Ordered Resolution

Resolution Strategies as Decision Procedures

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Resolution Decision Procedures

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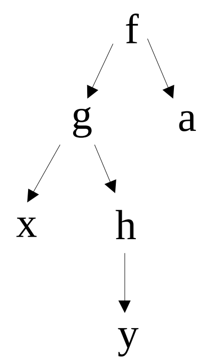
Contents

Definitions

- Term
  Variable, constant and function
- Atom
  If t_1, ...t_n are terms and P is n−place predicate
  then $P(t_1, ..., t_n)$
- Literal
  An atom or an atom preceded by a negation sign
- Clause
  Finite set of literals. Empty clause: $\perp$

Definitions – contd...

- Term depth
  - Example: f(a, g(x, h(y))) –> 3



- Substitution
  - V=set of variables, T=set of terms
  - Mapping $\sigma: V \rightarrow T$
  - σ is ground substitution if there are no variables in T

Definitions – contd...

- Instance
  - E_1 is an instance of E_2 if $E_1 = E_2\sigma$
  - Ground instance: if $E_2\sigma$ has no variables
- More General Than (mgt)
  - E_1 is mgt E_2 if there exists a substitution σ, such that $E_1\sigma = E_2$
  - Substitution ρ is mgt θ, if for some σ, $\rho\sigma = \theta$

Definitions – contd...

- Unification
  - A set of expressions M is unifiable by σ if $E_i\sigma = E_j\sigma$ for all $E_i, E_j \in M$.
  - σ is called most general unifier (mgu) if for every other unifier ρ of M: σ mgt ρ

Definitions – contd...

- Condensation/Condensed

$$p(x,a) \vee p(x,y)$$
$$\Downarrow \sigma$$
$$\subseteq \quad \Uparrow$$
$$p(x,a) \vee p(x,a)$$
$$\Leftrightarrow$$
$$p(x,a)$$

  - C' is a condensation of C, if
    - $C\sigma$ = C' (up to elimination of duplicates) and
    - C' $\subset$ C
  - If no such condition then C is condensed

Converting to clause form

- Why do this?
  - Because resolution works on clauses
- An arbitrary class of formulae is converted to set of clauses by
  - Moving all Negations inside
  - Moving all ORs inside and ANDs outside (conjunctive normal form)
  - Skolemization (Removing $\exists$ quantifier)
- A=formulae, c(A) is clause form of A

Example

- $\neg(p\wedge/\vee q) -> \neg p \vee/\wedge \neg q$
- $\neg\exists$ x,y,... p...–> $\forall$x,y,... $\neg$p...
- $\neg\forall$ x,y,... p...–> $\exists$x,y,... $\neg$p...
- $(p \wedge q) \vee r -> (p \vee r) \wedge (q \vee r)$
- Skolemization:
  - $\exists x,\forall y,\exists z\ p(x,y) \vee q(y,z) -> \forall y\ p(a,y) \vee q(y,f(y))$

Resolution

- Resolution R operates on set of clauses S by forming new clauses (resolvents)
- Resolution inference rule
  - If A, A', C & D are clauses then:

$$\frac{C \vee A \qquad D \vee \neg A}{C \vee D}$$

Ground clauses

$$\frac{C \vee A \qquad D \vee \neg A'}{(C \vee D)\sigma}$$

σ=mgu(A, A')

Resolution

- $R^0(S)$ is the condensation of S (sometimes =S).
- R(S) is S $\cup$ some resolvents of S
- $R^{n+1}(S)$ is defined as $R(R^n(S))$.
- Goal: generate an empty clause if S unsatisfiable.

Completeness

- Completeness of R:
  produces $\bot$ if S is unsatisfiable

$\bot \in R^n(S)$ for some $n \geq 0 \leftrightarrow$ S is unsatisfiable

- Halting conditions: if R produces
  - $\bot$, S is unsatisfiable
  - no new clause, S is satisfiable

$[(R^n(S) \approx R^{n+1}(S))$ and $\bot \notin R^n(S))]$ for some $n \geq 0 \rightarrow$
S is satisfiable

Decision Procedures

- Decision problem for a class C of formulae is the problem of finding an algorithm, called a decision procedure, which determines for each formula F in C whether F is satisfiable.
- C is solvable if it has a decision procedure
- Example of solvable classes

Solvable classes − Examples

- Monadic
  - Only unary predicates
  - Example: $\forall x, \exists y, \forall z\ p(x) \vee p(y) \rightarrow p(z),\ p(x) \rightarrow q(y)$
- Herbrand
  - Conjunction of one literal
- Bernays−Schoenfinkel
  - Prefix $\exists^* \forall^*$
  - Example: $\exists x_1 x_2 y_1 y_2\ \forall z_1 z_2$ ....

Solvable classes – Examples

- Ackermann
  - Prefix ∃* ∀∃*
- Gödel
  - Prefix ∃* ∀ ∀∃*
- All solvable classes have either
  - restriction on prefixes (quantifiers) or
  - restriction on what comes after the quantifiers

Till now...

- Definitions
- Resolution
- Solvable classes − Examples
- Resolution decision procedure ←
- Restrictions on resolution
- Ordered Resolution
- Ordered Resolution on one solvable class

Resolution Decision Procedure

- A resolution decision procedure R on class C of first order formula is complete if

  $[F \in C \ \& F \text{ satisfiable}] \rightarrow [R^{p}(c(F)) = R^{p+1}(c(F))$ for some $p \geq 0$

- R will always halt if
  - it produces empty clause or
  - fails to generate new clauses

Examples

- Are all resolution strategies decision procedure?
- Example 1:

$F = \forall x \exists y \, (P(x) \vee P(y)) \wedge (\neg P(x) \vee \neg P(y)))$

After skolemization

$c(F) = \{P(x), P(f(x))\}, \{\neg P(x), \neg P(f(x))\}$

Example 1

$$c(F)=\{P(x),P(f(x))\} , \{\neg P(x),\neg P(f(x))\}$$

- Applying R to c(F) produces following resolvents:

$\{P(x),\neg P(f(f(x)))\}, \{P(x),P(f(f(f(x))))\}, ...$

- R will never halt and therefore it is not a decision procedure

Examples

- Example 2:

$E=\forall x_1, x_2, x_3 ((P(x_1,x_2) \vee Q(x_2,x_3) \vee R(x_1,x_3)) \wedge (\neg R(x_1,x_2) \vee Q(x_2,x_3) \vee \neg P(x_1,x_3)))$

- After Skolemization

$c(E)=\{P(x_1,x_2),Q(x_2,x_3),R(x_1,x_3)\} , \{\neg R(x_1,x_2), Q(x_2,x_3),\neg P(x_1,x_3)\}$

Example 2

$$c(E)=\{P(x_1,x_2),Q(x_2,x_3),R(x_1,x_3)\}\ ,\ \{\neg R(x_1,x_2),\ Q(x_2,x_3),\neg P(x_1,x_3)\}$$

- Applying R to c(E) produces following resolvents:

$\{P(x_1,x_2), Q(x_2,x_3), Q(x_3,x_4), Q(x_4, x_5), R(x_1, x_5)\}, \ldots,$

$\{P(x_1,x_2), Q(x_2,x_3), Q(x_3,x_4), \ldots, Q(x_{n-1}, x_n), R(x_1, x_n)\}$

Result from Examples

- If there is no restriction on resolution then, we have resolvents which have:
  - unbounded nesting of functions (Depth)
  - unbounded number of arguments (Span)
- Solution ?
  - Apply restriction to depth and span

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- Solvable classes − Examples
- Resolution decision procedure
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Restrictions

- Restriction on depth − Ordering
  - Define an ordering on literals in the clause set
  - Unify those literals first which are maximal
- Properties of ordering
  - Noetherian
  - Stable under substitution

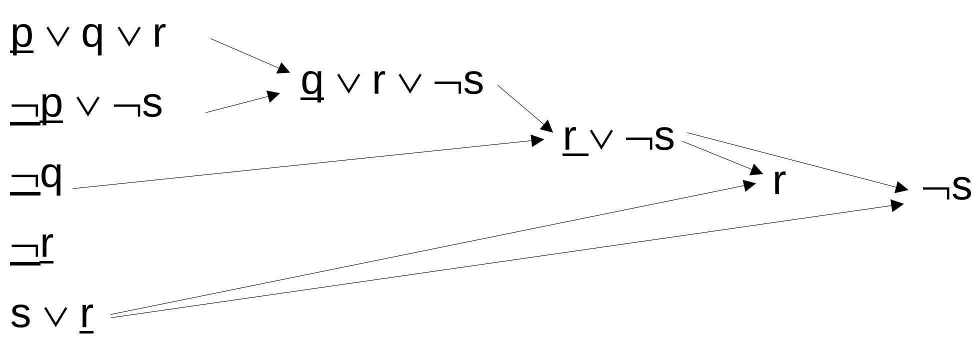
Ordering

- Select an ordering satisfying both
- Definition
  - $p(f(x_1, ..., x_m)) < q(g(x_1, ..., x_n))$ if $m < n$
  - $p(x) < q(f(x_1, ..., x_n))$
  - $p(a) < q(g(x_1, ..., x_n))$

Ordering

- Example:
  - Let p,q,r,s be atoms with ordering

    $(\neg p > p > \neg q > q > \neg r > r > \neg s > s)$



Condensation

- Restriction on span
  - Keep only condensed clauses
  - Example:

    p(c)$\vee$p(b)

    p(a)$\vee$p(b) → p(b)$\vee$p(c)$\vee$p(e)

    $\neg$p(a)$\vee$p(c)$\vee$p(e) → p(b)$\vee$p(c)$\vee$p(e)
  - can drop the resolvent as its condensed form p(c)$\vee$p(b) is already in the clause set

Ordered Resolution

- So what is ordered resolution?
- It is a resolution decision procedure with
  - ordering on literals and restrictions on inference rules
  - condensation
- It is proved that ordered resolution is complete

Solvable class

- Why ordered resolution terminates for solvable classes?
- Example: Ackermann class (Prefix $\exists^* \forall\exists^*$)
  - $\exists x_1 x_2 \forall x_3 \exists x_4 x_5 \;\; p(x_1,x_2) \vee q(x_3,x_4) \wedge \neg p(x_1,x_3) \vee r(x_5) \vee \neg q(x_2,x_4)$
  - After Skolemization (at most one variable)
  - $\forall x_3 \;\; p(a,b) \vee q(x3,f(x_3)) \wedge \neg p(a,x_3) \vee r(g(x_3)) \vee \neg q(b,f(x_3))$

Ackermann Class

- Various conditions
  - p(a,b) ¬p(a,b) − simple
  - p(x, f(x)) ¬r(y, f(y)) − not unifiable
  - p(x, f(x)) ¬p(f(y), y) − not unifiable
  - p(f(x), x) ¬p(f(y), y) − $y \rightarrow x$
  - <u>p(x)</u>∨..^{x}..∨q(g(x))..^{x}. ..^{y}..<u>¬p(f(y))</u>..^{y}.. − $x \rightarrow f(y)$=>..^{y}.. ∨q(g(f(y))∨..^{y}.. (increase in term depth)

End