Automata Theory for Presburger Arithmetic Logic

References from

Introduction to Automata Theory, Languages & Computation

and

Constraints in Computational Logic

Theory & Application

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Structure of Presentation

- Introduction of notions and automata theory
  - Formal definitions and examples
- Presburger Arithmetic
- Translation from Presburger formulas to finite automata
  - Automata associated with equalities and inequalities
- Extensions….

Introduction to notions

- Strings
  - Prefix
  - Suffix
  - Concatenation
    - Empty string

Alphabets

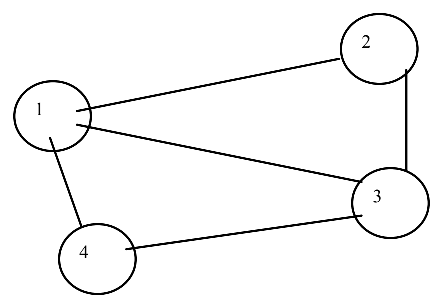
- An Alphabet is a finite set of symbols

Language

- Σ ={a} then Σ*={e, a, aa,aaa,.,.,.,.,.,}

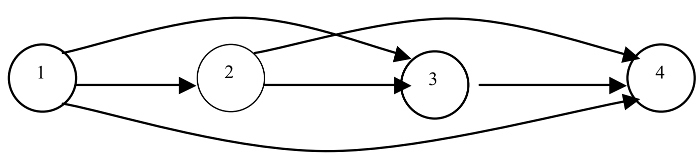
Graph

- Mathematically written as G (V, E)
- Path
  - v1, v2,.,.,., vk, k>=1, such that there is an edge (vi,vi+1) for each i, 1 <= i < k
- Cycle
  - if vi=vk the path is a cycle



Directed Graph

- Also called as digraph is a sequence of vertices and an ordered pair of edges called arcs
- vi→v(i+1) is an arc for each i, 1<=i<k
- An arc from x to y is denoted by x → y. In the arc x → y, x is the predecessor of y and y is the successor of x



Tree

- There is one vertex called the root that has no predecessors and from which there is a path to every vertex
- Each vertex other than the root has exactly one predecessor
- The successors of each vertex are ordered from the left

Inductive Proofs

- Principle of Mathematical induction is as follows
  - Basic Step P(0)
  - Inductive Step P(n) implies P(n+1) for n>=0
- Example
  - To prove that $0+1+2+3+\ldots+n=n(n+1)/2$

Proof

- We could argue like this: For n = 0, the result is clearly true i.e., 0=0.2/2=0

Now suppose that for integer n >= 1,

0+1+2+….+n = n(n+1)/2

By induction hypothesis

$$(1+2+\ldots n)+(n+1)=n\frac{(n+1)}{2} + (n+1)$$

$$= \frac{n(n+1)+2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2} = \frac{(n+1)(n+1+1)}{2}$$

and so the result follows by induction.

Sets

- A set is a collection of objects without repetition.

Set notation

- {x in A | p(x)} "The set of objects x such that p (x) is true".

Operations On Sets

- Union '$\cup$' $\{x | x \in A \text{ or } x \in B\}$
- Intersection '$\cap$' $\{x | x \in A \text{ and } x \in B\}$
- Difference '—' $\{x | x \in A \text{ and } x \notin B\}$
- Product ' X' $\{(a,b) \mid a \in A \text{ and } b \in B\}$
- Power Set is the set of all subsets of A. Denoted by 2^A

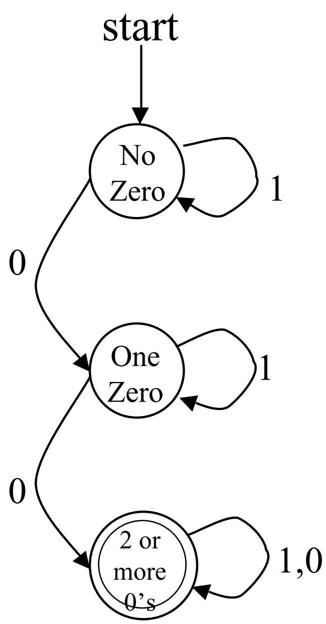
Relation

- Domain
- Range

Properties of Relations

Reflexive	if aRa for all a in set	{(a,a),(b,b)}
Irreflexive	if aRa is false for all a in set	{(a,b)}
Transitive	if aRb and bRc imply aRc	{(a,b),(b,c),(a,c)}
Symmetric	if aRb implies bRa	{(a,b),(b,a)}
Asymmetric	if aRb implies bRa is false	{(a,b),(b,c)}

Finite State System



- Linear Control Systems
  - Control Mechanism of an elevator
- Telecommunication
  - Pay phone controller
- Computer Science
  - Compilers
  - Switching circuit (control unit of a computer)

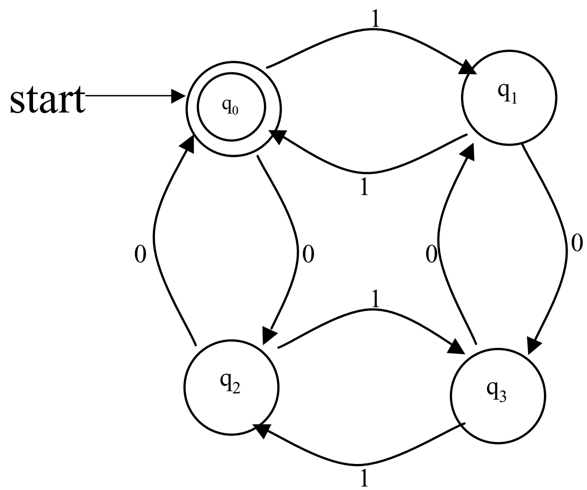
Finite Automaton

- Formally a finite automaton is defined by a 5-tuple $(Q, A, \delta, q_{0,} Q_f)$
  - Q is the finite set of states.
  - A is the input alphabet.
  - δ is the transition relation mapping Q x A to Q,
    - i.e., $\delta(q , a)$ is a state for each state q and input symbol a.
  - $q_0 \in Q$ is the initial state,
  - $Q_f \subseteq Q$ is the set of Final states.

Transition Diagram

- A directed graph that represents finite automaton is called Transition Diagram.
- States of the automaton
- Transition from one state to another

An Example



- In this transition diagram of finite automaton, the control is at q_0 if and only if there are both an even number of 0's and even number of 1's in the input sequence

$$M=(Q, A, \delta, q_0, Q_f).$$

- Where $Q = \{q_0, q_1, q_2, q_3\}$ and $A=\{0,1\}$, $Q_f=\{q_0\}$ and δ is shown in the following table.

Applying 110101 as input sequence to M. We get q_0 .

States	Inputs 0	1
q_0	q_2	q_1
q_1	q_3	q_0
q_2	q_0	q_3
q_3	q_1	q_2

Finite Automaton Continued…

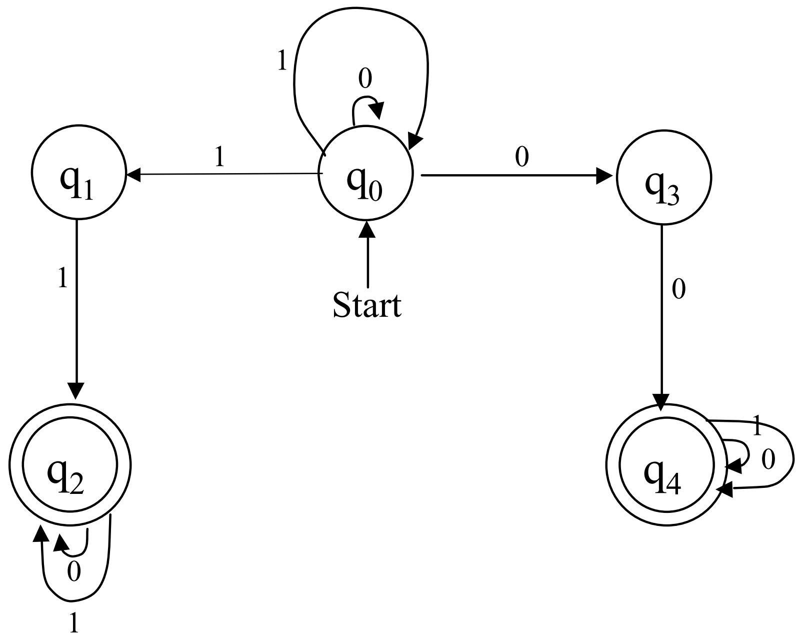
- For every word $w \in A^*$ if i is a natural number which is smaller or equal to the length of w we can say $w(i)$ is the i^{th} letter of word w
- A run of the automaton (Q, Q_f, q_0, δ) on a word $w \in A^*$ is a state sequence $\rho \in Q^*$ such that $\rho(0) = q_0$, and if $\rho(i) = q$, $\rho(i+1) = q'$ then $(q, w(i), q') \in \delta$

- Successful Run
  - A successful run ρ on w is a run such that $\rho(n+1) \in Q_f$ where n is length of w

Non Deterministic Finite Automata

- Definition
- An Example

I/P sequence 01001 is accepted by the NFA. The sequence of transitions is through the states q_0, q_0, q_0, q_3, q_4, q_4



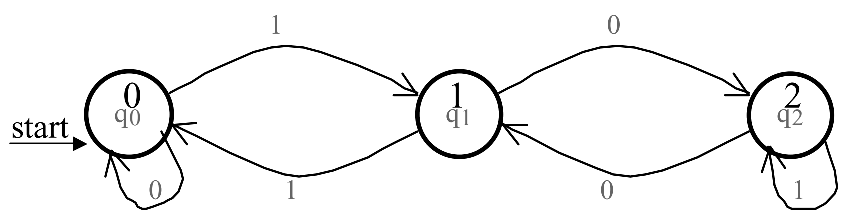
- Difference

Finite Automata with output

- **Moore Machine**

Represented by tuple (Q, A, δ, q_0 ,Q_f,λ, Δ,) where Q, A, δ, q_0, Q_f have their usual meanings as in FA. Δ is the output alphabet and λ is mapping from Q to Δ giving the output associated with each state

Machine calculating residue mod 3



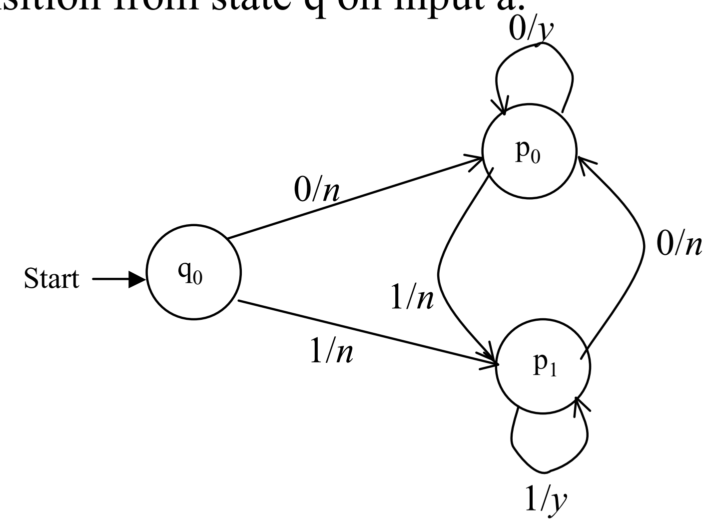
On input 1010 number of states entered is q_0, q_1, q_2 , q_2 , q_1 giving output sequence 01221.

i.e., *e* has residue 0, 1 has 1, 2_{10} has 2, 5_{10} has 2 and 10_{10} has 1.

Mealy Machine

Represented by $(Q, A, \delta, q_0, Q_f, \Delta, \lambda)$ where Q, A, δ, q_0, Q_f have their usual meanings as in Moore machine. Except that λ is mapping from Q X A to Δ. i.e., λ (q,a) gives the output associated with the transition from state q on input a.

Response of M to input 01100 is *n, n, y, n, y* with the sequence of states being entered q_0, p_0, p_1, p_1, p_0, p_0,



Machine $M(\{q_0,p_0,p_1\}, \{0,1\}, \delta, \{q_0\}, \{y,n\}, \lambda)$

Presburger Arithmetic

- Basic terms
  - $x+x+1+1+1$ is an example which is a basic term and can be abbreviated as $2x+3$.
- Atomic formula
  - are equalities and inequalities between basic terms. For instance $x+2y=3z+1$ is an atomic formula
- Formulas
  - of the logic are the first-order formula built on the atomic formulas.

- <u>Connectives</u>
  - Conjunction ($\wedge$)
  - Disjunction ($\vee$)
  - Negation ($\neg$)
  - Existential Quantification ($\exists$)
  - Universal Quantification ($\forall$)
- An example of a logical formula is $\forall x\ (\exists y\ x=2y \vee \exists y\ x=2y+1)$ Expressing in words as, every integer is even or odd.

- Free Variables
  - Variables which are not quantified are called free variables

- Solution
  - A solution of a formula $\varphi(x_1, x_2, \ldots, x_n)$ is an assignment of $x_1, x_2, \ldots, x_n$ in N which satisfies the formula. For Instance $\{x= 0; y = 2; z =1\}$ is a solution of $x+2y=3z+1$ and every assignment $\{x= n\}$ is a solution of, $\exists y\ (x=2y \vee x=2y+1)$

Translation from Presburger Formula to Finite Automata

- Automata associated with equalities

For every basic formula

$$a_1x_1+a_2x_2+\ldots\ldots+a_nx_n = b \text{ (where } a_1,a_2,\ldots\ldots,a_n, b\in \mathbf{Z})$$

The automaton is constructed by saturating the set of transition and the set of states, initially set to $\{q_b\}$ using the inference rule:

$$\frac{q_c\in Q}{q_d\in Q,\ (q_c,\theta, q_d)\in \delta} \quad \text{If } \begin{cases} a_1\theta_1+\ldots\ldots a_n\theta_n =_2 c \\ d =(c- a_1\theta_1 \ldots - a_n\theta_n)/2 \\ \theta\in \{0,1\}^n \text{ encodes } (\theta_1,\ldots\ldots, \theta_n) \end{cases}$$

For every state $q_c \in Q$, computing the solutions $(\theta_1,\ldots,\theta_n)$ of $a_1x_1+a_2x_2+\ldots\ldots+a_nx_n = c$ modulo 2 and add the state q_d and the rule $q_c \xrightarrow{\theta} q_d$ where $d =(c- a_1\theta_1 \ldots - a_n\theta_n)/2$

- Considering the equation $x+2y = 3z+1$

Here $b=1$ we have $q_1 \in Q$. Computing the solution modulo 2 of the above equation we get $\{(0,0,1), (0,1,1), (1,0,0), (1,1,0)\}$.

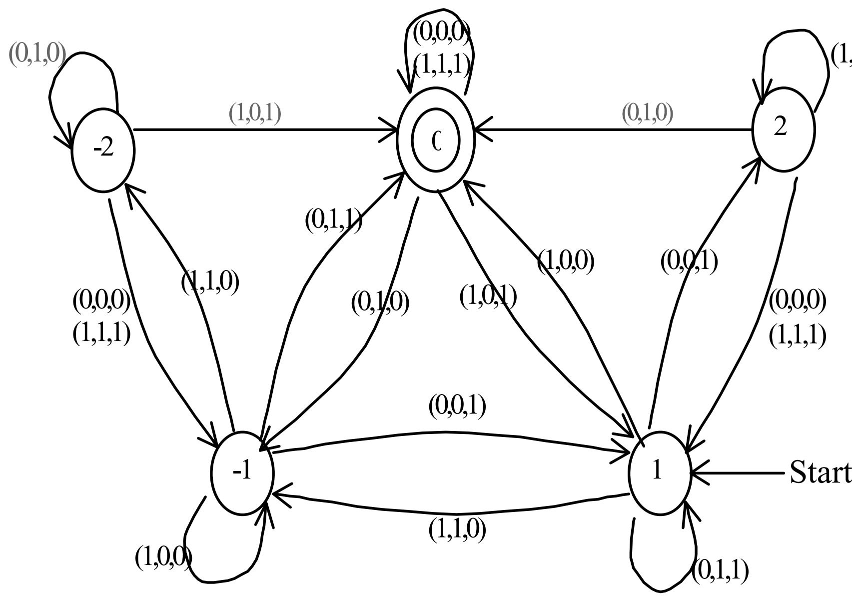
Computing the new states by

- $d= (1-0-0+3)/2=2$ i.e., q_2
- $d= (1-0-2+3)/2=1$ i.e., q_1
- $d= (1-1-0-0)/2= 0$ i.e., q_0
- $d= (1-1-2-0)/2= -1$ i.e., q_{-1}
- q_2, q_1, q_0, q_{-1} are $\in$ of Q.

- The new transitions

$q_{1\,(0,0,1)} \rightarrow q_2$ $q_{1(0,1,1)} \rightarrow q_1$ $q_{1(1,0,0)} \rightarrow q_0$ $q_{1(1,1,0)} \rightarrow q_{-1}$

- Automaton for equation x+2y = 3z+1



- In this automaton our initial state is q_1 and final state is q_0 indicated by double circle. Traversing the transition diagram from initial state q_1 through q_{-1}, q_{-2}, q_0, q_1, q_2 to the final state q_0 we get the word

#	$q_1 \rightarrow q_{-1}$	$q_{-1} \rightarrow q_{-2}$	$q_{-2} \rightarrow q_0$	$q_0 \rightarrow q_1$	$q_1 \rightarrow q_2$	$q_2 \rightarrow q_0$
1	1	1	1	1	0	0
2	1	1	0	0	0	1
3	0	0	1	1	1	0

#1,2 & 3 gives $x=15_d$, $y=35_d$ & $z=28_d$

The word is accepted by the automaton and verifies the equation $x+2y=3z+1 \Rightarrow 15+2(35)=3(28)+1$ $85=85$

Automata associated with inequalities

- Computing an automaton for inequality

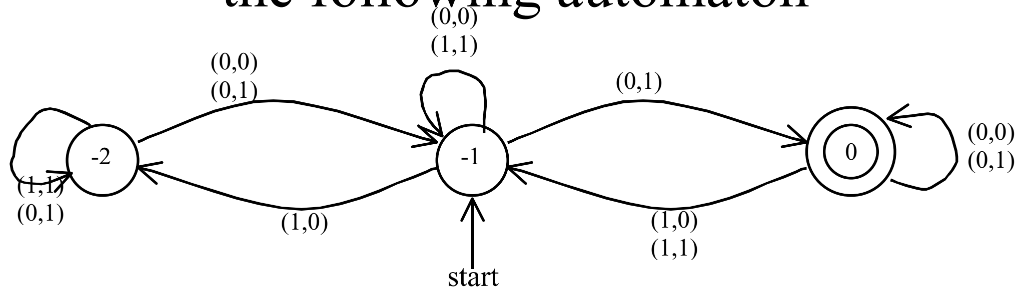
$$a_1x_1+a_2x_2+\ldots\ldots+a_nx_n \leq b$$

Starting from state q_b as initial state and from a state q_c we compute the transitions and states as:

$$q_c \xrightarrow{\theta_1\ldots\ldots\theta_n} q_d$$

with $d=\lfloor (c-\sum_{i=1}^{n} a_i\theta_i)/2 \rfloor$ $\quad Q_f = \{q_c \mid c \geq 0\}$

- Considering the in equality $2x-y \le -1$ we get the following automaton



Traversing the machine from initial state q_{-1} through q_{-1}, q_{-2}, q_{-2}, q_{-1}, q_0 to the final state q_0 we get the word

#	$q_{-1}\rightarrow q_{-1}$	$q_{-1}\rightarrow q_{-2}$	$q_{-2}\rightarrow q_{-1}$	$q_{-1}\rightarrow q_0$	$q_0\rightarrow q_0$
1	1	1	0	0	0
2	1	0	1	1	1

#1 & 2 gives $x=3_d$ & $y=29_d$

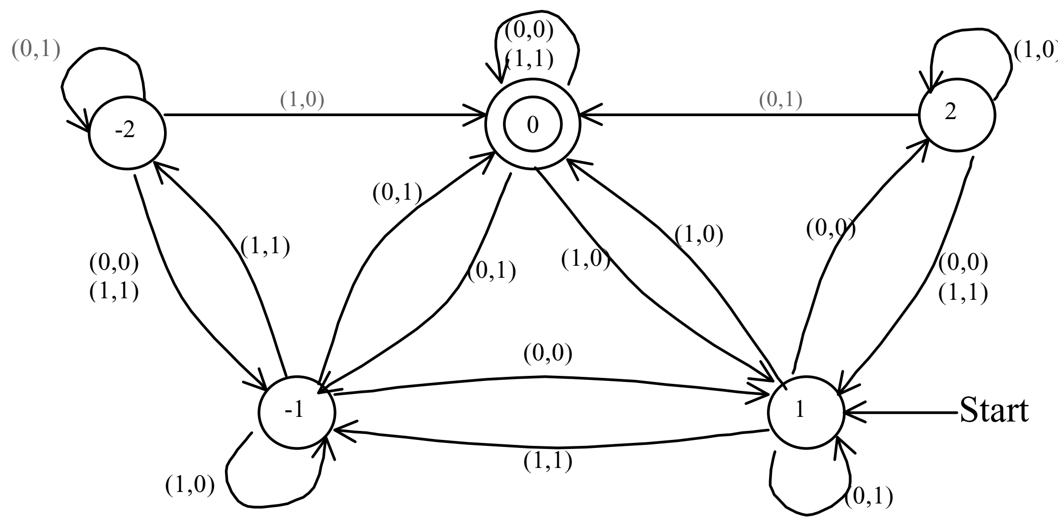
The word is accepted by the automaton and verifies the equation $2x-y \le -1 \Rightarrow 2(3) - 29 = 6 - 29 = -23 \le -1$

Closure properties

- A class of languages closed under a particular operation
- Union and intersection i.e., $A_{\varphi 1} \vee A_{\varphi 2}$, $A_{\varphi 1} \wedge A_{\varphi 2}$ of two automata $A_{\varphi 1}(\vec{x})$ and $A_{\varphi 2}(\vec{x})$ can be computed by the classical constructions.
- Negation corresponds to complement. For quantifier $\forall x\ \varphi$ the complement is $\neg\exists x \neg\ \varphi$. We only have to consider existential quantification. $A_{\exists x\ \varphi}$ is computed from A_{φ} by projection. Transitions, Initial and Final states are identical to those of A_{φ} .

Example

Automaton accepting solution for
$\exists z\ x+2y=3z+1$



Extensions...

- More expressive automata models are
  - Push down automata (Stacks)
  - Turing Machines
  - Rabin Automata