Algorithms and Data Structures **Summer 2008**
K. Mehlhorn and R. Seidel **due on Mo. Sept. 8th, morning**
Homework 1

You are supposed to work on this homework on you own. If you use other sources, please say so. The homework is due on Monday, September 8th. We will collect the solutions in class. Please send an email to Victor Alvarez (alvarez@cs.uni-sb.de) so that Victor has your email address.

1. (15 points) We study another variant of the partition step in quicksort. Show that the following code partitions $a[\ell..r]$ into subarrays $a[\ell..k-1]$ and $a[k+1..r]$ where $\ell \le k \le r$ and all elements in the first subarray are smaller than $a[k]$ and all elements in the second subarray are at least as large as $a[k]$. $pickPivotPos(a,\ell,r)$ picks some number between ℓ and r (inclusive).

   $h := pickPivotPos(a,\ell,r)$ // Pick a pivot element and
   $swap(a[\ell],a[h])$ // bring it to the first position.
   $p := a[\ell]; k := \ell$ // p is the pivot and sits in $a[k]$
   for $i := \ell+1$ **to** r **do**
       /* $i > k$, $k < r$, $a[k] = p$, $a[j] < p$ for $\ell \le j < k$, $a[j] \ge p$ for $k+1 \le j < i$ */
       if $a[i] < p$ **then** // need to swap
           simultaneously move $a[i]$ *to* $a[k]$, $a[k]$ *to* $a[k+1]$, *and* $a[k+1]$ *to* $a[i]$
           k++;
       /* $i \ge k$, $k \le r$, $a[k] = p$, $a[j] < p$ for $\ell \le j < k$, $a[j] \ge p$ for $k+1 \le j \le i$ */

   You may want to use the invariants stated in the program. Don't forget that i could be equal to $k+1$, when the three-way swap is executed.

2. Solve the following recurrences. The base case is $f(x) = 1$ for $x < b$ in the first case, $x \le 2$ in the other cases.

   (a) (10 points) $f(x) = af(x/b) + x^c$ for integers a, b, and c; $b > 1$.

   (b) (10 points) $f(x) = 2f(x/2) + x\ln x$.

   (c) (10 points) $f(x) = f(x/2) + f(2x/3) + x$

   Solve one recurrence by repeated substitution and summation and the others by appealing to the general theorem treated in class.

3. (10 points) We toss a random unfair coin (head comes up with probability p and tail comes up with probability $q = 1 - p$) until head comes up. What is the expected number of tosses? Hint: you may use the identity $\sum_{i \geq 1} iq^i = q/p^2$.

4. (20 points) Consider the following chance experiment. We have m bins B_1 to B_m.

   - We throw a single ball into a random bin, i.e., the ball ends up in B_i with probability $1/m$ for any i. Let B_i be the bin in which our first ball ends up.
   - We throw more balls at random. We throw balls until we hit a bin B_j with $j \leq i$. Let X be the number of balls thrown.

   Is the following argument correct? The expected value of i is $(m+1)/2$. Therefore, the probability that a ball hits a bin B_j with $j \leq i$ in step 2 is $p = \frac{(m+1)/2}{m} = 1/2$. So by the previous exercise $\mathrm{E}[X] = 2$.

   What is the expected value of X? Hint: first compute the conditional expectation of X under the assumption that the first ball ends up in bin B_i. Then compute the expectation.

5. (20 points) Given are n items from a linearly ordered set. We want to determine the largest and the second largest. Show that this can be done with $n + \log n + O(1)$ comparisons. Prove (this is difficult) that $n + \log n - O(1)$ comparisons are necessary. Hint (for the upper bound): organize a tournament among the n items to determine the largest. Which elements can be the second largest?

Have fun with the solution!