More on Light Graphs in 3-Connected Plane Graphs Without Triangular or Quadrangular Faces

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Abstract

Intuitively, a graph $H$ is light in a class $G$ of graphs when $H$ is a subgraph of some member of $G$ and for every $G \in G$ if $G$ contains $H$ as a subgraph, there is also an occurrence $H'$ of $H$ in $G$ such that the degrees of vertices of $H'$ in $G$ are bounded by some constant. Characterizing light graphs in various subclasses of plane graphs attracted many researchers in recent years. We focus on the class $\mathcal{P}(3, 5)$ of 3-connected plane graphs with faces of length at least 5. The class was previously studied by S. Jendrol’ and P. J. Owens [6] who showed that every $k$-cycle, $k > 5$ and $k \neq 14$, is not light in $\mathcal{P}(3, 5)$. They also proved that every block of any light graph in $\mathcal{P}(3, 5)$ has at most 18 vertices.

In this paper we show that every block of a light graph in $\mathcal{P}(3, 5)$ is a bridge, a cycle $C_5$ or a cycle $C_{14}$. Very recently, T. Madaras [9] showed that $K_{1, 3}$ is light in $\mathcal{P}(3, 5)$. This is an immediate corollary from more general result stating that in every planar graph with faces of length $\geq 5$ and vertices of degree $\geq 3$ there exists a vertex of degree at most 3 with neighbors $a, b, c$ such that $\deg(a) = \deg(b) = 3$ and $\deg(c) \leq 4$. We give a new, much shorter proof of the latter result.

1 Introduction

One of the most well known facts concerning planar graphs states that every planar graph contains a vertex of degree at most 5. In 1955 Kotzig [7] showed that every 3-connected planar graph contains an edge for which the sum of degrees of its end-vertices is at most 13. Erdős conjectured that this property is valid also for planar graphs with vertices of degrees at least 3 and it turned out to be true due to Borodin [1]. We can put these results in a more general context. Consider the following definition.

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Definition. Let $\mathcal{G}$ be a class of graphs and let $H$ be a graph isomorphic to a subgraph of at least one member of $\mathcal{G}$. We say that $H$ is light in class $\mathcal{G}$ if there exists a constant $c$ such that every graph $G \in \mathcal{G}$ which has a subgraph isomorphic to $H$ also contains a subgraph $H'$ isomorphic to $H$ and such that, for every vertex $v \in V(H')$ we have $\deg_G(v) \leq c$. The smallest constant $c$ for which this condition holds is denoted by $\varphi(H, \mathcal{G})$.

Thus, we can say that a vertex is light in the class of all planar graphs and an edge is light in the class of 3-connected planar graphs. Star $K_{1,n}$ shows that an edge is not light in planar graphs. Moreover, apart from a single vertex, there is no more connected light graphs in the class of all planar graphs. Surprisingly the situation is totally different in the class of 3-connected planar graphs. Fabrici and Jendrol’ [3] showed that every path is light and any other graph is not light in this class. It is an interesting problem to characterize light graphs in other classes.

Among most natural subclasses of 3-connected planar graphs are classes $\mathcal{P}(\delta, \rho)$ of 3-connected planar graphs with minimum degree $\delta$ and minimum face size $\rho$. The classes are nonempty only for $(\delta, \rho) \in \{(3, 3), (3, 4), (4, 3), (3, 5), (5, 3)\}$. Except for paths, there is no more light graphs neither in class $\mathcal{P}(3, 4)$ [4] nor in class $\mathcal{P}(4, 3)$ [2]. The main open problem in this area is to characterize light graphs in $\mathcal{P}(3, 5)$ and $\mathcal{P}(5, 3)$. There are some partial results. Jendrol’ and Madaras [5] proved that star $K_{1,r}$ is light in $\mathcal{P}(5, 3)$ if and only if $r \in \{3, 4\}$. For $\mathcal{P}(5, 3)$ there is a classical result of Lebesgue [8] stating that $C_5$ is light in this class. More precisely, he showed that every graph in $\mathcal{P}(5, 3)$ contains a face of length 5 with four vertices of degree 3 and one vertex of degree 5. Jendrol’ and Owens [6] showed that every $k$-cycle, $k > 5$ and $k \neq 14$, is not light in $\mathcal{P}(3, 5)$. They also proved that every block of any light graph in $\mathcal{P}(3, 5)$ has at most 18 vertices. We continue their work by showing that every block of a light graph in $\mathcal{P}(3, 5)$ is a bridge, cycle $C_5$ or cycle $C_{14}$. Nevertheless, it is still not known whether $C_{14}$ is light in $\mathcal{P}(3, 5)$. We also prove that $K_{1,3}$ is light in $\mathcal{P}(3, 5)$ and $\varphi(K_{1,3}, \mathcal{P}(3, 5)) = 4$. This is a corollary from a more general result stating that in every planar graph with faces of lengths $\geq 5$ and vertices of degrees $\geq 3$ there exists a vertex of degree at most 3 with neighbors $a, b, c$ such that $\deg(a) = \deg(b) = 3$ and $\deg(c) \leq 4$. The latter result was proved independently by T. Madaras [9]. In this paper we give a new, much shorter proof. Let us note that the theorem is best possible in the sense that there is an example of a graph in $\mathcal{P}(3, 5)$ with every vertex of degree 3 adjacent to certain vertex of degree 4. Moreover it follows from the paper [6] that for every $r \geq 4$ the star $K_{1,r}$ is not light in our class.

2 On Blocks of Light Graphs in $\mathcal{P}(3, 5)$

In this section we prove the following theorem.

Theorem 1. Let $H$ be a connected light graph in $\mathcal{P}(3, 5)$. Every block in $H$ is either a bridge, a cycle of length 5 or a cycle of length 14.

The proof follows from lemmas below. We start from describing an operation of connecting a graph to a triangle. Let $G$ be a plane graph with selected face $f$ and let $ABC$
be a triangle. We start from putting $G$ inside $ABC$. Then we add a new vertex $X$ in the interior of $f$ and add edges from $X$ to all vertices of degree 2 in face $f$. Finally each of $k$ vertices of degree 2 in the outer face of $G$ is connected by an edge to one of vertices $A$, $B$ or $C$ in such a way that each of triangle vertices is connected to at least $\lfloor \frac{k}{3} \rfloor$ vertices of degree 2 and the resulting graph $G'$ stays plane. We call $G'$ as $G$ connected to a triangle.

![Figure 1: Infinite class of graphs $\{A_n\}$. Each graph $A_n$ is compound of $n$ copies of bold graph joined together to form a cycle (see $A_{15}$ in Fig. 2).](image1)

**Example.** Figure 1 shows an infinite class of graphs $\{A_n\}$. Each graph $A_n$ has two faces of length $3n$. One of them is the external face. We select the other face of length $3n$. Figure 2 shows a graph $A_{15}$ from Fig. 1 connected the triangle.

![Figure 2: Connecting the graph $A_{15}$ described in Fig. 1 to triangle $ABC$. Degrees of vertices $A, B, C, X$ can be arbitrarily large when we take $A_{3k}$ for $k$ large enough.](image2)

**Lemma 1.** Let $H$ be a light subgraph in $\mathcal{P}(3, 5)$. Every block of $H$ is an outerplanar graph with every face of length $5 + 3k$ for certain $k = 0, 1, 2, \ldots$

**Proof.** Let us choose an arbitrary $M > 3 \cdot \varphi(H, \mathcal{P}(3, 5))$. Denote by $T$ the graph $A_M$ (see Fig. 1) connected to a triangle. Let $H''$ be an arbitrary triangulation of $H$. After
replacing each triangular face of $H'$ by $T$ we obtain a new graph $G$. Clearly $G \in \mathcal{P}(3,5)$. Subsequently $H$ is a subgraph of the graph $A_M$. Moreover, $H \neq A_M$ since $H$ is also a subgraph of $A_k$ for any $k > M$. That ends the proof.

Jendrol’ and Owens showed that every block of a light graph in $\mathcal{P}(3,5)$ has at most 18 vertices. Now we show a little stronger result using similar methods:

**Lemma 2.** Every block of a light subgraph in $\mathcal{P}(3,5)$ has at most 16 vertices.

![Figure 3: Infinite class of graphs $\{B_n\}$. Each graph $B_n$ is compound of $n$ copies of bold graph joined together to form a cycle.](image)

Proof. Consider a family of graphs $\{B_n\}$ described in Fig. 3. We proceed similarly as in the proof of Lemma 1 but instead of graph $A_M$ from Fig. 1 we use $B_M$ from Fig. 3. As a result we get that every light graph in $\mathcal{P}(3,5)$ is a subgraph of $B_M$. Subsequently, every block of a light graph is a subgraph of the bold graph from Fig. 3 without the vertex of degree one.

**Lemma 3.** Light subgraphs in $\mathcal{P}(3,5)$ do not contain $C_8$.

![Figure 4: Infinite class of graphs $\{D_n\}$. Each graph $D_n$ is compound of $n$ copies of bold graph joined together to form a cycle.](image)

Proof. Consider graphs $D_n$ from Fig. 4. By replacing $A_M$ by $D_M$ in the proof of Lemma 1 we get that arbitrary light graph in $\mathcal{P}(3,5)$ is a subgraph of $D_M$. It suffices to observe that for any $M > 2$ graph $D_M$ does not contain a cycle of length 8.

**Lemma 4.** Let $H$ be a light subgraph in $\mathcal{P}(3,5)$. $H$ does not contain $C_{11}$.

Proof. The proof is actually the same as the proof of lemma 3, but instead graph $D_M$ from Fig. 4 we use $E_M$ from Fig. 5. For any $M > 3$ graph $E_M$ does not contain a cycle of length 11.

It is easy to observe that Theorem 1 follows immediately from Lemmas 1, 2, 3 and 4.
3 Vertices of Low Degree with Neighbors of Low Degree

It is well known that every planar graph contains a vertex of degree at most 5. The following theorem shows that for planar graphs satisfying certain conditions much stronger invariant holds. The result was proved independently by T. Madaras [9]. Below we give much shorter proof.

Theorem 2. Let $G$ be a planar graph with every vertex of degree at least 3 and every face of length at least 5. Then $G$ contains a vertex of degree at most 3 with neighbors $a, b, c$ such that $\deg_G(a) = \deg_G(b) = 3$ and $\deg_G(c) \leq 4$.

Proof. We use the well-known discharging technique. We put a charge of $\deg_G(v) - 4$ on every vertex $v$ of $G$. Moreover, each face $q$ of $G$ obtains a charge of $|q| - 4$. Let $n, m, f$ denote the number of vertices, edges and faces of graph $G$, respectively and let $V, F$ be the set of vertices and faces of $G$ respectively. Using Euler’s formula we can easily calculate the total charge on $G$:

$$\sum_{v \in V} (\deg_G(v) - 4) + \sum_{q \in F} (|q| - 4) = 2m - 4n + 2m - 4f = -8$$

Now we move the charge from faces to vertices in such a way that each face $q$ sends $\frac{|q| - 4}{|q|}$ units of charge to every edge incident with $q$. If such an edge is incident with only one face it receives double charge. As $|q| \geq 5$, $\frac{|q| - 4}{|q|} \geq \frac{1}{5}$. Then each edge divides its charge between its ends. Let $\phi(v)$ denote the amount of charge put so far on a vertex $v$. Subsequently, every vertex $v$ has got $\phi(v) \geq \deg(v) - 4 + \frac{\deg(v)}{5}$ units of charge. We see that $\phi(v) \geq -\frac{2}{5}$ when $v$ has degree 3 and $\phi(v) \geq \frac{4}{5}$ when $v$ has degree at least 4. Finally, every vertex $v$ with charge $\phi(v) > 0$ sends $\frac{\phi(v)}{\deg(v)}$ units of charge to each of its neighbors.

Since $\frac{\phi(v)}{\deg(v)} = (\frac{8}{5} \deg(v) - 4) / \deg(v) = \frac{8}{5} - \frac{4}{5 \deg(v)}$, we see that every vertex of degree 4 sends at least $\frac{1}{5}$ units of charge and every vertex of degree at least 5 sends at least $\frac{2}{5}$ units of charge to each of its neighbors.

Let us assume that the desired vertex does not exist in $G$. Let $u$ be an arbitrary vertex of degree 3 in $G$. Either $u$ has two neighbors of degree at least 4 or $u$ has a neighbor of degree at least 5. In the first case, $u$ has got at least $-\frac{2}{5} + 2 \cdot \frac{1}{5} = 0$ units of charge. In the other case $u$ stores at least $-\frac{2}{5} + \frac{2}{5} = 0$ units of charge. Thus, every vertex in $G$ stores now nonnegative amount of charge – a contradiction. \qed
Corollary 1. $K_{1,3}$ is light in $P(3,5)$.

Let us note that T. Madaras [9] shows an example of a graph in $P(3,5)$ with every vertex of degree 3 adjacent with a vertex of degree 4. It follows that $\varphi(K_{1,3}, P(3,5)) = 4$ and that the above theorem is best possible, i.e. the constants 3, 4 cannot be improved. It is also worthwhile to point out that S. Jendrol’ and P. J. Owens [6] showed that there is no light graph in $P(3,5)$ containing a vertex of degree $\geq 4$. In particular, for any $r \geq 4$ the star $K_{1,r}$ is not light in $P(3,5)$.

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References


