Optimising for Scale in Globally Multiply-Linked Gravitational Point Set Registration Leads to Singularities

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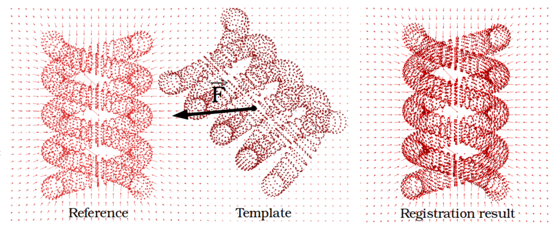
GRAVITATIONAL METHODS FOR POINT SET ALIGNMENT

$$\mathbf{E}(\mathbf{R},\mathbf{t}) = -G\sum_{i,j}\frac{m_{\mathbf{y}_i}m_{\mathbf{x}_j}}{\|\mathbf{R}\,\mathbf{y}_i+\mathbf{t}-\mathbf{x}_j\|_2+\epsilon}$$

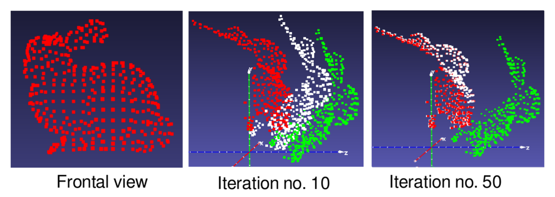
$$\vec{\mathbf{f}}_i = -Gm_{\mathbf{y}_i}\sum_j m_{\mathbf{x}_j}\left(\|\mathbf{y}_i-\mathbf{x}_j\|^2+\epsilon^2\right)^{-3/2}\hat{\mathbf{n}}_{i,j} - \eta v_i^t$$



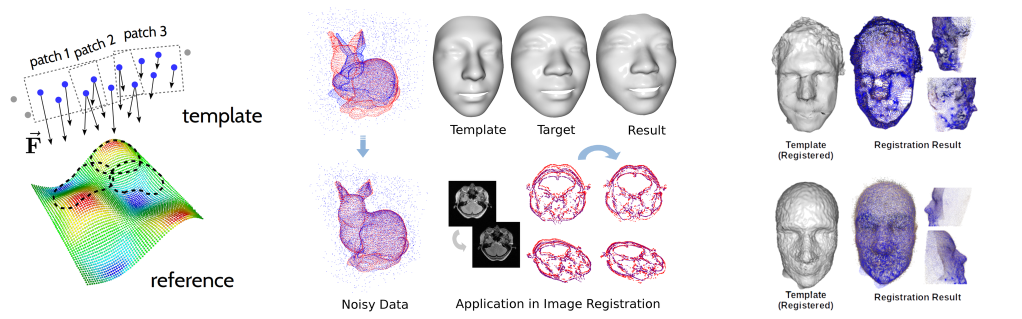
Rigid Gravitational Approach (Second-Order ODEs) [3]

$$\mathbf{f}^{Yi} \propto g(s^{Xj,Yi})$$

$$\mathbf{f}^{Yi} = -G\sum_{j=1}^{N} g(s^{Xj,Yi})h(\|r^{Yi}-r^{Xj}\|)\mathbf{n}_{ij}$$



Modified Rigid Gravitational Approach with Shape Descriptors [4]

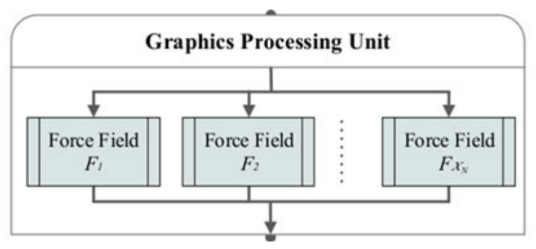


Non-Rigid Gravitational Approach (Second-Order ODEs) [5]

$$F_i^{(ex)} = -\gamma\sum_{j=1}^{M}\frac{m_i m_j}{\|\boldsymbol{q}_i-\boldsymbol{p}_j\|^3}(\boldsymbol{q}_i-\boldsymbol{p}_j)$$

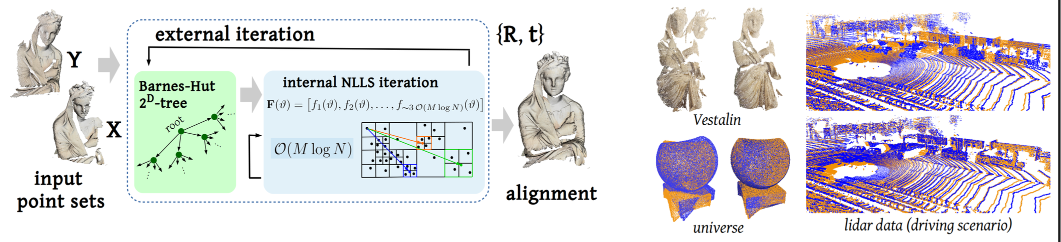
$$F_i^{(ex)} = \kappa\sum_{j=1}^{M}\Delta_i\frac{(\boldsymbol{q}_i-\boldsymbol{p}_j)}{\|\boldsymbol{q}_i-\boldsymbol{p}_j\|^3}$$

$$\Delta_i = 1-\frac{\|f_i^q-f_j^p\|}{\sqrt{D}} \qquad \Delta_i = 2\left(0.5-\frac{\|f_i^q-f_j^p\|}{\sqrt{D}}\right)$$



acceleration on a GPU

Feature-Enhanced Physics-Based Approach with Rigid Body Dynamics (Physical Heuristics and Monte-Carlo Simulation) [6]



$$\xi^-\left(\mathbf{E}(\mathbf{R},\mathbf{t})\right) = \sum_{i,j}\frac{1}{G\,m_{\mathbf{y}_i}m_{\mathbf{x}_j}}\|\mathbf{R}\,\mathbf{y}_i+\mathbf{t}-\mathbf{x}_j\|_2 \qquad \sum_{k=1}^{K}m_{\mathbf{y}_i}m_{\mathbf{x}_k}\|\hat{\mathbf{y}}_i-\mathbf{x}_k\|_2 \approx m_{\mathbf{y}_i}\left[\sum_{k=1}^{K}m_{\mathbf{x}_k}\right]\|\hat{\mathbf{y}}_i-\tilde{\mathbf{x}}\|_2$$

$$\mathbf{E}(\mathbf{R},\mathbf{t}) = \sum_i\sum_j m_{\mathbf{y}_i}m_{\mathbf{x}_j}\|\mathbf{R}\,\mathbf{y}_i+\mathbf{t}-\mathbf{x}_j\|_2 \qquad \mathbf{E}(\mathbf{R},\mathbf{t}) = \sum_{\forall\mathbf{y}_i}m_{\mathbf{y}_i}\sum_{\mathbf{k}_j\in\mathcal{K}(\mathbf{y}_i)}m_{\mathbf{k}_j}\|\mathbf{R}\,\mathbf{y}_i+\mathbf{t}-\mathbf{k}_j\|_2$$

Rigid Gravitational Approach (Altered Physical Laws) [8]

ANALYSIS OF THE ENERGY LANDSCAPE WITH CALCULUS

Hollow Primitives

Unit Circle S^1

Singularity

$$\mathbf{E}_s = rNM = NM$$

Alignment

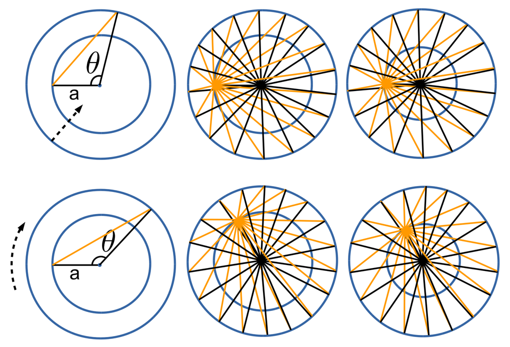
$$d = \sqrt{2\,(1-\cos\theta)} = 2\,\sin\frac{\theta}{2}.$$

$$\bar{d} = \frac{1}{\pi}\int_0^{\pi}2\,\sin\frac{\theta}{2}\,d\theta = -\frac{4}{\pi}\left(\cos\frac{\theta}{2}\right)\Big|_0^{\pi} = \frac{4}{\pi}$$

$$\mathbf{E}_a = \bar{d}\,NM = \frac{4NM}{\pi}$$

General Case

$$\boxed{\bar{d}(a) = \frac{1}{\pi}\int_0^{\pi}\sqrt{a^2+1^2-2a\cos\theta}\,d\theta = \begin{cases}\frac{4}{\pi}, ★ & \text{if } a=1,\\ \frac{2|a-1|}{\pi}E\left(\frac{\theta}{2}\,\middle|\,-\frac{4a}{(a-1)^2}\right)\Big|_{\pi}^{0}, ◆ & \text{if } a\neq 1.\end{cases}}$$



Unit Sphere S^2

Singularity

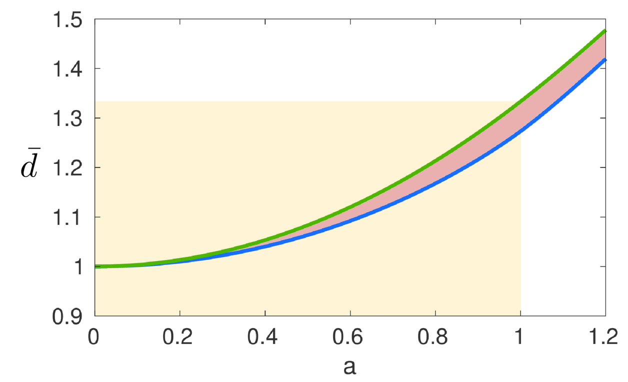
$$\mathbf{E}_s = NM$$

Alignment

$$\bar{d} = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}2\sin\frac{\theta}{2}\sin\theta\,d\theta\,d\phi =$$

$$\int_0^{\pi}\sin\frac{\theta}{2}\sin\theta\,d\theta = \frac{4\sin^3\frac{\theta}{2}}{3}\Big|_0^{\pi} = \frac{4}{3}$$

$$\mathbf{E}_a = \bar{d}\,NM = \frac{4NM}{3}$$



average distances for unit circle and unit sphere

General Case

$$\boxed{\bar{d}(a) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\sqrt{a^2+1-2a\cos\theta}\,sin\theta\,d\theta\,d\phi = \begin{cases}\frac{1}{2}\int_0^{\pi}\sin\theta\,d\theta, & \text{if } a=0,\\ \frac{1}{2}\int_0^{\pi}\sqrt{a^2+1-2a\cos\theta}\,\sin\theta\,d\theta, & \text{if } a>0.\end{cases} = \begin{cases}1, ⬟ & \text{if } a=0,\\ \frac{(a^2+1-2a\cos\theta)^{\frac{3}{2}}}{6a}\Big|_0^{\pi}, & \text{if } a>0.\end{cases}}$$

Volumetric Primitives

Unit Disk

Singularity

$$\bar{d} = \frac{1}{\pi}\int_0^1 2r\int_0^{\pi}r\,d\theta\,dr = \int_0^1 2r^2\,dr = \frac{2}{3}$$

Alignment

$$\bar{d} = \frac{1}{\pi}\int_0^1 2r\int_0^1 2q\int_0^{\pi}\sqrt{r^2+q^2-2rq\cos\theta}\,d\theta\,dq\,dr =$$

$$\frac{8}{\pi}\int_0^1 r\int_0^1 q\,|q-r|\,E\left(\frac{\theta}{2}\,\middle|\,-\frac{4rq}{(q-r)^2}\right)\Big|_{\pi}^{0}dq\,dr \approx \underline{0.9062}$$

$$\bar{d} = \frac{125}{45\pi} \approx \underline{0.9054} \quad \textbf{[6]}$$

General Case

$$\boxed{\begin{aligned}\bar{d}(a) &= \frac{8}{\pi}\int_0^1 r\int_0^a\frac{q}{a^2}|q-r|\,E\left(\frac{\theta}{2}\,\middle|\,-\frac{4rq}{(q-r)^2}\right)\Big|_{\pi}^{0}dq\,dr\\ \bar{d} &\approx \frac{8a}{\pi nm}\sum_{r=0}^{1}r\sum_{q=0}^{a}\frac{q}{a^2}|q-r|\,E\left(\frac{\theta}{2}\,\middle|\,-\frac{4rq}{(q-r)^2}\right)\Big|_{\pi}^{0}\end{aligned}}$$

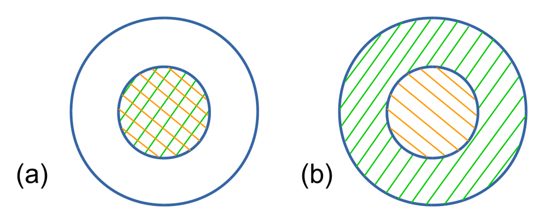
Unit Ball

Singularity

$$\bar{d} = \int_0^1 3r^2\int_0^{\pi}r\,\frac{\sin\theta}{2}\,d\theta\,dr = \int_0^1 3r^3\,dr = \boxed{\frac{3}{4}}$$

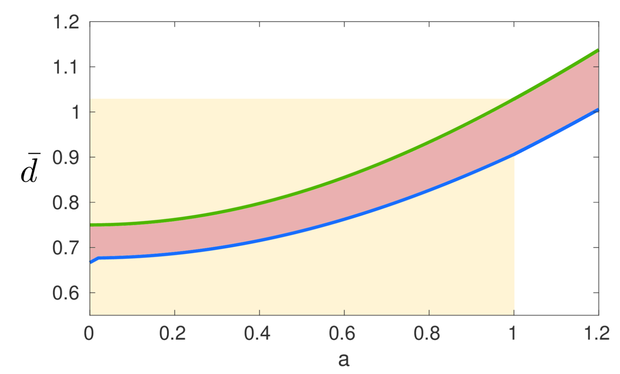
Alignment

$$\bar{d} = \int_0^1 3r^2\int_0^1 3q^2\int_0^{\pi}\frac{\sqrt{r^2+q^2-2rq\cos\theta}\,\sin\theta}{2}\,d\theta\,dq\,dr$$



$$\int_0^{\pi}\frac{\sqrt{r^2+q^2-2rq\cos\theta}\,\sin\theta}{2}\,d\theta \overset{(r,q\geq 0)}{=} \frac{(r+q)^3-(r-q)^2|r-q|}{6rq} \overset{(r\geq q)}{=} \frac{6r^2q+2q^3}{6rq}$$

$$\bar{d} = 2\int_0^1\int_0^1\frac{9q^2r^2(6r^2q+2q^3)}{6rq}\,dq\,dr = \boxed{\frac{36}{35}}$$



average distances for unit disk and unit ball

General Case

$$\bar{d}(a) = \int_0^1 3r^2\int_0^a\frac{3q^2}{a^3}\int_0^{\pi}\frac{\sqrt{r^2+q^2-2rq\cos\theta}\,\sin\theta}{2}\,d\theta\,dq\,dr$$

$$\bar{d}(a) = d_a(a) + d_{a,1}(a)$$

$$d_a(a) = 2\int_0^a\int_0^a\frac{9q^2r^2}{a^3}\frac{(6r^2q+2q^3)}{6rq}\,dq\,dr = \frac{36}{35}a^4$$

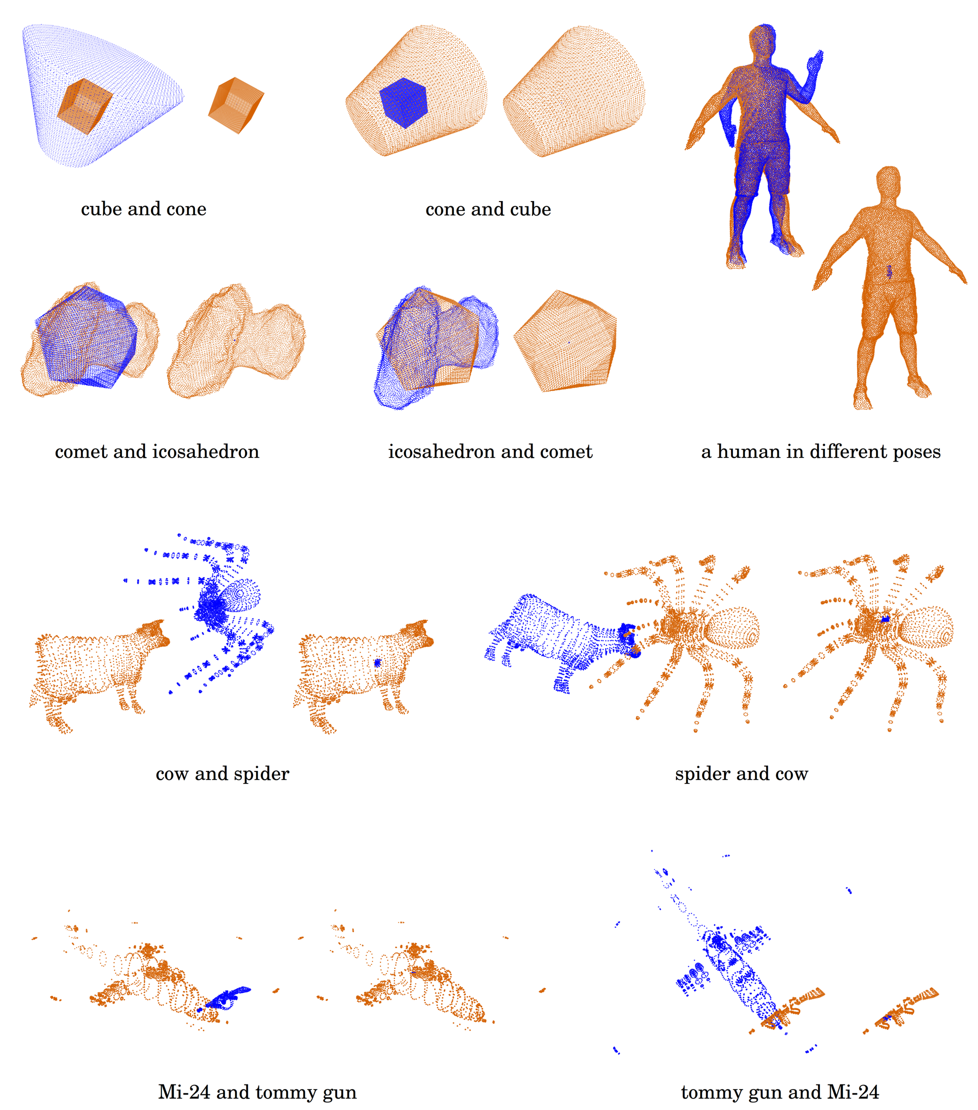
$$d_{a,1}(a) = \int_a^1\int_0^a\frac{3rq}{2a^3}(6r^2q+2q^3)\,dq\,dr =$$

$$\int_a^1\frac{3}{5}r\,(5r^2+a^2)\,dr = -\frac{21a^4-6a^2-15}{20}$$

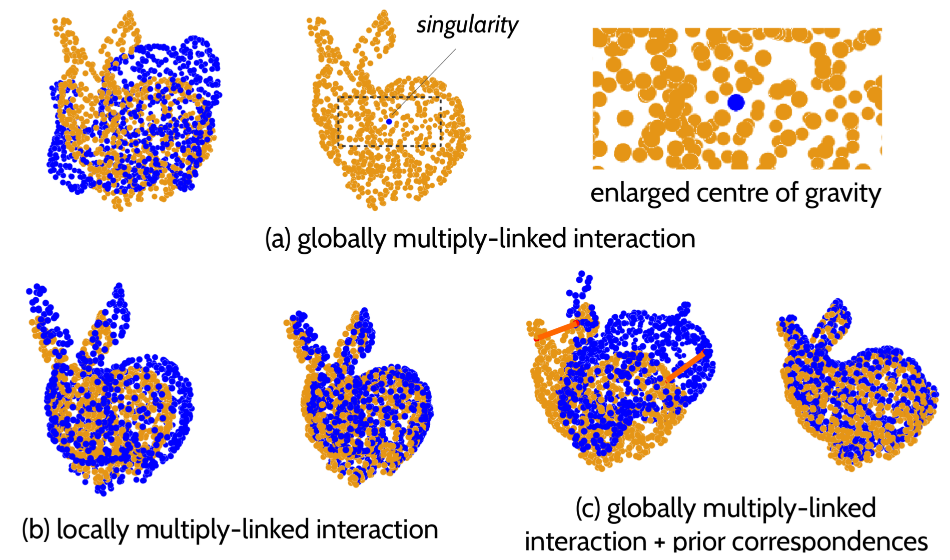
$$\boxed{\bar{d}(0) = \frac{3}{4}} \qquad \boxed{\bar{d}(1) = \frac{36}{35}}$$

$$\boxed{\bar{d}(a) = \frac{36}{35}a^4 - \frac{21a^4-6a^2-15}{20} = \frac{26\frac{1}{4}+10\frac{1}{2}a^2-\frac{3}{4}a^4}{35}}$$

OTHER SHAPES (TESTS WITH REAL DATA)



Selected registration results by a 7 DoF version of BH-RGA [8] with different shape combinations. For every pair of shapes, initialisation is on the left, and the alignment is on the right.



If template scaling is allowed, globally multiply-linked gravitational point set alignment results in a singularity (a). To remedy scaling resolution, point interactions can be restricted to local neighbourhoods (b) or prior matches with high masses can be embedded (c).

WHAT IS A SCALE SINGULARITY?

- A singularity is a state when a template collapses to a single point, and scale ~0.
- A singularity can arise in 7DoF point set alignment approaches due to numerical reasons (*e.g.*, it can be observed in CPD [2]).
- In gravitational approaches, a singularity arises because of the underlying form of the gravitational potential energy (GPE) functional. We found that the GPE of a singularity is always smaller than the GPE of the optimal alignment under allowed scaling, and there are no equienergetic states with singularities.

ELLIPTIC INTEGRALS

… of the second kind in the Legendre form are integrals of type

$$E(\phi,k) = \int_0^{\phi}\sqrt{1-k\sin^2\theta}\,d\theta$$

- Elliptic integrals arise in the study of the arc length problem for ellipses.
- As a rule, they cannot be simplified and analytically evaluated.

APPENDIX

★
$$\lim_{a\to 1}\frac{1}{\pi}\int_0^{\pi}\sqrt{a^2+1^2-2a\cos\theta}\,d\theta = \frac{4}{\pi}$$

⬟
$$\lim_{a\to 0}\frac{1}{2}\int_0^{\pi}\sqrt{1+a^2+2a\cos\theta}\,\sin\theta\,d\theta = 1$$

◆
$$\int\sqrt{a^2+1-2a\cos\theta}\,d\theta = \int\sqrt{a^2+1+\frac{4a(1-\cos\theta)}{2}-2a}\,d\theta =$$

$$\int\sqrt{4a\sin^2\frac{\theta}{2}+\frac{(a-1)^4}{(a-1)^2}}\,d\theta = \sqrt{(a-1)^2}\int\sqrt{\frac{4a\sin^2\frac{\theta}{2}}{(a-1)^2}+1}\,d\theta = 2|a-1|\,E\left(\frac{\theta}{2},-\frac{4a}{(a-1)^2}\right)$$

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