Algorithms and Data Structures **Summer 2008**
K. Mehlhorn and R. Seidel **due on Tuesday, Sept. 30th, morning**
Homework 4

You are supposed to work on this homework on you own. If you use other sources, please say so. The homework is due on Tuesday, September 30th. We will collect the solutions in class.

1. (20 points) Mirjam Neu-Weigand, Jochen Reuter, Steffen Lsch und Sigurd Schneider observed an error on the formulation of the first exercise and provided me with the following correction.

   Let $G = (V,E)$ be an undirected graph and let $w : E \to R$ be a weight function on the edges. For this exercise, we assume that the weights are pairwise distinct. For a path p, the bottleneck edge is the ~~smallest~~ biggest weight edge on p. For a pair (u,v) of nodes, their bottleneck edge is the ~~largest~~ smallest weight bottleneck edge of any path connecting u and v.

   (a) Draw a small graph (four nodes) and indicate the bottleneck edge for any pair of nodes.

   (b) For any real λ , let $E_\lambda = \{e \in E; w(e) \leq \lambda\}$ (was $\geq$ before) be the set of edges of weight ~~at least~~ up to λ. For any pair (u,v) of nodes, let

   $$\lambda(u,v) = \cancel{\max}\min\{\lambda; \text{there is a path from } u \text{ to } v \text{ using only edges in } E_\lambda\}$$

   What is the connection between $\lambda(u,v)$ and the weight bottleneck edge of u,v of the bottleneck edge of u and v.

   (c) Prove that for any pair (u,v) of nodes, their bottleneck edge is contained in the minimum spanning tree of G.

   (d) Prove that for any pair (u,v) of nodes, their bottleneck edge is contained in the minimum spanning tree of G.

   (e) Give an algorithm that for any pair (u,v) of nodes finds their bottleneck edge.

2. (30 points) Let $G = (V,E)$ be a directed graph and w an integral weight function on the edges. For a cycle C, let $w(C)$ be the weight of C and $|C|$ be the number of edges of C. Then $w(C)/|C|$ is the mean edge weight of C. A <u>minimum mean cycle</u> is a cycle C_0 such that $w(C_0)/|C_0| \leq w(C)/|C|$ for any cycle C. Let $\lambda_0 = w(C_0)/|C_0|$.

   (a) Draw a small graph (four nodes) and indicate the minimum mean cycle.

   (b) Show that there is a minimum mean cycle that is a simple cycle.

(c) Let W be the largest absolute value of any edge weight. Derive upper and lower bounds on λ_0 in terms of W and the number of nodes n of G.

(d) Let λ be an arbitrary real number. Define the weight function w_λ by $w_\lambda(e) = w(e) - \lambda$. What is the weight of C_0 with respect to the cost function w_{λ_0}?

Show:

- If $\lambda > \lambda_0$, then (G, w_λ) has a negative cycle.
- If $\lambda = \lambda_0$, then (G, w_λ) has a zero cost cycle but no negative cycle.
- If $\lambda < \lambda_0$, then (G, w_λ) has only positive cycles.

(e) use the preceding item to derive a binary search algorithm for finding a minimum mean cycle.

3. (20 points) Let $G = (V, E)$ be a directed graph, with source $s \in V$, sink $t \in V$, and nonnegative edge capacities $c : E \to \mathbb{R}_{\geq 0}$. Give an algorithm that decides whether G has a unique minimum (s,t)-cut, i.e., an (s,t)-cut of capacity strictly less than any other (s,t)-cut.

4. (30 points) Let $G = (V, E)$ be an undirected graph. We want to label the nodes with integers in $\{1, \ldots, r\}$. There are two kinds of costs: $a_{v,k}$ is the cost of assigning k to v. For any edge $e = (u, v)$, if k is assigned to u and ℓ is assigned to v, a cost $b_e|k - \ell|$ arises. Show how to compute an assignment of minimum cost.

   Motivation: G could be a grid graph and integers represent grey values. We have an estimate for the grey value at each node and the $a_{v,k}$'s penalize unwanted grey values and reward wanted grey values. We also want some kind of consistency between adjacent pictures and hence penalize differences in adjacent pixels.

Have fun with the solution!