Exercise 2.1:
Three tools for proving termination have been presented in the lecture: monotone mappings, lexicographic products, and multisets. Try to prove termination of the following programs with each of them. The type of all variables is \( \mathbb{N}_+ \).

a) \begin{verbatim}
while x \neq y do
  if x > y then
    x := x - y;
  else
    y := y - x;
\end{verbatim}

b) \begin{verbatim}
while x + y > 1 do
  print(x, y);
  if x = 1 then
    y := y - 1; x := y;
  else
    x := x - 1;
\end{verbatim}

What do the programs compute?

Exercise 2.2:
Let \( p_1, p_2, \ldots \) denote the sequence of the prime numbers 2, 3, \ldots; and let the reduction system \((\mathbb{N}_+, \rightarrow)\) consist of all the reductions \( n \cdot p_{k+1} \rightarrow n \cdot p_k \cdot \ldots \cdot p_1 \) where \( k \) and \( n \) are in \( \mathbb{N}_+ \). Show that \( \rightarrow \) is terminating.

Exercise 2.3:
Given a partial ordering \((A, >)\), prove the following properties of \( >_{\text{mul}} \) on \( M(A) \):

a) Singletons compare like their elements: \( \{a\} >_{\text{mul}} \{b\} \) iff \( a > b \)

b) \( >_{\text{mul}} \) is stable under insertion of equal elements: \( M >_{\text{mul}} M' \) implies \( M \cup N >_{\text{mul}} M' \cup N \).

Exercise 2.4:
Consider two terminating reduction relations \( \leadsto \) and \( \rightarrow \) over some set \( A \), and their union \( \rightarrow := \leadsto \cup \rightarrow \). Assume that \( \leadsto \) commutes over \( \rightarrow \); that is, whenever \( u \leadsto v \leadsto w \), then there exists some \( v' \) such that \( u \leadsto v' \rightarrow^* w \). Show the following:
a) More generally, \( u \rightarrow^* v \rightsquigarrow w \) implies \( u \rightsquigarrow v' \rightarrow^* w \) with some suitable \( v' \).
   Hint: induction on the length \( i \) of the \( \rightarrow \)-chain

b) The combined relation \( \rightarrow \) is terminating.
   Hint: induction with \( > = \rightsquigarrow^+ \)

Put your solution into the mail box at the door of room 617 in the MPI building (46.1) before the upcoming lecture, or submit it to the lecturer at the beginning of that lecture.

Note: In case of group work, write the names of all group members (not more than three!) on a single solution sheet. Do not submit several identical solution sheets.