking the inferred concept hierarchy. Also, computing this hierarchy once and storing it can speed up further queries.

- Classification can be reduced to checking entailment of GCIs.

- While this requires quadratically many checks, one can often do much better in practice by applying optimizations and exploiting that subsumption is a preorder.
Instance Retrieval

Instance retrieval task is to find all named individuals that are known to be in a certain concept (role).

- \( \hat{\text{retrieve}}(C, \mathcal{K}) = \{ a \in N_I \mid a^I \in C^I \text{ for every model } \mathcal{I} \text{ of } \mathcal{K} \} \)
- \( \hat{\text{retrieve}}(r, \mathcal{K}) = \{(a, b) \in N_I^2 \mid (a^I, b^I) \in r^I \text{ for every model } \mathcal{I} \text{ of } \mathcal{K} \} \)

- It can be reduced to checking entailment of as many individual assertions as there are named individuals in \( \mathcal{K} \) i.e., test \( \mathcal{K} \models C(a) \) for each \( a \) occurring in \( \mathcal{K} \) (excluding pathologic cases).

- Depending on the system used and the inference algorithm, this can be done in a much more efficient way (e.g. by a transformation into a database query, in SQL or Datalog).
Novel Reasoning Problems

In recent years, other reasoning tasks have been gaining attention.
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- **Conjunctive Query Answering**
  
  conjunctive queries allow to join pieces of information
  more expressive: union of CQAs (akin to SQL), rules
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- **Inconsistency Handling**

  repair inconsistent KB or avoid that $\mathcal{K} \models \alpha$ for every $\alpha$ (‘knowledge explosion’) if $\mathcal{K}$ is inconsistent by introducing, e.g., new semantics
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  Conjunctive queries allow to join pieces of information more expressive: union of CQAs (akin to SQL), rules

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  Repair inconsistent KB or avoid that $K \models \alpha$ for every $\alpha$ (‘knowledge explosion’) if $K$ is inconsistent by introducing, e.g., new semantics

- **Entailment Explanation**
  
  Identify axioms in the knowledge base that support a conclusion $K \models \alpha$, typically a smallest $K' \subseteq K$ such that $K' \models \alpha$ (‘axiom pinpointing’)

Conjunctive Query Answering

Generalize Instance Retrieval by allowing joins and projections:

\[ q(Z) = \exists Y \exists X. \text{childOf}(Z,Y) \land \text{childOf}(Z,X) \land \text{marriedWith}(Y,X) \]

- In databases:
  - just one model (the DB itself) by *Closed World Assumption* (R. Reiter, 1978: if atom \( A \) is not provable from DB, \( \neg A \) is true).

- this is rather easy
Conjunctive Query Answering, cont’d

Generalize Instance Retrieval by allowing joins and projections:

\[ q(Z) = \exists Y \exists X. \text{childOf}(Z, Y) \land \text{childOf}(Z, X) \land \text{marriedWith}(Y, X) \]

- In Description Logics:
  - one knowledge base, many models (*Open World Assumption*)
  - not so easy
  - the \( \exists \)-variables must be suitably mapped in *every* model
Types of Reasoning Procedures

- Roughly, DL inference algorithms can be separated into two groups:
  - **model-based algorithms**: show satisfiability by constructing a model (or a representation of it).
    - **Examples**: tableau, automata, type elimination algorithms
  - **proof-based algorithms**: apply deduction rules to the KB to infer new axioms.
    - **Examples**: resolution, consequence-based algorithms

**Note:**
- both strategies are known from first-order logic theorem proving
- additional care is needed to ensure decidability (in particular completeness and termination of algorithms).
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Tableau Algorithm for DLs

Tableau-based techniques

They try to decide the satisfiability of a formula (or theory) by using rules to construct (a representation of) a model.

- Tableau-based techniques have been used in FOL and modal logics for many years.

- For DLs, they have been extensively explored since the late 1990s [Smolka, 1990], [Baader and Sattler, 2001].

- They are considered well-suited for implementation.

- In fact, many of the most successful DL reasoners implement tableau techniques or variations of them, e.g.: RACER, FaCT++, Pellet, Hermit, etc.
Tableau Algorithm for Deciding KB Satisfiability

- Perform a "bottom-up" construction of a model:
Tableau Algorithm for Deciding KB Satisfiability

- Perform a "bottom-up" construction of a model:
  - Initialize an interpretation by all explicitly known (i.e., named) individuals and their known properties.
Tableau Algorithm for Deciding KB Satisfiability

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  - Initialize an interpretation by all explicitly known (i.e., named) individuals and their known properties.
  - Most probably, this "model draft" will violate some of the axioms.
  - Iteratively "repair" it by adding new information about concept or role memberships and/or introducing new (i.e., anonymous) individuals, this may require case distinction and backtracking.
  - If we arrive at an interpretation satisfying all axioms, satisfiability has been shown.
  - If every repairing attempt eventually results in an overt inconsistency, unsatisfiability has been shown.

Note: as the finite model property does not hold in general, not a full model is constructed but a finite representation of it (cf. "blocking").
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**Note:** as the finite model property does not hold in general, not a full model is constructed but a *finite* representation of it (cf. “blocking”).
### Tableau Overview Example: Happy Cat Owner

**RBox** $R$

- $\text{owns} \sqsubseteq \text{caresFor}$
  
  "If somebody owns something, s/he cares for it."

**TBox** $T$

- $\text{Healthy} \sqsubseteq \neg \text{Dead}$
  
  "Healthy beings are not dead."

- $\text{Cat} \sqsubseteq \text{Dead} \sqcup \text{Alive}$
  
  "Every cat is dead or alive."

- $\text{HappyCatOwner} \sqsubseteq \exists \text{owns}. \text{Cat} \land \forall \text{caresFor}. \text{Healthy}$
  
  "A happy cat owner owns a cat and all beings he cares for are healthy."

**ABox** $A$

- $\text{HappyCatOwner} (\text{schroedinger})$
  
  "Schrödinger is a happy cat owner."

Is $\mathcal{K}$ satisfiable?
### Tableau Overview Example: Happy Cat Owner

<table>
<thead>
<tr>
<th>RBox ( \mathcal{R} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{owns} \sqsubseteq \text{caresFor} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TBox ( \mathcal{T} )</th>
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</thead>
<tbody>
<tr>
<td>( \text{Healthy} \sqsubseteq \neg \text{Dead} )</td>
</tr>
<tr>
<td>( \text{Cat} \sqsubseteq \text{Dead} \sqcup \text{Alive} )</td>
</tr>
<tr>
<td>( \text{HappyCatOwner} \sqsubseteq \exists \text{owns} \cdot \text{Cat} \sqcap \forall \text{caresFor} \cdot \text{Healthy} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ABox ( \mathcal{A} )</th>
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<tbody>
<tr>
<td>( \text{HappyCatOwner}(s) )</td>
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Is \( \mathcal{K} \) satisfiable?
# Tableau Overview Example: Happy Cat Owner

<table>
<thead>
<tr>
<th>RBox $\mathcal{R}$</th>
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</thead>
<tbody>
<tr>
<td><code>owns</code> ⊑ <code>caresFor</code></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TBox $\mathcal{T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>Healthy</code> ⊑ ¬Dead</td>
</tr>
<tr>
<td><code>Cat</code> ⊑ Dead ⊔ Alive</td>
</tr>
<tr>
<td><code>HappyCatOwner</code> ⊑ ∃owns.Cat ⊓ ∀caresFor.Healthy</td>
</tr>
</tbody>
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<th>ABox $\mathcal{A}$</th>
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<td><code>HappyCatOwner(s)</code></td>
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Is $\mathcal{K}$ satisfiable?

$s \bullet \quad \mathcal{L}(s) = \{\text{HappyCatOwner}\}$
Tableau Overview Example: Happy Cat Owner

\[ RBox \mathcal{R} \]
\[
\begin{align*}
\text{owns} & \sqsubseteq \text{caresFor} \\
\end{align*}
\]

\[ TBox \mathcal{T} \]
\[
\begin{align*}
\text{Healthy} & \sqsubseteq \lnot \text{Dead} \\
\text{Cat} & \sqsubseteq \text{Dead} \sqcup \text{Alive} \\
\text{HappyCatOwner} & \sqsubseteq \exists \text{owns}. \text{Cat} \sqcap \forall \text{caresFor}. \text{Healthy} \\
\end{align*}
\]

\[ ABox \mathcal{A} \]
\[
\begin{align*}
\text{HappyCatOwner}(s) \\
\end{align*}
\]

Is \( \mathcal{K} \) satisfiable?

\[
\begin{align*}
s & \bullet \\
\text{owns} & \\
\bullet & \text{c} \\
\end{align*}
\]

\[
\mathcal{L}(s) = \{ \text{HappyCatOwner} \}
\]

\[
\mathcal{L}(c) = \{ \text{Cat} \} 
\]
### Tableau Overview Example: Happy Cat Owner

**RBox R**

- \( \text{owns} \subseteq \text{caresFor} \)

**TBox T**

- \( \text{Healthy} \sqsubseteq \neg \text{Dead} \)
- \( \text{Cat} \sqsubseteq \text{Dead} \sqcap \text{Alive} \)
- \( \text{HappyCatOwner} \sqsubseteq \exists \text{owns} \cdot \text{Cat} \sqcap \forall \text{caresFor} \cdot \text{Healthy} \)

**ABox A**

- \( \text{HappyCatOwner}(s) \)

---

**Is \( \mathcal{K} \) satisfiable?**

\[
\begin{align*}
\mathcal{L}(s) &= \{ \text{HappyCatOwner} \} \\
\mathcal{L}(c) &= \{ \text{Cat} \}
\end{align*}
\]
**Tableau Overview Example: Happy Cat Owner**

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<tr>
<td>Cat $\sqsubseteq$ Dead $\sqcup$ Alive</td>
</tr>
<tr>
<td>HappyCatOwner $\sqsubseteq$ $\exists$owns.Cat $\sqcap\forall$caresFor.Healthy</td>
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<tbody>
<tr>
<td>HappyCatOwner($s$)</td>
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**Is $\mathcal{K}$ satisfiable?**

- $\mathcal{L}(s) = \{\text{HappyCatOwner}\}$
- $\mathcal{L}(c) = \{\text{Cat, Healthy}\}$
### Tableau Overview Example: Happy Cat Owner

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<td>$\text{HappyCatOwner}(s)$</td>
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Is $\mathcal{K}$ satisfiable?

\[ s \quad \quad \mathcal{L}(s) = \{ \text{HappyCatOwner} \} \]

\[ \text{owns} \quad \text{caresFor} \quad c \quad \mathcal{L}(c) = \{ \text{Cat}, \text{Healthy}, \neg \text{Dead} \} \]
Tableau Overview Example: Happy Cat Owner

**RBox** \( \mathcal{R} \)
- owns \( \sqsubseteq \) caresFor

**TBox** \( \mathcal{T} \)
- Healthy \( \sqsubseteq \neg \text{Dead} \)
- Cat \( \sqsubseteq \text{Dead} \sqcup \text{Alive} \)
- HappyCatOwner \( \sqsubseteq \exists \text{owns}. \text{Cat} \sqcap \forall \text{caresFor}. \text{Healthy} \)

**ABox** \( \mathcal{A} \)
- HappyCatOwner(s)

Is \( \mathcal{K} \) satisfiable?

\[ s \]
\[ \mathcal{L}(s) = \{ \text{HappyCatOwner} \} \]

\[ \text{owns} \quad \text{caresFor} \]
\[ c \]
\[ \mathcal{L}(c) = \{ \text{Cat}, \text{Healthy}, \neg \text{Dead}, \text{Alive} \} \]
### Tableau Overview Example: Happy Cat Owner

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Is $\mathcal{K}$ satisfiable? Yes!

Let $s$

\[ \mathcal{L}(s) = \{ \text{HappyCatOwner} \} \]

\[
\text{owns} \quad \text{caresFor}
\]

Let $c$

\[ \mathcal{L}(c) = \{ \text{Cat, Healthy, } \neg \text{Dead, Alive} \} \]
Naive Tableau Algorithm for $\mathcal{ALC}$

Given a KB in NNF we construct a tableau, which for $\mathcal{ALC}$ KBs consists of

- a set of nodes, labeled with individual names or variable names
- directed edges between some pairs of nodes
- for each node labeled $x$, a set $\mathcal{L}(x)$ of class expressions and
- for each pair of nodes $x$ and $y$, a set $\mathcal{L}(x, y)$ of role names.

Provisos:

- omit edges which are labeled with the empty set
- assume $\top$ is contained in $\mathcal{L}(x)$ for any $x$.
- concept expressions should be in negation normal form
### Recall \( ALC \) Syntax

<table>
<thead>
<tr>
<th>Construct</th>
<th>Syntax</th>
<th>Example</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>atomic concept</td>
<td>( A )</td>
<td>Doctor</td>
<td>( A^I \subseteq \Delta^I )</td>
</tr>
<tr>
<td>atomic role</td>
<td>( r )</td>
<td>hasChild</td>
<td>( r^I \subseteq \Delta^I \times \Delta^I )</td>
</tr>
<tr>
<td>conjunction</td>
<td>( C \cap D )</td>
<td>Human ( \cap ) Male</td>
<td>( C^I \cap D^I )</td>
</tr>
<tr>
<td>unqual. exist. res.(^2)</td>
<td>( \exists r )</td>
<td>( \exists ) hasChild</td>
<td>( { o \mid \exists o'. (o,o') \in r^I } )</td>
</tr>
<tr>
<td>value res.</td>
<td>( \forall r.C )</td>
<td>( \forall ) hasChild. Male</td>
<td>( { o \mid \forall o'. (o,o') \in r^I \rightarrow o' \in C^I } )</td>
</tr>
<tr>
<td>full negation</td>
<td>( \neg C )</td>
<td>( \neg \forall ) hasChild. Male</td>
<td>( \Delta^I \setminus C^I )</td>
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\(^2\)Unqualified existential restriction
Negation Normal Form

Negation normal form (NNF)

A concept expression $C$ is in negation normal form, if negation occurs in $C$ only in front of atomic concepts, nominal concepts and self-restrictions.
Negation Normal Form

Negation normal form (NNF)

A concept expression $C$ is in negation normal form, if negation occurs in $C$ only in front of atomic concepts, nominal concepts and self-restrictions.

Given a KB $K$ to construct $\text{nnf}(K)$ we need to

- replace every $C \equiv D$ by $C \sqsubseteq D$ and $D \sqsubseteq C$;
- replace every $C \sqsubseteq D$ by $\neg C \sqcup D$;
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Negation normal form (NNF)
A concept expression $C$ is in negation normal form, if negation occurs in $C$ only in front of atomic concepts, nominal concepts and self-restrictions.

Given a KB $K$ to construct $nnf(K)$ we need to
- replace every $C \equiv D$ by $C \subseteq D$ and $D \subseteq C$;
- replace every $C \subseteq D$ by $\neg C \sqcup D$;
- recursively translate every $C$ into $nnf(C)$:

$nnf(C) \leadsto C$ if $C \in \{A, \neg A, \{a_1 \ldots a_n\}, \neg\{a_1 \ldots a_n\}, \exists r.\text{Self}, \neg \exists r.\text{Self}, \top, \bot\}$

$nnf(\neg \neg C) \leadsto nnf(C)$
$nnf(C \sqcap D) \leadsto nnf(C) \sqcap nnf(D)$
$nnf(C \sqcup D) \leadsto nnf(C) \sqcup nnf(D)$
$nnf(\neg(C \sqcup D)) \leadsto nnf(\neg C \sqcup \neg D)$
$nnf(\neg(C \sqcap D)) \leadsto nnf(\neg C \sqcap \neg D)$
$nnf(\forall r. C) \leadsto \forall r. nnf(C)$
$nnf(\exists r. C) \leadsto \exists r. nnf(C)$

$nnf(\neg \forall r. C) \leadsto \exists r. nnf(\neg C)$
$nnf(\neg \exists r. C) \leadsto \forall r. nnf(\neg C)$
$nnf(\leq kr.C) \leadsto \leq kr.nnf(C)$
$nnf(\geq kr.C) \leadsto \geq kr.nnf(C)$
$nnf(\neg \leq kr.C) \leadsto \geq (k + 1) r.nnf(C)$
$nnf(\neg \geq kr.C) \leadsto \leq (k - 1) r.nnf(C)$
$nnf(\neg \top) \leadsto \bot$
Negation Normal Form

Negation normal form (NNF)

A concept expression $C$ is in negation normal form, if negation occurs in $C$ only in front of atomic concepts, nominal concepts and self-restrictions.

Given a KB $\mathcal{K}$ to construct $\text{nnf}(\mathcal{K})$ we need to

- replace every $C \equiv D$ by $C \sqsubseteq D$ and $D \sqsubseteq C$;
- replace every $C \sqsubseteq D$ by $\neg C \sqcup D$;
- recursively translate every $C$ into $\text{nnf}(C)$:

\[
\begin{align*}
\text{nnf}(C) & \rightsquigarrow C & \text{if } C \in \{ A, \neg A, \{a_1 \ldots a_n\}, \neg\{a_1 \ldots a_n\}, \exists r.\text{Self}, \neg\exists r.\text{Self}, \top, \bot \} \\
\text{nnf}(\neg\neg C) & \rightsquigarrow \text{nnf}(C) \\
\text{nnf}(C \cap D) & \rightsquigarrow \text{nnf}(C) \cap \text{nnf}(D) \\
\text{nnf}(C \sqcup D) & \rightsquigarrow \text{nnf}(C) \sqcup \text{nnf}(D) \\
\text{nnf}(\neg (C \sqcup D)) & \rightsquigarrow \text{nnf}(\neg C \sqcup \neg D) \\
\text{nnf}(\neg (C \cap D)) & \rightsquigarrow \text{nnf}(\neg C \cap \neg D) \\
\text{nnf}(\forall r.C) & \rightsquigarrow \forall r.\text{nnf}(C) \\
\text{nnf}(\exists r.C) & \rightsquigarrow \exists r.\text{nnf}(C) \\
\text{nnf}(\neg \forall r.C) & \rightsquigarrow \exists r.\text{nnf}(\neg C) \\
\text{nnf}(\neg \exists r.C) & \rightsquigarrow \forall r.\text{nnf}(\neg C) \\
\text{nnf}(\neg \leq k r.C) & \rightsquigarrow \leq (k + 1) r.\text{nnf}(C) \\
\text{nnf}(\neg \geq k r.C) & \rightsquigarrow \geq (k - 1) r.\text{nnf}(C) \\
\text{nnf}(\neg \sqsubseteq kr.C) & \rightsquigarrow \geq (k - 1) \text{nnf}(C) \\
\text{nnf}(\neg \sqsupseteq kr.C) & \rightsquigarrow \leq (k + 1) \text{nnf}(C) \\
\text{nnf}(\neg \top) & \rightsquigarrow \bot
\end{align*}
\]

$C$ and $\text{nnf}(C)$ are logically equivalent, i.e., $\mathcal{I}(C) = \mathcal{I}(\text{nnf}(C))$, for every interpretation $\mathcal{I}$, and the translation process terminates in linear time.
Example: Negation Normal Form

\[
\text{FilmActor} \subseteq (\exists \text{actedIn} \cap \text{Artist}) \cup \neg (\neg \exists \text{actedIn} \cup \exists \text{playsIn. Theater})
\]
Example: Negation Normal Form

Negation Normal Form

\[ \text{FilmActor} \subseteq (\exists \text{actedIn} \cap \text{Artist}) \sqcup \neg (\neg \exists \text{actedIn} \sqcup \exists \text{playsIn. Theater}) \]

1. \[ \neg \text{FilmActor} \sqcup (\exists \text{actedIn} \cap \text{Artist}) \sqcup \neg (\neg \exists \text{actedIn} \sqcup \exists \text{playsIn. Theater}) \]
Example: Negation Normal Form

**Negation Normal Form**

\[ \text{FilmActor} \subseteq (\exists \text{actedIn} \land \text{Artist}) \lor (\neg \exists \text{actedIn} \lor \exists \text{playsIn. Theater}) \]

1. \( \neg \text{FilmActor} \lor (\exists \text{actedIn} \land \text{Artist}) \lor (\neg \exists \text{actedIn} \lor \exists \text{playsIn. Theater}) \)

2. \( \neg \text{FilmActor} \lor (\exists \text{actedIn} \land \text{Artist}) \lor (\exists \text{actedIn} \land \neg \exists \text{playsIn. Theater}) \)
### Example: Negation Normal Form

Negation Normal Form

\[ \text{FilmActor} \subseteq (\exists \text{actedIn} \land \exists \text{Artist}) \lor \neg(\neg \exists \text{actedIn} \lor \exists \text{playsIn. Theater}) \]

1. \( \neg \text{FilmActor} \lor (\exists \text{actedIn} \land \exists \text{Artist}) \lor \neg(\neg \exists \text{actedIn} \lor \exists \text{playsIn. Theater}) \)

2. \( \neg \text{FilmActor} \lor (\exists \text{actedIn} \land \exists \text{Artist}) \lor (\exists \text{actedIn} \land \neg \exists \text{playsIn. Theater}) \)

3. \( \neg \text{FilmActor} \lor (\exists \text{actedIn} \land \exists \text{Artist}) \lor (\exists \text{actedIn} \land \forall \text{playsIn. Theater}) \)
Initial Tableau

For an \(\mathcal{ALC}\) knowledge base \(\mathcal{K}\) in negation normal form, the initial tableau is defined as follows:

1. For each individual \(a\) occurring in \(\mathcal{K}\), create a node labeled \(a\) and set \(L(a) = \emptyset\).

2. For all pairs \(a, b\) of individuals, set \(L(a, b) = \emptyset\).

3. For each ABox statement \(C(a)\) in \(\mathcal{K}\), set \(L(a) \leftarrow C\).

4. For each ABox statement \(r(a, b)\) in \(\mathcal{K}\) set \(L(a, b) \leftarrow r\).

Note: “\(\leftarrow\)” means “update with”, “add to”
Example: Initial Tableau

\[ \mathcal{K} = \{ A(a), \ (\exists r. B)(a), \ r(a, b), \ r(a, c), \ s(b, b), \ (A \sqcup B)(c), \ \neg A \sqcup (\forall s. B) \} \]
Example: Initial Tableau

\[ \mathcal{K} = \{ A(a), (\exists r. B)(a), r(a, b), r(a, c), s(b, b), (A \sqcup B)(c), \neg A \sqcup (\forall s. B) \} \]

\[ \mathcal{L}(a) = \{ A, \} \]

\[ b \bullet \]

\[ a \bullet \]

\[ c \bullet \]
Example: Initial Tableau

\[ \mathcal{K} = \{ A(a), (\exists r. B)(a), r(a, b), r(a, c), s(b, b), (A \sqcup B)(c), \neg A \sqcup (\forall s. B) \} \]

\[ \mathcal{L}(a) = \{ A, \exists r. B \} \]
Example: Initial Tableau

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**Example: Initial Tableau**

\[ \mathcal{K} = \{ A(a), \; (\exists r. B)(a), \; r(a, b), \; r(a, c), \; s(b, b), \; (A \sqcup B)(c), \; \neg A \sqcup (\forall s. B) \} \]

**Diagram:**

- **\( L(b) = \emptyset \)**
- **\( L(a) = \{ A, \exists r. B \} \)**
- **\( L(c) = \{ A \sqcup B \} \)**
Example: Initial Tableau

\[ \mathcal{K} = \{ A(a), (\exists r . B)(a), r(a,b), r(a,c), s(b,b), (A \sqcup B)(c), \neg A \sqcup (\forall s . B) \} \]

\[ \mathcal{L}(b) = \emptyset \]

\[ \mathcal{L}(a) = \{ A, \exists r . B \} \]

\[ \mathcal{L}(c) = \{ A \sqcup B \} \]
Expansion Rules for the Naive Tableau

- **⊓-rule:** If \( C \cap D \in \mathcal{L}(x) \) and \( \{C, D\} \not\subseteq \mathcal{L}(x) \), then set \( \mathcal{L}(x) \leftarrow \{C, D\} \).
Expansion Rules for the Naive Tableau

- **⊓-rule:** If $C \cap D \in \mathcal{L}(x)$ and $\{C, D\} \not\subseteq \mathcal{L}(x)$, then set $\mathcal{L}(x) \leftarrow \{C, D\}$.

- **⊔-rule:** If $C \sqcup D \in \mathcal{L}(x)$ and $\{C, D\} \cap \mathcal{L}(x) = \emptyset$, then set $\mathcal{L}(x) \leftarrow C$ or $\mathcal{L}(x) \leftarrow D$. 

- **∃-rule:** If $\exists r. C \in \mathcal{L}(x)$ and there is no $y$ with $r \in \mathcal{L}(x, y)$ and $C \in \mathcal{L}(y)$, then
  1. add a new node with label $y$ (where $y$ is a new node label),
  2. set $\mathcal{L}(x, y) \leftarrow \{r\}$, and
  3. set $\mathcal{L}(y) \leftarrow \{C\}$.

- **∀-rule:** If $\forall r. C \in \mathcal{L}(x)$ and there is a node $y$ with $r \in \mathcal{L}(x, y)$ and $C \not\in \mathcal{L}(y)$, then set $\mathcal{L}(y) \leftarrow C$.

- **TBox-rule:** If $C$ is a (rewritten and normalized) TBox statement and $C \not\in \mathcal{L}(x)$, then set $\mathcal{L}(x) \leftarrow C$. 

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Expansion Rules for the Naive Tableau

- **¬-rule**: If \( C \sqcap D \in \mathcal{L}(x) \) and \( \{C, D\} \not\subseteq \mathcal{L}(x) \), then set \( \mathcal{L}(x) \leftarrow \{C, D\} \).

- **∪-rule**: If \( C \sqcup D \in \mathcal{L}(x) \) and \( \{C, D\} \cap \mathcal{L}(x) = \emptyset \), then set \( \mathcal{L}(x) \leftarrow C \) or \( \mathcal{L}(x) \leftarrow D \).

- **∃-rule**: If \( \exists r.C \in \mathcal{L}(x) \) and there is no \( y \) with \( r \in \mathcal{L}(x, y) \) and \( C \in \mathcal{L}(y) \), then
  1. add a new node with label \( y \) (where \( y \) is a new node label),
  2. set \( \mathcal{L}(x, y) = \{r\} \), and
  3. set \( \mathcal{L}(y) = \{C\} \).

- **∀-rule**: If \( \forall r.C \in \mathcal{L}(x) \) and there is a node \( y \) with \( r \in \mathcal{L}(x, y) \) and \( C \not\in \mathcal{L}(y) \), then set \( \mathcal{L}(y) \leftarrow C \).

- **TBox-rule**: If \( C \) is a (rewritten and normalized) TBox statement and \( C \not\in \mathcal{L}(x) \), then set \( \mathcal{L}(x) \leftarrow C \).
Expansion Rules for the Naive Tableau

- $\sqcap$-rule: If $C \sqcap D \in \mathcal{L}(x)$ and $\{C, D\} \not\subseteq \mathcal{L}(x)$, then set $\mathcal{L}(x) \leftarrow \{C, D\}$.

- $\sqcup$-rule: If $C \sqcup D \in \mathcal{L}(x)$ and $\{C, D\} \cap \mathcal{L}(x) = \emptyset$, then set $\mathcal{L}(x) \leftarrow C$ or $\mathcal{L}(x) \leftarrow D$.

- $\exists$-rule: If $\exists r.C \in \mathcal{L}(x)$ and there is no $y$ with $r \in L(x, y)$ and $C \in \mathcal{L}(y)$, then
  1. add a new node with label $y$ (where $y$ is a new node label),
  2. set $\mathcal{L}(x, y) = \{r\}$, and
  3. set $\mathcal{L}(y) = \{C\}$.

- $\forall$-rule: If $\forall r.C \in \mathcal{L}(x)$ and there is a node $y$ with $r \in \mathcal{L}(x, y)$ and $C \not\in \mathcal{L}(y)$, then set $\mathcal{L}(y) \leftarrow C$. 
Expansion Rules for the Naive Tableau

- **⊓-rule:** If \( C \sqcap D \in \mathcal{L}(x) \) and \( \{C, D\} \not\subseteq \mathcal{L}(x) \), then set \( \mathcal{L}(x) \leftarrow \{C, D\} \).

- **⊔-rule:** If \( C \sqcup D \in \mathcal{L}(x) \) and \( \{C, D\} \cap \mathcal{L}(x) = \emptyset \), then set \( \mathcal{L}(x) \leftarrow C \) or \( \mathcal{L}(x) \leftarrow D \).

- **∃-rule:** If \( \exists r. C \in \mathcal{L}(x) \) and there is no \( y \) with \( r \in L(x, y) \) and \( C \in \mathcal{L}(y) \), then
  1. add a new node with label \( y \) (where \( y \) is a new node label),
  2. set \( \mathcal{L}(x, y) = \{r\} \), and
  3. set \( \mathcal{L}(y) = \{C\} \).

- **∀-rule:** If \( \forall r. C \in \mathcal{L}(x) \) and there is a node \( y \) with \( r \in L(x, y) \) and \( C \not\in \mathcal{L}(y) \), then set \( \mathcal{L}(y) \leftarrow C \).

- **TBox-rule:** If \( C \) is a (rewritten and normalized) TBox statement and \( C \not\in \mathcal{L}(x) \), then set \( \mathcal{L}(x) \leftarrow C \).
Naive Tableau Algorithm

**Input:** $\mathcal{ALC}$ knowledge base $\mathcal{K}$ in negation normal form

**Output:** “yes” if $\mathcal{K}$ is satisfiable

1. Initialize the tableau;
2. While some expansion rule is applicable:
   2.1. nondeterministically apply an applicable rule;
   2.2. if for some node $x$, there exists some $C \in \mathcal{L}(x)$ such that $\neg C \in \mathcal{L}(x)$, output “no” and terminate;
3. Output “yes”.

A nondeterministic run of the algorithm terminates, if either

- for some node $x$, $\mathcal{L}(x)$ contains a contradiction (“no”; attempt to find a model for $\mathcal{K}$ was unsuccessful), or
- no expansion rule is applicable (“yes”; attempt was successful)
- $\mathcal{K}$ is satisfiable if some run outputs “yes”, and unsatisfiable if every run outputs “no”
- only the $\sqcup$-rule creates true branching regarding yes/no output
Example: Expansion Rules

\[ \mathcal{K} = \{ C(a), \neg C \sqcup \exists r. D, \neg D \sqcup E, \forall r. \neg E(a) \} \]
Example: Expansion Rules

\( \mathcal{K} = \{ C(a), \neg C \sqcup \exists r. D, \neg D \sqcup E, \forall r. \neg E(a) \} \)

\[ a \bullet \quad \mathcal{L}(a) = \{ C, \} \]
Example: Expansion Rules

\[ \mathcal{K} = \{ C(a), \neg C \sqcup \exists r. D, \neg D \sqcup E, \forall r. \neg E(a) \} \]

\[ a \bullet \quad \mathcal{L}(a) = \{ C, \forall r. \neg E, \} \]
Example: Expansion Rules

\[ \mathcal{K} = \{ C(a), \neg C \sqcup \exists r.D, \neg D \sqcup E, \forall r.\neg E(a) \} \]

\[ a \bullet \quad \mathcal{L}(a) = \{ C, \forall r.\neg E, \neg C \sqcup \exists r.D, \} \]
Example: Expansion Rules

\[ \mathcal{K} = \{ C(a), \neg C \sqcup \exists r. D, \neg D \sqcup E, \forall r. \neg E(a) \} \]

\[ a \bullet \quad \mathcal{L}(a) = \{ C, \forall r. \neg E, \neg C \sqcup \exists r. D, \} \]
**Example: Expansion Rules**

\[ \mathcal{K} = \{C(a), \neg C \sqcup \exists r. D, \neg D \sqcup E, \forall r. \neg E(a)\} \]

\[ \mathcal{L}(a) = \{C, \forall r. \neg E, \neg C \sqcup \exists r. D, \exists r. D\} \]

\[ \mathcal{L}(x) = \{D, \} \]
Example: Expansion Rules

\[ \mathcal{K} = \{ C(a), \neg C \sqcup \exists r.D, \neg D \sqcup E, \forall r.\neg E(a) \} \]

\[ \mathcal{L}(a) = \{ C, \forall r.\neg E, \neg C \sqcup \exists r.D, \exists r.D \} \]

\[ \mathcal{L}(x) = \{ D, \neg D \sqcup E, \} \]
Example: Expansion Rules

\[ \mathcal{K} = \{ C(a), \neg C \sqcup \exists r. D, \neg D \sqcup E, \forall r. \neg E(a) \} \]

\[ \mathcal{L}(a) = \{ C, \forall r. \neg E, \neg C \sqcup \exists r. D, \exists r. D \} \]

\[ \mathcal{L}(x) = \{ D, \neg D \sqcup E, \} \]
Example: Expansion Rules

\[ \mathcal{K} = \{C(a), \neg C \sqcup \exists r. D, \neg D \sqcup E, \forall r. \neg E(a)\} \]

\[ \mathcal{L}(a) = \{C, \forall r. \neg E, \neg C \sqcup \exists r. D, \exists r. D\} \]

\[ \mathcal{L}(x) = \{D, \neg D \sqcup E, E\} \]
Example: Expansion Rules

\[ \mathcal{K} = \{ C(a), \neg C \sqcup \exists r. D, \neg D \sqcup E, \forall r. \neg E(a) \} \]

\[ \mathcal{L}(a) = \{ C, \forall r. \neg E, \neg C \sqcup \exists r. D, \exists r. D \} \]

\[ \mathcal{L}(x) = \{ D, \neg D \sqcup E, E, \neg E \} \]
Example: Expansion Rules

\[ \mathcal{K} = \{ C(a), \neg C \sqcup \exists r.D, \neg D \sqcup E, \forall r.\neg E(a) \} \]

\[ \mathcal{L}(a) = \{ C, \forall r.\neg E, \neg C \sqcup \exists r.D, \exists r.D \} \]

\[ \mathcal{L}(x) = \{ D, \neg D \sqcup E, E, \neg E \} \]

Clash is obtained, KB is unsatisfiable!
Tableau Algorithm with Blocking for $\mathcal{ALC}$

- Naive tableau algorithm does not always terminate
  
  Example: $\mathcal{K} = \{\neg Person \sqcup \exists \text{hasParent}, \ Person(a_1)\}$.

- Modify the naive tableau algorithm to ensure termination

- Use blocking

- A node with label $x$ is **directly blocked** by a node with label $y$ if
  - $x$ is a variable (i.e., not an individual),
  - $y$ is an ancestor of $x$, and
  - $\mathcal{L}(x) \subseteq \mathcal{L}(y)$

- A node with label $x$ is **blocked**, if it is directly blocked or one of its ancestors is blocked

- Expansion rules may only be applied if $x$ is not blocked
Example: Blocking

\[ \mathcal{K} = \{ B(t), \lnot H \cup \exists P.H, \ H(t) \} \]
Example: Blocking

\[ \mathcal{K} = \{B(t), \neg H \sqcup \exists P.H, H(t)\} \]

\[ t \bullet \quad \mathcal{L}(t) = \{B, H, \} \]
Example: Blocking

\[ \mathcal{K} = \{ B(t), \neg H \sqcup \exists P.H, H(t) \} \]

\[ t \bullet \quad \mathcal{L}(t) = \{ B, H, \neg H \sqcup \exists P.H, \} \]
Example: Blocking

\[ \mathcal{K} = \{ B(t), \neg H \sqcup \exists P.H, H(t) \} \]

\[ t \bullet \quad \mathcal{L}(t) = \{ B, H, \neg H \sqcup \exists P.H, \} \]
Example: Blocking

\[ \mathcal{K} = \{B(t), \neg H \sqcup \exists P.H, H(t)\} \]

\[ t \quad \mathcal{L}(t) = \{B, H, \neg H \sqcup \exists P.H, \exists P.H\} \]

\[ x \quad \mathcal{L}(x) = \{H\} \]
Example: Blocking

\[ \mathcal{K} = \{ B(t), \neg H \sqcup \exists P.H, H(t) \} \]

\[ \mathcal{L}(t) = \{ B, H, \neg H \sqcup \exists P.H, \exists P.H \} \]

\[ \mathcal{L}(x) = \{ H \} \subseteq \mathcal{L}(t) \]
Computational Properties

• The simple tableau algorithm is expensive in general (worst case):
  • the worst case complexity is double exponential
  • testing KB consistency in $\mathcal{ALC}$ is EXPTIME-complete
  • testing concept satisfiability in $\mathcal{ALC}$ is PSPACE-complete

• Still in practice, (optimized) tableaux algorithms work well
  • other notions of blocking might be used, e.g. “cross-path blocking”
    (blocking node need not be an ancestor)
  • single exponential time tableaux algorithms are available

• For many other description logics, also tableaux algorithms exist
  (e.g. $\mathcal{SHIQ}$, $\mathcal{SHOIQ}$, ...)
  Some algorithms are quite involved!
Summary

1. Satisfaction and Entailment
   - Notions
   - Decidability

2. Reasoning Problems
   - Knowledge base consistency
   - Entailment checking
   - Concept satisfiability
   - Classification
   - Instance retrieval

3. Algorithmic Approaches to DL Reasoning
   - Types of reasoning procedures
   - Tableaux

4. Novel reasoning problems
   - Conjunctive query answering
Franz Baader, Diego Calvanese, Deborah McGuinness, Daniele Nardi, and Peter Patel-Schneider, editors.

*The Description Logic Handbook: Theory, Implementation and Applications.*

Pascal Hitzler, Markus Krötzsch, and Sebastian Rudolph.

*Foundations of Semantic Web Technologies.*

Sebastian Rudolph.

Foundations of description logics.