Stabilizing Consensus with Many Opinions

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joint work with
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Stabilizing Almost-Consensus

A set of nodes each having one color out of a set $\Sigma$.

Initial colors are called *valid*. 
At the start of each round, nodes update their color, according to the given communication model and protocol.
Stabilizing Almost-Consensus

At the end of each round, an $F$-dynamic adversary can change the color of $F$ nodes, possibly choosing different subsets of nodes over different rounds.
Stabilizing Almost-Consensus

Except for a small number of nodes we want to reach consensus (almost consensus),

on any valid color (almost validity),
Almost-consensus has to be preserved for any poly\((n)\) rounds,

even if the adversary changes colors at each round (almost stability).
Stabilizing Almost-Consensus

A stabilizing almost-consensus protocol guarantees that, w.h.p., for some $\gamma < 1$, from any initial conf., in a finite number of rounds, the system reaches a set of conf.s where $n - \mathcal{O}(n^\gamma)$ nodes

- hold the same color (almost consensus),
- the color was in the initial conf. (almost validity),
- and the convergence hold for any poly($n$) rounds (almost stability).
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- and the convergence hold for any poly($n$) rounds (almost stability).

Cf. classical byzantine agreement: agreement, validity and termination.
The Setting

**Communication model.** Uniform Gossip model: Each node in one round can communicate with one node chosen u.a.r.

**Protocol constraints.** *simple* rule (*dynamics*): Anonymous, \( O(\log |\Sigma|) \) local memory and message size, counters of non-constant length, ...

**Motivations.** Biological systems, chemical reaction networks, social networks, sensor networks.
Previous Work: 3-Median Dynamics

Each node observes the color of three other nodes chosen u.a.r....
Previous Work: 3-Median Dynamics

...and changes its color according to the median of these three...
Previous Work: 3-Median Dynamics

Colors are totally ordered: ...< < < 

...
The 3-Median Process

Almost consensus?  Almost validity?  Almost stability?
Theorem (Doerr, Goldberg, Minder, Sauerwald, Scheideler ’11). For any $\sqrt{n}$-bounded adversary, the 3-median computes an almost stable value between the $(n/2 - c\sqrt{n\log n})$-largest and the $(n/2 + c\sqrt{n\log n})$-largest of the initial values, in $O(\log k \cdot \log \log n + \log n)$ rounds w.h.p.
3-Median Dynamics

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Does 3-median guarantee Stabilizing Almost Consensus?

- Almost consensus
- Almost validity
- Almost stability
3-Median Dynamics

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- **Almost consensus** ✓
- Almost validity
- **Almost stability** ✓
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Does 3-median guarantee Stabilizing Almost Consensus?

- Almost consensus ✓
- Almost validity ?
- Almost stability ✓
3-Median Dynamics
3-Median Dynamics

\[ o(\sqrt{n}) \text{ changed by adversary} \]
3-Median Dynamics

No almost validity!

\[ o(\sqrt{n}) \text{ changed by adversary} \]

The adversary can manipulate the system.
Each node observes the color of three other nodes chosen u.a.r....
...and changes its color according to the majority of these three (breaking ties u.a.r.).
The Majority Process

3-Majority

Almost consensus? Almost validity? Almost stability?
3-Majority for Plurality Consensus

\[ c_i^{(t)} := |\{i\text{-colored nodes}\}| \quad \text{color 1 is the plurality} \]

**Initial bias** \( s \): For all \( i \neq 1 \), \( c_1 - c_i \geq s \)

**Theorem** (Becchetti, Clementi, Natale, Pasquale, Silvestri, Trevisan ’14).

- From any configuration with \( k < 3\sqrt{n} \) colors, with bias \( s = \Omega(\sqrt{kn \log n}) \), the 3-majority converges to the plurality color in \( O(k \log n) \) rounds w.h.p., against a \( O(\sqrt{n}) \)-bounded dynamic adversary.

- From configurations where every color is supported by almost \( n/k \) nodes, convergence takes \( \Omega(k) \) rounds w.h.p.
3-Majority vs 3-Median

Supporting nodes

plurality

median

$\leq$ $\leq$ $\leq$ $\leq$ $\leq$ $\leq$ $\leq$ $\leq$ $\leq$ $\leq$ $\leq$ $\leq$
3-Majority with Bias

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Our Contribution: 3-Majority without Bias

What if we start from any initial configuration, i.e. there may be no initial bias?

**Theorem.** Let $k \leq n^\alpha$, for a suitable constant $\alpha < 1$, and $F = O(\sqrt{n}/(k^{5/2} \log n))$. The 3-majority dynamics is a stabilizing almost-consensus protocol against any $F$-dynamic adversary, with convergence time $O((k^2 \sqrt{\log n} + k \log n)(k + \log n))$, w.h.p.
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What if we start from any initial configuration, i.e. there may be no initial bias?

**Theorem.** Let $k \leq n^\alpha$, for a suitable constant $\alpha < 1$, and $F = \mathcal{O}(\sqrt{n}/(k^{5/2} \log n))$. The 3-majority dynamics is a stabilizing almost-consensus protocol against any $F$-dynamic adversary, with convergence time $\mathcal{O}((k^2 \sqrt{\log n} + k \log n)(k + \log n))$, w.h.p.

- First solution of the almost-stabilizing consensus problem in the uniform gossip model.
- Closes open question on convergence of 3-majority for $|\Sigma| > 2$. 

The Problem without Bias

Supporting nodes

Very small gap between the plurality colors and second colors: one of the second colors may become plurality.

Plurality may not be unique.
Analysis of 3-Majority

$C_i^{(t)} :=$ number of nodes supporting color $i$ at round $t$.

$\mu_j(c) = E[C_j^{(t+1)} | C^{(t)} = c]$ 

**Lemma 1.** For any color $j$ it holds

$$\mu_j(c) = c_j \left( 1 + \frac{c_j}{n} - \frac{1}{n^2} \sum_{h \in [k]} c_h^2 \right).$$

**Lemma 2.** Let 1 be a plurality color and $j$ be a second-most-frequent color, then

$$\mu_1 - \mu_j \geq s(c) \left( 1 + \frac{c_1}{n} \left( 1 - \frac{c_1}{n} \right) \right).$$
Analysis of 3-Majority

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Indices are random variables: without bias, cannot concentrate on who is the plurality.
Lemma. Let $c$ be the conf. at round $t$ with $j$ supported colors. For any color $i$ it holds,

$$\mathbb{E}[C_i^{(t+1)} \mid C^{(t)} = c] \leq c_i \left(1 + \frac{c_i}{n} - \frac{1}{j}\right).$$
A “dying phase”

**Lemma.** Let \( c \) be any conf. with \( j \leq n^{1/3-\varepsilon} \) supported colors (\( \forall \varepsilon > 0 \) const), and such that an color \( i \) exists with \( c_i \leq n/j - \sqrt{jn \log n} \). Within \( t = \mathcal{O}(j \log n) \) rounds color \( i \) becomes \( \mathcal{O}(j^2 \log n) \) w.h.p.

\[ c_i \leq n/j - \sqrt{jn \log n} \quad t = \mathcal{O}(j \log n) \quad \text{w.h.p.} \quad c_i = \mathcal{O}(j^2 \log n) \]
A “dying phase”

**Lemma.** Let $c$ be any conf. with $j \leq n^{1/3-\varepsilon}$ supported colors ($\forall \varepsilon > 0$ const), and such that an color $i$ exists with $c_i \leq n/j - \sqrt{jn \log n}$. Within $t = \mathcal{O}(j \log n)$ rounds color $i$ becomes $\mathcal{O}(j^2 \log n)$ w.h.p.

$c_i \leq n/j - \sqrt{jn \log n} \quad t = \mathcal{O}(j \log n) \quad \Rightarrow \quad c_i = \mathcal{O}(j^2 \log n)$

How to reach such imbalance from any configuration?
Simmetry Breaking

Folklore example:

\[ \Omega(m^2) \text{ steps to “escape”} \]
Simmetry Breaking

jump of expected length $\lambda$

$\mathcal{O}((m/\lambda)^2)$ steps to “escape”
Lemma 42.

\( \{X_t\}_t \) a Markov chain with finite state space \( \Omega \),
\( f : \Omega \to \mathbb{N}, \ Y_t = f(X_t), \)
m \( \in [n] \) a “target value” and \( \tau = \inf \{t \in \mathbb{N} : Y_t \geq m\} \).

If \( \forall x \in \Omega \) with \( f(x) \leq m - 1 \), it holds

1. Positive drift: \( \mathbb{E}[Y_{t+1} | X_t = x] \geq f(x) + \lambda \) \( (\lambda > 0) \),
2. Bounded jumps: \( \Pr\{Y_{\tau} \geq \alpha m\} \leq \alpha m/n \), \( (\alpha > 1) \),

then

\[ \mathbb{E}[\tau] \leq 2\alpha \frac{m}{\lambda}. \]
Lemma. Let $c$ be any configuration with $j$ supported colors. Within $t = \mathcal{O}(j^2 \sqrt{\log n})$ rounds it holds that

$$\Pr(\exists i \text{ such that } C_i^{(t)} \leq n/j - \sqrt{jn \log n}) \geq \frac{1}{2}$$

Proof. Let $m(t)$ be the index of minimum-size color and apply Lemma 42 with $f(c) = C_{m(t)}$. 
Handling the Adversary

Supporting nodes

$C_1$, $C_2$, ...
Handling the Adversary

The adversary may introduce small new colors and keep alive dying minorities.
Handling the Adversary

The adversary may introduce small new colors and keep alive dying minorities.

\[ E[C_i^{(t+1)} | C^{(t)} = c] \leq c_i \left( 1 + \frac{c_i}{n} - \frac{1}{j} \right) \]

Supported colors

\[ F = O\left( \frac{\sqrt{n}}{(k^{5/2} \log n)} \right) \]

\[ E[\tilde{C}_i^{(t+1)} | C^{(t)} = c] \leq c_i \left( 1 + \frac{c_i}{n} - \frac{1}{j} + \sqrt{\frac{k}{n}} \right) \]

“Big” colors
Handling the Adversary

jump of expected length $\lambda$
Handling the Adversary

Action of length $o(\lambda)$

jump of expected length $\lambda$
Handling the Adversary

Action of length $o(\lambda)$

jump of expected length $\lambda$
Open Problems

• Convergence in time $\mathcal{O}(k \log n)$?

• Stabilizing consensus on random/expander graphs?
Thank you!