Knowledge Representation for the Semantic Web
Lecture 5: Description Logics IV

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slides based on Reasoning Web 2011 tutorial “Foundations of Description Logics and OWL” by S. Rudolph

Max Planck Institute for Informatics
D5: Databases and Information Systems group

WS 2017/18
Unit Outline

Satisfaction and Entailment

Other Reasoning Problems

Algorithmic Approaches to DL Reasoning
Satisfaction and Satisfiability
Satisfaction and Satisfiability of Knowledge Bases

Satisfaction of a KB by an interpretation

An interpretation $\mathcal{I}$ satisfies (or is a model of) a knowledge base $\mathcal{K} = \langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$, if $\mathcal{I}$ satisfies every axiom of $\mathcal{K}$, i.e., $\mathcal{I} \models \alpha$ for $\alpha \in \mathcal{K}$. 
### Satisfaction and Satisfiability of Knowledge Bases

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#### KB (un)satisfiability / (in)consistency

A KB $\mathcal{K}$ is satisfiable (also: consistent), if it has some model; otherwise it is unsatisfiable (also: inconsistent or contradictory).
Satisfaction and Satisfiability of Knowledge Bases

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KB (un)satisfiability / (in)consistency

A KB $\mathcal{K}$ is satisfiable (also: consistent), if it has some model; otherwise it is unsatisfiable (also: inconsistent or contradictory).

- unsatisfiability of a KB hints at a design bug
- unsatisfiable axioms carry no information:

  $\alpha$ is unsatisfiable $\iff$ $\neg \alpha$ is tautologic (if negation is applicable), i.e., $\mathcal{I} \models \neg \alpha$ for every interpretation $\mathcal{I}$
Example: KB Satisfiability

<table>
<thead>
<tr>
<th>RBox R</th>
</tr>
</thead>
<tbody>
<tr>
<td>owns ⊑ caresFor</td>
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<td>”If somebody owns something, s/he cares for it.”</td>
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<tr>
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</tr>
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<table>
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<th>ABox A</th>
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<tr>
<td>HappyCatOwner(schroedinger)</td>
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<tr>
<td>”Schrödinger is a happy cat owner.”</td>
</tr>
</tbody>
</table>

Is $\mathcal{K} = \langle R, T, A \rangle$ satisfiable?
### Example: KB Satisfiability

**RBox** $\mathcal{R}$
- $\text{owns} \sqsubseteq \text{caresFor}$
  - "If somebody owns something, s/he cares for it."

**TBox** $\mathcal{T}$
- $\text{Healthy} \sqsubseteq \neg \text{Dead}$
  - "Healthy beings are not dead."
- $\text{Cat} \sqsubseteq \text{Dead} \sqcup \text{Alive}$
  - "Every cat is dead or alive."
- $\text{HappyCatOwner} \sqsubseteq \exists \text{owns}.\text{Cat} \sqcap \forall \text{caresFor}.\text{Healthy}$
  - "A happy cat owner owns a cat and all beings he cares for are healthy."

**ABox** $\mathcal{A}$
- $\text{HappyCatOwner}(\text{schroedinger})$
  - "Schrödinger is a happy cat owner."

Is $\mathcal{K} = \langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$ satisfiable? **Yes!**
Example: KB Satisfiability

\[ TBox \ T \]
\[
\begin{align*}
\text{Deer} & \sqsubseteq \text{Mammal} \\
"\text{Deers are mammals.}" \\
\text{Mammal} \sqcap \text{Flies} & \sqsubseteq \text{Bat} \\
"\text{Mammals, who fly are bats.}" \\
\text{Bat} & \sqsubseteq \forall \text{worksFor}. \{\text{batman}\} \\
"\text{Bats work only for Batman}" \\
\end{align*}
\]

\[ ABox \ A \]
\[
\begin{align*}
\text{Deer} \sqcap \exists \text{hasNose}. \text{Red}(\text{rudolph}) \\
"\text{Rudolph is a deer with a red nose.}" \\
\forall \text{worksFor}^-. (\neg \text{Deer} \sqcap \text{Flies})(\text{santa}) \\
"\text{Only non-deers or fliers work for Santa.}" \\
\text{worksFor}(\text{rudolph}, \text{santa}) \\
"\text{Rudolph works for Santa.}" \\
\text{santa} \not\approx \text{batman} \\
"\text{Santa is different from Batman.}" \\
\end{align*}
\]

Is \( \mathcal{K} \) satisfiable?
**Example: KB Satisfiability**

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<td>Deer ⊑ Mammal</td>
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| "Deers are mammals."
| Mammal ∩ Flies ⊑ Bat |
| "Mammals, who fly are bats."
| Bat ⊑ ∀ worksFor. {batman} |
| "Bats work only for Batman" |

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<td>Deer ∩ ∃ hasNose. Red (rudolph)</td>
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| "Rudolph is a deer with a red nose."
| ∀ worksFor. ¬ (Deer ∩ Flies) (santa) |
| "Only non-deers or fliers work for Santa." |
| worksFor (rudolph, santa) |
| "Rudolph works for Santa." |
| santa ≠ batman |
| "Santa is different from Batman." |

Is $\mathcal{K}$ satisfiable? No!
Entailment of Axioms

**Entailment checking**

A knowledge base $\mathcal{K}$ entails an axiom $\alpha$ (in symbols, $\mathcal{K} \models \alpha$), if every model $\mathcal{I}$ of $\mathcal{K}$ satisfies $\alpha$.

- Informally, $\mathcal{K} \models \alpha$ elicits implicit knowledge
- If $\alpha$ occurs in $\mathcal{K}$, then trivially $\mathcal{K} \models \alpha$
- If $\mathcal{K}$ is unsatisfiable, then $\mathcal{K} \models \alpha$ for every axiom $\alpha$
Example: Entailment

**RBox \( R \)**

- \( \text{owns} \sqsubseteq \text{caresFor} \)
  
  "If somebody owns something, s/he cares for it."

**TBox \( T \)**

- \( \text{Healthy} \sqsubseteq \neg \text{Dead} \)
  
  "Healthy beings are not dead."

- \( \text{Cat} \sqsubseteq \text{Dead} \sqcup \text{Alive} \)
  
  "Every cat is dead or alive."

- \( \text{HappyCatOwner} \sqsubseteq \exists \text{owns} . \text{Cat} \sqcap \forall \text{caresFor. Healthy} \)
  
  "A happy cat owner owns a cat and all beings he cares for are healthy."

**ABox \( A \)**

- \( \text{HappyCatOwner}(\text{schroedinger}) \)

  "Schrödinger is a happy cat owner."
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**RBox** $\mathcal{R}$
- `owns □ caresFor`
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- `Cat ⊑ Dead ⊔ Alive`
- "Every cat is dead or alive."
- `HappyCatOwner ⊑ ∃owns.Cat ⊓ ∀caresFor.Healthy`
- "A happy cat owner owns a cat and all beings he cares for are healthy."

**ABox** $\mathcal{A}$
- `HappyCatOwner(schroedinger)`
- "Schrödinger is a happy cat owner."

\[ \mathcal{K} \models \exists caresFor. (Cat \sqcap Alive)(schroedinger) ? \]
### Example: Entailment

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- \( \mathcal{K} \models \exists \text{caresFor}.(\text{Cat} \sqcap \text{Alive})(\text{schroedinger}) \)
- \( \mathcal{K} \models \forall \text{owns}.\neg \text{Cat} \sqsubseteq \neg \text{HappyCatOwner} ? \)
### Example: Entailment

#### RBox $\mathcal{R}$

```latex
\begin{align*}
\text{owns} & \sqsubseteq \text{caresFor} \\
\text{"If somebody owns something, s/he cares for it."}
\end{align*}
```

#### TBox $\mathcal{T}$

```latex
\begin{align*}
\text{Healthy} & \sqsubseteq \neg \text{Dead} \\
\text{"Healthy beings are not dead."} \\
\text{Cat} & \sqsubseteq \text{Dead} \sqcup \text{Alive} \\
\text{"Every cat is dead or alive."} \\
\text{HappyCatOwner} & \sqsubseteq \exists \text{owns}. \text{Cat} \sqcap \forall \text{caresFor}. \text{Healthy} \\
\text{"A happy cat owner owns a cat and all beings he cares for are healthy."}
\end{align*}
```

#### ABox $\mathcal{A}$

```latex
\begin{align*}
\text{HappyCatOwner}(\text{schroedinger}) \\
\text{"Schrödinger is a happy cat owner."}
\end{align*}
```

- $\mathcal{K} \models \exists \text{caresFor}. (\text{Cat} \sqcap \text{Alive})(\text{schroedinger})$
- $\mathcal{K} \models \forall \text{owns}. \neg \text{Cat} \sqsubseteq \neg \text{HappyCatOwner}$
- $\mathcal{K} \models \text{Cat} \sqsubseteq \text{Healthy}$
Example: Entailment

**RBox R**

| owns ⊑ caresFor |
| "If somebody owns something, s/he cares for it." |

**TBox T**

| Healthy ⊑ ¬Dead |
| "Healthy beings are not dead." |
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| HappyCatOwner ⊑ ∃owns.Cat ⊓ ∀caresFor.Healthy |
| "A happy cat owner owns a cat and all beings he cares for are healthy." |

**ABox A**

| HappyCatOwner(schroedinger) |
| "Schrödinger is a happy cat owner." |

- $\mathcal{K} \models \exists caresFor.(Cat \sqcap Alive)(schroedinger)$
- $\mathcal{K} \models \forall owns.\neg Cat \sqsubseteq \neg HappyCatOwner$
- $\mathcal{K} \not\models Cat \sqsubseteq Healthy$
Decidability of DLs

DLs are decidable, i.e., there exists an algorithm that

**Given:** a KB and an axiom $\alpha$,

**Output:** “yes” iff $\text{KB} \models \alpha$ and no otherwise.

- Likewise, there is a similar algorithm that decides whether an input KB is satisfiable
- Just ask $\text{KB} \models \top \sqsubseteq \bot$: if the answer is “yes”, then KB is unsatisfiable, otherwise it is satisfiable.
Standard Reasoning Problems
Standard DL Reasoning Problems

- **KB Satisfiability:** verify whether the KB is satisfiable

- **Entailment:** verify whether the KB entails a certain axiom
e.g., $\mathcal{K} \models \text{CatOwner}(\text{Schroedinger})$

- **Concept Satisfiability:** verify whether a given concept is (un)satisfiable, e.g., $\mathcal{K} \models \text{Dead} \sqcap \text{Alive} \sqsubseteq \bot$

- **Coherence:** verify whether none of the concepts in the KB is unsatisfiable

- **Classification:** compute the subsumption hierarchy of all atomic concepts, e.g. $\mathcal{K} \models \text{Healthy} \sqsubseteq \neg \text{Dead}$, etc.

- **Instance Retrieval:** retrieve all the individuals known to be instances of a certain concept, e.g., find all $a$, s.t. $\exists \text{caresFor}(a)$
Deciding KB Satisfiability

- deciding KB satisfiability is a basic inference task (the “mother” of all standard reasoning tasks)
- directly needed in the process of KB engineering
  - detect severe modeling errors
- other reasoning tasks can be reduced to checking KB (un)satisfiability (and vice versa)

**Theorem 1: Reducing reasoning problems to KB satisfiability**

Let $\mathcal{K}$ be a KB and $a$ an individual name not in $\mathcal{K}$. Then

1. $C$ is satisfiable w.r.t. $\mathcal{K}$ iff $\mathcal{K} \cup C(a)$ is satisfiable;
2. $\mathcal{K}$ is coherent iff, for each concept name $C$, $\mathcal{K} \cup C(a)$ is satisfiable;
3. $\mathcal{K} \models A \sqsubseteq B$ iff $\mathcal{K} \cup (A \sqcap \neg B)(a)$ is unsatisfiable;
4. $\mathcal{K} \models B(b)$ iff $\mathcal{K} \cup \neg B(b)$ is unsatisfiable.
Entailment Checking

- used in the KB modeling process to check, whether the specified knowledge has the intended consequences
- used for querying the KB if certain propositions are necessarily true

Reduction of entailment problem $\mathcal{K} \models \alpha$ to checking KB inconsistency (see Th.1) follows the idea of proof by contradiction:

- negate the axiom $\alpha$
- add the negated axiom $\neg \alpha$ to $\mathcal{K}$
- check for inconsistency of the resulting KB $\mathcal{K} \cup \{\neg \alpha\}$

If an axiom cannot be negated directly, its negation can be emulated ($\{\neg \alpha\} \leadsto A_\alpha$).
### Entailment Checking, cont’d

Axiom sets $A_\alpha$ such that $\mathcal{K} \models \alpha$ exactly if $\mathcal{K} \cup A_\alpha$ is unsatisfiable:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$A_\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1 \circ \ldots \circ r_n \sqsubseteq r$</td>
<td>${\neg r(c_0, c_1), r_1(c_0, c_n), \ldots, r_n(c_{n-1}, c_n)}$</td>
</tr>
<tr>
<td>$\text{Dis}(r, r')$</td>
<td>${r(c_1, c_2), r'(c_1, c_2)}$</td>
</tr>
<tr>
<td>$C \sqsubseteq D$</td>
<td>${(C \sqcap \neg D)(c)}$ or $\top \sqsubseteq \exists u(C \sqcap \neg D)$</td>
</tr>
<tr>
<td>$C(a)$</td>
<td>${\neg C(a)}$</td>
</tr>
<tr>
<td>$\neg C(a)$</td>
<td>${C(a)}$</td>
</tr>
<tr>
<td>$r(a, b)$</td>
<td>${\neg r(a, b)}$</td>
</tr>
<tr>
<td>$\neg r(a, b)$</td>
<td>${r(a, b)}$</td>
</tr>
<tr>
<td>$a \approx b$</td>
<td>${a \not\approx b}$</td>
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</table>

- Individual names $c$ with possible subscripts are supposed to be fresh\(^1\).
- For GCI (third line), the first variant is normally employed; the second is logical equivalent instead of just emulating.

---

\(^1\)Fresh individuals are those not appearing in the given KB $\mathcal{K}$. 
Concept Satisfiability

A concept expression $C$ is called **satisfiable** with respect to a knowledge base $\mathcal{K}$, if there exists a model $\mathcal{I}$ of $\mathcal{K}$ such that $C^\mathcal{I} \neq \emptyset$.

- Unsatisfiable atomic concepts normally indicate KB modeling errors.
- Concept satisfiability can be reduced to KB consistency (Th. 1) and non-entailment resp.:
  - $C$ is **satisfiable** wrt. $\mathcal{K} \iff \mathcal{K} \cup \{ C(a) \}$ is consistent, where $a$ is a fresh individual name
  - $C$ is **satisfiable** wrt. $\mathcal{K} \iff \mathcal{K} \not\models C \sqsubseteq \bot$
Concept Satisfiability, cont’d

Entailment of general concept inclusions $C \sqsubseteq D$ and equivalences $C \equiv D$ can be reduced to both concept (un)satisfiability and KB (un)satisfiability.

- $C \sqsubseteq D \iff C \cap \lnot D$ is unsatisfiable
  $\iff$ KB $\{C(a), \lnot D(a)\}$ is unsatisfiable

- $C \equiv D \iff (C \cap \lnot D) \cup (D \cap \lnot C)$ is unsatisfiable
  $\iff$ both $C \cap \lnot D$ and $D \cap \lnot C$ are unsatisfiable
  $\iff$ both KBs $\{C(a), \lnot D(a)\}$ and $\{D(a), \lnot C(a)\}$ are unsatisfiable
Classification

**KB Classification**

Classification of a knowledge base $\mathcal{K}$ is to determine for any two concept names $A, B$, whether $\mathcal{K} \models A \sqsubseteq B$ holds.

- This is useful at KB design time for checking the inferred concept hierarchy. Also, computing this hierarchy once and storing it can speed up further queries.

- Classification can be reduced to checking entailment of GCIs.

- While this requires quadratically many checks, one can often do much better in practice by applying optimizations and exploiting that subsumption is a preorder.
Instance Retrieval

Instance retrieval task is to find all named individuals that are known to be in a certain concept (role).

- \( \text{retrieve}(C, \mathcal{K}) = \{ a \in N_I \mid a^I \in C^I \text{ for every model } I \text{ of } \mathcal{K} \} \)
- \( \text{retrieve}(r, \mathcal{K}) = \{ (a, b) \in N_I^2 \mid (a^I, b^I) \in r^I \text{ for every model } I \text{ of } \mathcal{K} \} \)

- It can be reduced to checking entailment of as many individual assertions as there are named individuals in \( \mathcal{K} \) i.e., test \( \mathcal{K} \models C(a) \) for each \( a \) occurring in \( \mathcal{K} \) (excluding pathologic cases).

- Depending on the system used and the inference algorithm, this can be done in a much more efficient way (e.g. by a transformation into a database query, in SQL or Datalog).
Novel Reasoning Problems

In recent years, other reasoning tasks have been gaining attention.
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- **Conjunctive Query Answering**

  conjunctive queries allow to join pieces of information
  more expressive: union of CQAs (akin to SQL), rules
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  conjunctive queries allow to join pieces of information
  
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- **Inconsistency Handling**
  
  repair inconsistent KB or avoid that $\mathcal{K} \models \alpha$ for every $\alpha$ (‘knowledge explosion’) if $\mathcal{K}$ is inconsistent by introducing, e.g., new semantics
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- **Entailment Explanation**
  
  identify axioms in the knowledge base that support a conclusion
  
  $\mathcal{K} \models \alpha$, typically a smallest $\mathcal{K}' \subseteq \mathcal{K}$ such that $\mathcal{K}' \models \alpha$ (‘axiom pinpointing’)

Conjunctive Query Answering

Generalize Instance Retrieval by allowing joins and projections:

\[ q(Z) = \exists Y \exists X. \text{childOf}(Z, Y) \land \text{childOf}(Z, X) \land \text{marriedWith}(Y, X) \]

- In databases:
  - just one model (the DB itself) by Closed World Assumption
    (R. Reiter, 1978: if atom \( A \) is not provable from DB, \( \neg A \) is true).

- this is rather easy
Conjunctive Query Answering, cont’d

Generalize Instance Retrieval by allowing joins and projections:

\[ q(Z) = \exists Y \exists X. \text{childOf}(Z, Y) \land \text{childOf}(Z, X) \land \text{marriedWith}(Y, X) \]

- In Description Logics:
  - one knowledge base, many models (*Open World Assumption*)
  - not so easy
  - the \( \exists \)-variables must be suitably mapped in every model
Algorithmic Approaches to DL Reasoning
Types of Reasoning Procedures

- Roughly, DL inference algorithms can be separated into two groups:
  - **model-based algorithms**: show satisfiability by constructing a model (or a representation of it).
    - Examples: tableau, automata, type elimination algorithms
  - **proof-based algorithms**: apply deduction rules to the KB to infer new axioms.
    - Examples: resolution, consequence-based algorithms

**Note:**

- both strategies are known from first-order logic theorem proving
- additional care is needed to ensure decidability (in particular completeness and termination of algorithms).
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Tableau Algorithm for DLs

Tableau-based techniques

They try to decide the satisfiability of a formula (or theory) by using rules to construct (a representation of) a model.

- Tableau-based techniques have been used in FOL and modal logics for many years.
- For DLs, they have been extensively explored since the late 1990s [Smolka, 1990], [Baader and Sattler, 2001].
- They are considered well-suited for implementation.
- In fact, many of the most successful DL reasoners implement tableau techniques or variations of them, e.g.: RACER, FaCT++, Pellet, Hermit, etc.
Tableau Algorithm for Deciding KB Satisfiability

- Perform a ”bottom-up” construction of a model:
Tableau Algorithm for Deciding KB Satisfiability

- Perform a "bottom-up" construction of a model:
  - Initialize an interpretation by all explicitly known (i.e., named) individuals and their known properties.
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  - Iteratively "repair" it by adding new information about concept or role memberships and/or introducing new (i.e., anonymous) individuals, this may require case distinction and backtracking.

Note: as the finite model property does not hold in general, not a full model is constructed but a finite representation of it (cf. "blocking").
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  - Iteratively “repair” it by adding new information about concept or role memberships and/or introducing new (i.e., anonymous) individuals, this may require case distinction and backtracking.
  - If we arrive at an interpretation satisfying all axioms, satisfiability has been shown.
  - If every repairing attempt eventually results in an overt inconsistency, unsatisfiability has been shown.

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### Tableau Overview Example: Happy Cat Owner

#### RBox $\mathcal{R}$

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<tr>
<th>Class</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>HappyCatOwner $\sqsubseteq$ $\exists$ owns.Cat $\sqcap$ $\forall$ caresFor.Healthy</td>
<td>&quot;A happy cat owner owns a cat and all beings he cares for are healthy.&quot;</td>
</tr>
</tbody>
</table>

#### ABox $\mathcal{A}$

- HappyCatOwner(schroedinger)
  - "Schrödinger is a happy cat owner."

Is $\mathcal{K}$ satisfiable?
**Tableau Overview Example: Happy Cat Owner**

<table>
<thead>
<tr>
<th>RBox $\mathcal{R}$</th>
<th>owns $\sqsubseteq$ caresFor</th>
</tr>
</thead>
<tbody>
<tr>
<td>TBox $\mathcal{T}$</td>
<td></td>
</tr>
<tr>
<td>Healthy $\sqsubseteq$ $\neg$Dead</td>
<td></td>
</tr>
<tr>
<td>Cat $\sqsubseteq$ Dead $\sqcup$ Alive</td>
<td></td>
</tr>
<tr>
<td>HappyCatOwner $\sqsubseteq$ $\exists$owns.Cat $\sqcap$$\forall$caresFor.Healthy</td>
<td></td>
</tr>
<tr>
<td>ABox $\mathcal{A}$</td>
<td></td>
</tr>
<tr>
<td>HappyCatOwner($s$)</td>
<td></td>
</tr>
</tbody>
</table>

Is $\mathcal{K}$ satisfiable?
Tableau Overview Example: Happy Cat Owner

<table>
<thead>
<tr>
<th>RBox ( \mathcal{R} )</th>
<th>owns ( \sqsubseteq ) caresFor</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>TBox ( \mathcal{T} )</th>
<th>Healthy ( \sqsubseteq ) \neg \text{Dead}</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cat ( \sqsubseteq ) \text{Dead} \sqcup \text{Alive}</td>
</tr>
<tr>
<td></td>
<td>\text{HappyCatOwner} \sqsubseteq \exists \text{owns}. \text{Cat} \sqcap \forall \text{caresFor}. \text{Healthy}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ABox ( \mathcal{A} )</th>
<th>\text{HappyCatOwner}(s)</th>
</tr>
</thead>
</table>

Is \( \mathcal{K} \) satisfiable?

\[ s \bullet \quad \mathcal{L}(s) = \{ \text{HappyCatOwner} \} \]
### Tableau Overview Example: Happy Cat Owner

**RBox R**

```
owns ⊑ caresFor
```

**TBox T**

```
Healthy ⊑ ¬ Dead
Cat ⊑ Dead ⊓ Alive
HappyCatOwner ⊑ ∃ owns.Cat ⊓ ∀ caresFor. Healthy
```

**ABox A**

```
HappyCatOwner(s)
```

**Is $\mathcal{K}$ satisfiable?**

```
\begin{align*}
L(s) &= \{ \text{HappyCatOwner} \} \\
L(c) &= \{ \text{Cat} \}
\end{align*}
```
Tableau Overview Example: Happy Cat Owner

\[ RBox \mathcal{R} \]
- \( \text{owns} \sqsubseteq \text{caresFor} \)

\[ TBox \mathcal{T} \]
- \( \text{Healthy} \sqsubseteq \neg \text{Dead} \)
- \( \text{Cat} \sqsubseteq \text{Dead} \sqcup \text{Alive} \)
- \( \text{HappyCatOwner} \sqsubseteq \exists \text{owns} . \text{Cat} \sqcap \forall \text{caresFor} . \text{Healthy} \)

\[ ABox \mathcal{A} \]
- \( \text{HappyCatOwner}(s) \)

Is \( \mathcal{K} \) satisfiable?

\[ s \]

\[ \text{owns} \quad \text{caresFor} \quad c \]

\[ \mathcal{L}(s) = \{ \text{HappyCatOwner} \} \]

\[ \mathcal{L}(c) = \{ \text{Cat} \} \]
Tableau Overview Example: Happy Cat Owner

RBox $\mathcal{R}$

$\text{owns} \sqsubseteq \text{caresFor}$

TBox $\mathcal{T}$

$\text{Healthy} \sqsubseteq \neg \text{Dead}$
$\text{Cat} \sqsubseteq \text{Dead} \sqcup \text{Alive}$
$\text{Happy Cat Owner} \sqsubseteq \exists \text{owns} \cdot \text{Cat} \sqcap \forall \text{caresFor} \cdot \text{Healthy}$

ABox $\mathcal{A}$

$\text{Happy Cat Owner}(s)$

Is $\mathcal{K}$ satisfiable?

$s$

$\text{owns} \; \text{caresFor}$

$\mathcal{L}(s) = \{\text{Happy Cat Owner}\}$

$c$

$\mathcal{L}(c) = \{\text{Cat, Healthy}\}$
### Tableau Overview Example: Happy Cat Owner

**RBox** \( \mathcal{R} \)

- `owns` \( \sqsubseteq \) `caresFor`

**TBox** \( \mathcal{T} \)

- `Healthy` \( \sqsubseteq \) `\neg Dead`
- `Cat` \( \sqsubseteq \) `Dead \sqcup Alive`
- `HappyCatOwner` \( \sqsubseteq \) `\exists owns.Cat \sqcap \forall caresFor.Healthy`

**ABox** \( \mathcal{A} \)

- `HappyCatOwner(s)`

Is \( \mathcal{K} \) satisfiable?

\[
\mathcal{L}(s) = \{ \text{HappyCatOwner} \}
\]

\[
\mathcal{L}(c) = \{ \text{Cat, Healthy, \neg Dead} \}
\]
Tableau Overview Example: Happy Cat Owner

RBox $\mathcal{R}$

- $\text{owns} \sqsubseteq \text{caresFor}$

TBox $\mathcal{T}$

- $\text{Healthy} \sqsubseteq \neg \text{Dead}$
- $\text{Cat} \sqsubseteq \text{Dead} \sqcap \text{Alive}$
- $\text{HappyCatOwner} \sqsubseteq \exists \text{owns}.\text{Cat} \sqcap \forall \text{caresFor}.\text{Healthy}$

ABox $\mathcal{A}$

- $\text{HappyCatOwner}(s)$

Is $\mathcal{K}$ satisfiable?

$s$

\[ \mathcal{L}(s) = \{ \text{HappyCatOwner} \} \]

\begin{itemize}
  \item owns
  \item caresFor
\end{itemize}

$c$

\[ \mathcal{L}(c) = \{ \text{Cat}, \text{Healthy}, \neg \text{Dead}, \text{Alive} \} \]
Tableau Overview Example: Happy Cat Owner

$RBox \; \mathcal{R}$

\begin{align*}
\text{owns} & \sqsubseteq \text{caresFor} \\
\end{align*}

$TBox \; \mathcal{T}$

\begin{align*}
\text{Healthy} & \sqsubseteq \neg\text{Dead} \\
\text{Cat} & \sqsubseteq \text{Dead} \sqcup \text{Alive} \\
\text{HappyCatOwner} & \sqsubseteq \exists \text{owns. Cat} \sqcap \forall \text{caresFor. Healthy} \\
\end{align*}

$ABox \; \mathcal{A}$

\begin{align*}
\text{HappyCatOwner}(s) \\
\end{align*}

Is $\mathcal{K}$ satisfiable? Yes!

$s$

\[ \mathcal{L}(s) = \{ \text{HappyCatOwner} \} \]

\begin{align*}
\text{owns} & \quad \text{caresFor} \\
\end{align*}

$c$

\[ \mathcal{L}(c) = \{ \text{Cat, Healthy, } \neg \text{Dead, Alive} \} \]
Naive Tableau Algorithm for $\mathcal{ALC}$

Given a KB in NNF we construct a tableau, which for $\mathcal{ALC}$ KBs consists of

- a set of nodes, labeled with individual names or variable names
- directed edges between some pairs of nodes
- for each node labeled $x$, a set $\mathcal{L}(x)$ of class expressions and
- for each pair of nodes $x$ and $y$, a set $\mathcal{L}(x, y)$ of role names.

Provisos:

- omit edges which are labeled with the empty set
- assume $\top$ is contained in $\mathcal{L}(x)$ for any $x$.
- concept expressions should be in negation normal form
## Recall ALC Syntax

<table>
<thead>
<tr>
<th>Construct</th>
<th>Syntax</th>
<th>Example</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>atomic concept</td>
<td>$A$</td>
<td><em>Doctor</em></td>
<td>$A^I \subseteq \Delta^I$</td>
</tr>
<tr>
<td>atomic role</td>
<td>$r$</td>
<td><em>hasChild</em></td>
<td>$r^I \subseteq \Delta^I \times \Delta^I$</td>
</tr>
<tr>
<td>conjunction</td>
<td>$C \sqcap D$</td>
<td><em>Human \sqcap Male</em></td>
<td>$C^I \sqcap D^I$</td>
</tr>
<tr>
<td>unqual. exist. res.(^2)</td>
<td>$\exists r$</td>
<td>$\exists \text{hasChild}$</td>
<td>${ o \mid \exists o'.(o, o') \in r^I }$</td>
</tr>
<tr>
<td>value res.</td>
<td>$\forall r. C$</td>
<td>$\forall \text{hasChild}. \text{Male}$</td>
<td>${ o \mid \forall o'.(o, o') \in r^I \rightarrow o' \in C^I }$</td>
</tr>
<tr>
<td>full negation</td>
<td>$\neg C$</td>
<td>$\neg \forall \text{hasChild}. \text{Male}$</td>
<td>$\Delta^I \setminus C^I$</td>
</tr>
</tbody>
</table>

\(^2\)Unqualified existential restriction
Negation Normal Form

Negation normal form (NNF)

A concept expression $C$ is in negation normal form, if negation occurs in $C$ only in front of atomic concepts, nominal concepts and self-restrictions.
Negation Normal Form

Negation normal form (NNF)

A concept expression $C$ is in negation normal form, if negation occurs in $C$ only in front of atomic concepts, nominal concepts and self-restrictions.

Given a KB $\mathcal{K}$ to construct $\text{nnf}(\mathcal{K})$ we need to

- replace every $C \equiv D$ by $C \subseteq D$ and $D \subseteq C$;
- replace every $C \subseteq D$ by $\neg C \sqcup D$;
Negation Normal Form

A concept expression $C$ is in negation normal form, if negation occurs in $C$ only in front of atomic concepts, nominal concepts and self-restrictions.

Given a KB $\mathcal{K}$ to construct $\text{nff}(\mathcal{K})$ we need to

- replace every $C \equiv D$ by $C \sqsubseteq D$ and $D \sqsubseteq C$;
- replace every $C \sqsubseteq D$ by $\neg C \sqcup D$;
- recursively translate every $C$ into $\text{nff}(C)$:

$$\text{nff}(C) \rightsquigarrow C \text{ if } C \in \{A, \neg A, \{a_1 \ldots a_n\}, \neg\{a_1 \ldots a_n\}, \exists r. \text{Self}, \neg \exists r. \text{Self}, \top, \bot\}$$

$$\begin{align*}
\text{nff}(\neg C) & \rightsquigarrow \text{nff}(C) \\
\text{nff}(C \sqcap D) & \rightsquigarrow \text{nff}(C) \sqcap \text{nff}(D) \\
\text{nff}(C \sqcup D) & \rightsquigarrow \text{nff}(C) \sqcup \text{nff}(D) \\
\text{nff}(\neg(C \sqcup D)) & \rightsquigarrow \text{nff}(\neg C \sqcap \neg D) \\
\text{nff}(\neg(C \sqcap D)) & \rightsquigarrow \text{nff}(\neg C \sqcup \neg D) \\
\text{nff}(\forall r. C) & \rightsquigarrow \forall r. \text{nff}(C) \\
\text{nff}(\exists r. C) & \rightsquigarrow \exists r. \text{nff}(C) \\
\text{nff}(\neg \forall r. C) & \rightsquigarrow \exists r. \text{nff}(\neg C) \\
\text{nff}(\neg \exists r. C) & \rightsquigarrow \forall r. \text{nff}(\neg C) \\
\text{nff}(\leq kr.C) & \rightsquigarrow \leq (k + 1) r. \text{nff}(C) \\
\text{nff}(\geq kr.C) & \rightsquigarrow \geq (k - 1) r. \text{nff}(C) \\
\text{nff}(\neg \top) & \rightsquigarrow \bot
\end{align*}$$
Negation Normal Form (NNF)

A concept expression $C$ is in negation normal form, if negation occurs in $C$ only in front of atomic concepts, nominal concepts and self-restrictions.

Given a KB $K$ to construct $nnf(K)$ we need to

- replace every $C \equiv D$ by $C \sqsubseteq D$ and $D \sqsubseteq C$;
- replace every $C \sqsubseteq D$ by $\neg C \sqcup D$;
- recursively translate every $C$ into $nnf(C)$:

$$nnf(C) \rightsquigarrow C \text{ if } C \in \{A, \neg A, \{a_1 \ldots a_n\}, \neg\{a_1 \ldots a_n\}, \exists r.\text{Self}, \neg\exists r.\text{Self}, \top, \bot\}$$

$$nnf(\neg\neg C) \rightsquigarrow nnf(C)$$
$$nnf(C \sqcap D) \rightsquigarrow nnf(C) \sqcap nnf(D)$$
$$nnf(C \sqcup D) \rightsquigarrow nnf(C) \sqcup nnf(D)$$
$$nnf(\neg(C \sqcup D)) \rightsquigarrow nnf(\neg(C \sqcap \neg D))$$
$$nnf(\neg(C \sqcap D)) \rightsquigarrow nnf(\neg(C \sqcup \neg D))$$
$$nnf(\forall r. C) \rightsquigarrow \forall r. nnf(C)$$
$$nnf(\exists r. C) \rightsquigarrow \exists r. nnf(C)$$

$C$ and $nnf(C)$ are logically equivalent, i.e., $C^\mathcal{I} = nnf(C)^\mathcal{I}$, for every interpretation $\mathcal{I}$, and the translation process terminates in linear time.
Example: Negation Normal Form

Negation Normal Form

\[ \text{FilmActor} \subseteq (\exists \text{actedIn} \cap \text{Artist}) \cup \neg(\neg \exists \text{actedIn} \cup \exists \text{playsIn. Theater}) \]
Example: Negation Normal Form

Negation Normal Form

\[ \text{FilmActor} \subseteq (\exists \text{actedIn} \land \text{Artist}) \lor \neg(\neg\exists \text{actedIn} \lor \exists \text{playsIn}.\text{Theater}) \]

1. \( \neg \text{FilmActor} \lor (\exists \text{actedIn} \land \text{Artist}) \lor \neg(\neg\exists \text{actedIn} \lor \exists \text{playsIn}.\text{Theater}) \)
Example: Negation Normal Form

Negation Normal Form

FilmActor ⊑ (∃actedIn ∩ Artist) ∪ ¬(¬∃actedIn ∪ ∃playsIn. Theater)

1. ¬FilmActor ∪ (∃actedIn ∩ Artist) ∪ ¬(¬∃actedIn ∪ ∃playsIn. Theater)

2. ¬FilmActor ∪ (∃actedIn ∩ Artist) ∪ (∃actedIn ∩ ¬∃playsIn. Theater)
Example: Negation Normal Form

Negation Normal Form

FilmActor ⊑ (∃actedIn ∩ Artist) ⊔ ¬(¬∃actedIn ⊔ ∃playsIn. Theater)

1. ¬FilmActor ⊔ (∃actedIn ∩ Artist) ⊔ ¬(¬∃actedIn ⊔ ∃playsIn. Theater)

2. ¬FilmActor ⊔ (∃actedIn ∩ Artist) ⊔ (∃actedIn ⊔ ¬∃playsIn. Theater)

3. ¬FilmActor ⊔ (∃actedIn ∩ Artist) ⊔ (∃actedIn ⊔ ∀playsIn. ¬Theater)
Initial Tableau

For an $\mathcal{ALC}$ knowledge base $\mathcal{K}$ in negation normal form, the initial tableau is defined as follows:

1. For each individual $a$ occurring in $\mathcal{K}$, create a node labeled $a$ and set $L(a) = \emptyset$.

2. For all pairs $a, b$ of individuals, set $L(a, b) = \emptyset$.

3. For each ABox statement $C(a)$ in $\mathcal{K}$, set $L(a) \leftarrow C$.

4. For each ABox statement $r(a, b)$ in $\mathcal{K}$ set $L(a, b) \leftarrow r$.

Note: “$\leftarrow$” means “update with”, “add to”
Example: Initial Tableau

\[ \mathcal{K} = \{ A(a), \ (\exists r. B)(a), \ r(a, b), \ r(a, c), \ s(b, b), \ (A \sqcup B)(c), \ \neg A \sqcup (\forall s. B) \} \]
Example: Initial Tableau

$\mathcal{K} = \{A(a), (\exists r. B)(a), r(a, b), r(a, c), s(b, b), (A \sqcup B)(c), \neg A \sqcup (\forall s. B)\}$

\[
\begin{align*}
    b & \quad \bullet \\
    a & \quad \bullet \\
    c & \quad \bullet
\end{align*}
\]

$\mathcal{L}(a) = \{A, \}$
Example: Initial Tableau

\[ K = \{ A(a), \ (\exists r.B)(a), \ r(a, b), \ r(a, c), \ s(b, b), \ (A \sqcup B)(c), \ \neg A \sqcup (\forall s.B) \} \]

\[ b \bullet \]

\[ a \bullet \quad \mathcal{L}(a) = \{ A, \exists r.B \} \]

\[ c \bullet \]
Example: Initial Tableau

\[ \mathcal{K} = \{ A(a), (\exists r. B)(a), r(a, b), r(a, c), s(b, b), (A \sqcup B)(c), \neg A \sqcup (\forall s. B) \} \]

\[ \mathcal{L}(a) = \{ A, \exists r. B \} \]
Example: Initial Tableau

\[ \mathcal{K} = \{ A(a), \ (\exists r. B)(a), \ r(a, b), \ r(a, c), \ s(b, b), \ (A \sqcup B)(c), \ \neg A \sqcup (\forall s. B) \} \]

\[ \mathcal{L}(a) = \{ A, \exists r. B \} \]
Example: Initial Tableau

\[ \mathcal{K} = \{ A(a), (\exists r. B)(a), \quad r(a, b), \quad r(a, c), \quad s(b, b), \quad (A \sqcup B)(c), \quad \neg A \sqcup (\forall s. B) \} \]

\[ \mathcal{L}(a) = \{ A, \exists r. B \} \]

\[ \mathcal{L}(c) = \{ A \sqcup B \} \]
**Example: Initial Tableau**

\[ \mathcal{K} = \{ A(a), (\exists r.B)(a), r(a,b), r(a,c), s(b,b), (A \sqcup B)(c), \neg A \sqcup (\forall s.B) \} \]

\[ L(b) = \emptyset \]

\[ L(a) = \{ A, \exists r.B \} \]

\[ L(c) = \{ A \sqcup B \} \]
Expansion Rules for the Naive Tableau

- **∩-rule**: If \( C \cap D \in \mathcal{L}(x) \) and \( \{C, D\} \not\subseteq \mathcal{L}(x) \), then set \( \mathcal{L}(x) \leftarrow \{C, D\} \).
Expansion Rules for the Naive Tableau

- **⊓-rule:** If $C \sqcap D \in \mathcal{L}(x)$ and $\{C, D\} \not\subseteq \mathcal{L}(x)$, then set $\mathcal{L}(x) \leftarrow \{C, D\}$.

- **⊔-rule:** If $C \sqcup D \in \mathcal{L}(x)$ and $\{C, D\} \cap \mathcal{L}(x) = \emptyset$, then set $\mathcal{L}(x) \leftarrow C$ or $\mathcal{L}(x) \leftarrow D$. 
Expansion Rules for the Naive Tableau

- **□-rule**: If $C \sqcap D \in \mathcal{L}(x)$ and $\{C, D\} \not\subseteq \mathcal{L}(x)$, then set $\mathcal{L}(x) \leftarrow \{C, D\}$.

- **⊔-rule**: If $C \sqcup D \in \mathcal{L}(x)$ and $\{C, D\} \cap \mathcal{L}(x) = \emptyset$, then set $\mathcal{L}(x) \leftarrow C$ or $\mathcal{L}(x) \leftarrow D$.

- **∃-rule**: If $\exists r.C \in \mathcal{L}(x)$ and there is no $y$ with $r \in \mathcal{L}(x, y)$ and $C \in \mathcal{L}(y)$, then
  1. add a new node with label $y$ (where $y$ is a new node label),
  2. set $\mathcal{L}(x, y) = \{r\}$, and
  3. set $\mathcal{L}(y) = \{C\}$.
Expansion Rules for the Naive Tableau

- **⊓-rule:** If \( C \sqcap D \in \mathcal{L}(x) \) and \( \{C, D\} \not\subseteq \mathcal{L}(x) \), then set \( \mathcal{L}(x) \leftarrow \{C, D\} \).

- **⊔-rule:** If \( C \sqcup D \in \mathcal{L}(x) \) and \( \{C, D\} \cap \mathcal{L}(x) = \emptyset \), then set \( \mathcal{L}(x) \leftarrow C \) or \( \mathcal{L}(x) \leftarrow D \).

- **∃-rule:** If \( \exists r.C \in \mathcal{L}(x) \) and there is no \( y \) with \( r \in \mathcal{L}(x, y) \) and \( C \in \mathcal{L}(y) \), then
  1. add a new node with label \( y \) (where \( y \) is a new node label),
  2. set \( \mathcal{L}(x, y) = \{r\} \), and
  3. set \( \mathcal{L}(y) = \{C\} \).

- **∀-rule:** If \( \forall r.C \in \mathcal{L}(x) \) and there is a node \( y \) with \( r \in \mathcal{L}(x, y) \) and \( C \not\in \mathcal{L}(y) \), then set \( \mathcal{L}(y) \leftarrow C \).
Expansion Rules for the Naive Tableau

- **\(\sqcap\)-rule:** If \(C \cap D \in \mathcal{L}(x)\) and \(\{C, D\} \not\subseteq \mathcal{L}(x)\), then set \(\mathcal{L}(x) \leftarrow \{C, D\}\).

- **\(\sqcup\)-rule:** If \(C \sqcup D \in \mathcal{L}(x)\) and \(\{C, D\} \cap \mathcal{L}(x) = \emptyset\), then set \(\mathcal{L}(x) \leftarrow C\) or \(\mathcal{L}(x) \leftarrow D\).

- **\(\exists\)-rule:** If \(\exists r.C \in \mathcal{L}(x)\) and there is no \(y\) with \(r \in L(x, y)\) and \(C \in \mathcal{L}(y)\), then
  1. add a new node with label \(y\) (where \(y\) is a new node label),
  2. set \(\mathcal{L}(x, y) = \{r\}\), and
  3. set \(\mathcal{L}(y) = \{C\}\).

- **\(\forall\)-rule:** If \(\forall r.C \in \mathcal{L}(x)\) and there is a node \(y\) with \(r \in L(x, y)\) and \(C \not\in \mathcal{L}(y)\), then set \(\mathcal{L}(y) \leftarrow C\).

- **TBox-rule:** If \(C\) is a (rewritten and normalized) TBox statement and \(C \not\in \mathcal{L}(x)\), then set \(\mathcal{L}(x) \leftarrow C\).
Naive Tableau Algorithm

**Input:** $\mathcal{ALC}$ knowledge base $\mathcal{K}$ in negation normal form  
**Output:** “yes” if $\mathcal{K}$ is satisfiable

1. Initialize the tableau;
2. While some expansion rule is applicable:
   2.1. nondeterministically apply an applicable rule;
   2.2. if for some node $x$, there exists some $C \in \mathcal{L}(x)$ such that $\neg C \in \mathcal{L}(x)$, output “no” and terminate;
3. Output “yes”.

A nondeterministic run of the algorithm terminates, if either

- for some node $x$, $\mathcal{L}(x)$ contains a contradiction (“no”; attempt to find a model for $\mathcal{K}$ was unsuccessful), or
- no expansion rule is applicable (“yes”; attempt was successful)
- $\mathcal{K}$ is satisfiable if some run outputs “yes”, and unsatisfiable if every run outputs “no’
- only the $\sqcup$-rule creates true branching regarding yes/no output
Example: Expansion Rules

\[ \mathcal{K} = \{ C(a), \neg C \sqcup \exists r.D, \neg D \sqcup E, \forall r.\neg E(a) \} \]
Example: Expansion Rules

\[ \mathcal{K} = \{ C(a), \neg C \sqcup \exists r. D, \neg D \sqcup E, \forall r. \neg E(a) \} \]

\[ a \bullet \quad \mathcal{L}(a) = \{ C, \} \]
Example: Expansion Rules

\[ \mathcal{K} = \{C(a), \neg C \sqcup \exists r.D, \neg D \sqcup E, \forall r.\neg E(a)\} \]

\[ a \bullet \]
\[ \mathcal{L}(a) = \{C, \forall r.\neg E\} \]
Example: Expansion Rules

\[ \mathcal{K} = \{C(a), \neg C \sqcup \exists r. D, \neg D \sqcup E, \forall r.\neg E(a)\} \]

\[ a \quad \mathcal{L}(a) = \{C, \forall r.\neg E, \neg C \sqcup \exists r. D, \} \]
Example: Expansion Rules

\[ \mathcal{K} = \{ C(a), \neg C \sqcap \exists r. D, \neg D \sqcap E, \forall r. \neg E(a) \} \]

\[ a \bullet \quad \mathcal{L}(a) = \{ C, \forall r. \neg E, \neg C \sqcap \exists r. D, \} \]
Example: Expansion Rules

\[ \mathcal{K} = \{ C(a), \neg C \sqcup \exists r. D, \neg D \sqcup E, \forall r. \neg E(a) \} \]

\[ \mathcal{L}(a) = \{ C, \forall r. \neg E, \neg C \sqcup \exists r. D, \exists r. D \} \]

\[ \mathcal{L}(x) = \{ D, \} \]
Example: Expansion Rules

\[ \mathcal{K} = \{ C(a), \neg C \sqcup \exists r. D, \neg D \sqcup E, \forall r. \neg E(a) \} \]

\[ \mathcal{L}(a) = \{ C, \forall r. \neg E, \neg C \sqcup \exists r. D, \exists r. D \} \]

\[ \mathcal{L}(x) = \{ D, \neg D \sqcup E, \} \]
Example: Expansion Rules

\[ \mathcal{K} = \{ C(a), \neg C \sqcup \exists r.D, \neg D \sqcup E, \forall r.\neg E(a) \} \]

\[ \mathcal{L}(a) = \{ C, \forall r.\neg E, \neg C \sqcup \exists r.D, \exists r.D \} \]

\[ \mathcal{L}(x) = \{ D, \neg D \sqcup E, \} \]
Example: Expansion Rules

\[ \mathcal{K} = \{ C(a), \neg C \sqcup \exists r.D, \neg D \sqcup E, \forall r.\neg E(a) \} \]

\[ \mathcal{L}(a) = \{ C, \forall r.\neg E, \neg C \sqcup \exists r.D, \exists r.D \} \]

\[ \mathcal{L}(x) = \{ D, \neg D \sqcup E, E, \} \]
Example: Expansion Rules

\[ \mathcal{K} = \{ C(a), \neg C \sqcup \exists r. D, \neg D \sqcup E, \forall r. \neg E(a) \} \]

\[ a \quad \mathcal{L}(a) = \{ C, \forall r. \neg E, \neg C \sqcup \exists r. D, \exists r. D \} \]

\[ x \quad \mathcal{L}(x) = \{ D, \neg D \sqcup E, E, \neg E \} \]
Example: Expansion Rules

\[ \mathcal{K} = \{ C(a), \neg C \sqcup \exists r. D, \neg D \sqcup E, \forall r. \neg E(a) \} \]

\[ a \]
\[ r \]
\[ x \]

\[ \mathcal{L}(a) = \{ C, \forall r. \neg E, \neg C \sqcup \exists r. D, \exists r. D \} \]

\[ \mathcal{L}(x) = \{ D, \neg D \sqcup E, E, \neg E \} \]

Clash is obtained, KB is unsatisfiable!
Tableau Algorithm with Blocking for $\mathcal{ALC}$

- Naive tableau algorithm does not always terminate
  
  Example: $\mathcal{K} = \{\neg\text{Person} \sqcup \exists \text{hasParent}, \text{Person}(a_1)\}$.

- Modify the naive tableau algorithm to ensure termination

- Use blocking

- A node with label $x$ is **directly blocked** by a node with label $y$ if
  - $x$ is a variable (i.e., not an individual),
  - $y$ is an ancestor of $x$, and
  - $\mathcal{L}(x) \subseteq \mathcal{L}(y)$

- A node with label $x$ is **blocked**, if it is directly blocked or one of its ancestors is blocked

- Expansion rules may only be applied if $x$ is not blocked
Example: Blocking

\[ \mathcal{K} = \{ B(t), \neg H \sqcup \exists P.H, H(t) \} \]

\[ t \bullet \]
Example: Blocking

\[ \mathcal{K} = \{ B(t), \neg H \sqcup \exists P.H, H(t) \} \]

\[ t \bullet \quad \mathcal{L}(t) = \{ B, H, \} \]
**Example: Blocking**

\[ \mathcal{K} = \{ B(t), \neg H \sqcup \exists P.H, H(t) \} \]

\[ t \bullet \quad \mathcal{L}(t) = \{ B, H, \neg H \sqcup \exists P.H, \} \]
Example: Blocking

\[ \mathcal{K} = \{ B(t), \neg H \sqcup \exists P.H, H(t) \} \]

\[ t \quad \mathcal{L}(t) = \{ B, H, \neg H \sqcup \exists P.H, \} \]
Example: Blocking

\[ \mathcal{K} = \{ B(t), \neg H \sqcup \exists \mathit{P.H}, H(t) \} \]

\[ \mathcal{L}(t) = \{ B, H, \neg H \sqcup \exists \mathit{P.H}, \exists \mathit{P.H} \} \]

\[ \mathcal{L}(x) = \{ H \} \]
**Example: Blocking**

\[ \mathcal{K} = \{B(t), \neg H \sqcup \exists P.H, H(t)\} \]

\[ \mathcal{L}(t) = \{B, H, \neg H \sqcup \exists P.H, \exists P.H\} \]

\[ \mathcal{L}(x) = \{H\} \subseteq \mathcal{L}(t) \]
Computational Properties

- The simple tableau algorithm is expensive in general (worst case):
  - the worst case complexity is double exponential
  - testing KB consistency in $\mathcal{ALC}$ is EXPTIME-complete
  - testing concept satisfiability in $\mathcal{ALC}$ is PSPACE-complete
- Still in practice, (optimized) tableaux algorithms work well
  - other notions of blocking might be used, e.g. “cross-path blocking” (blocking node need not be an ancestor)
  - single exponential time tableaux algorithms are available
- For many other description logics, also tableaux algorithms exist (e.g. $\mathcal{SHIQ}$, $\mathcal{SHOIQ}$, ...)
  Some algorithms are quite involved!
Summary

1. Satisfaction and Entailment
   - Notions
   - Decidability

2. Reasoning Problems
   - Knowledge base consistency
   - Entailment checking
   - Concept satisfiability
   - Classification
   - Instance retrieval

3. Algorithmic Approaches to DL Reasoning
   - Types of reasoning procedures
   - Tableaux

4. Novel reasoning problems
   - Conjunctive query answering
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