Algorithms and Data Structures **Summer 2008**
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Exercises 27 and 28

Motivation

We study variants of the maximum flow problem and applications of it. The book by Ahuja, Magnanti, and Orlin is a rich source for material about flows.
In class, the maximum flow problem was defined as follows. We are given a directed graph $G = (V,E)$, a nonnegative capacity function $cap : E \to \mathbb{R}_{\geq 0}$, and two special vertices s and t. An (s,t)-flow is a function $f : E \to \mathbb{R}$ such that

$$0 \leq f(e) \leq cap(e) \quad \text{for all } e \in E$$

and

$$excess(v) = 0 \quad \text{for all } v \in V \setminus \{s,t\},$$

where $excess(v)$ = flow into v − flow out of $v = \sum_{e=(u,v)} f(e) - \sum_{e=(v,w)} f(e)$. The value $val(f)$ of the flow is the excess of t. A maximum flow is a flow of maximum value.
An (s,t)-cut is a set S of nodes with $s \in S$ and $t \notin S$. The capacity $cap(S)$ of the cut is defined as $cap(S) = \sum_{e=(u,v);\, u \in S,\, v \notin S} cap(e)$. Then $val(f) \leq cap(S)$ for any flow f and any (s,t)-cut S. If f is a maximum flow, then there is a cut S with $val(f) = cap(S)$.

1. (integral flows) Show: If the capacities are integral, i.e., in $\mathbb{N}$, there is an integral maximum flow, i.e., $f(e) \in \mathbb{N}_0$ for all e. Hint: show that all augmentations increase the flow by an integral value.

2. (supplies and demands) Instead of the special vertices s and t, we have a function $b : V \to \mathbb{R}$ with $\sum_v b(e) = 0$. We call nodes v with $b(v) > 0$ *supply nodes* and nodes v with $b(v) < 0$, *demand nodes*. Instead of the flow conservation condition, we now have the modified flow conservation condition

   $$excess(v) + b(v) = 0 \quad \text{for all } v \in V.$$

   Show how to decide, whether a feasible flow exists? A flow is called feasible if it satisfies the capacity constraints and the modified flow conservation conditions. Hint: Add two new vertices a and t, an edge (s,v) of capacity $b(v)$ for any supply node, ...

   Formulate a cut-theorem. It should read something like the following. For a set S of nodes, let $b(S) := \sum_{v \in S} b(v)$ be the aggregated supply/demand of S. A feasible flow exists iff there is no set S of nodes with $b(S) > cap(S)$.

3. (lower and upper bounds) We now have in addition a lower capacity function $\ell : E \to \mathbb{R}_{\geq 0}$. We require that the flow f satisfies $\ell(e) \leq f(e) \leq u(e)$ for all $e \in E$. Show how to decide, whether a feasible flow exists. Hint: Use supplies and demands and modify upper and lower bounds.

   Formulate a cut theorem.

   Show: if all lower and upper bounds and all supply/demands are integral and there is a feasible flow, then there is an integral feasible flow.

4. (matrix rounding) We are given a $n \times m$ matrix M with nonnegative real entries. We want to round each entry to M_{ij} to either $\lceil M_{ij} \rceil$ or $\lfloor M_{ij} \rfloor$ such that all row and column sums are also rounded to an adjacent integer. For example,

$$\begin{array}{cc} 1.3 & 2.4 \\ 2.9 & 1.7 \end{array} \quad \text{could be rounded to} \quad \begin{array}{cc} 1 & 3 \\ 3 & 1 \end{array} \quad \text{but not to} \quad \begin{array}{cc} 2 & 3 \\ 3 & 1 \end{array}$$

   since in the latter array the sum of the first row 5. However, it should be either $\lceil 1.3 + 2.4 \rceil$ or $\lfloor 1.3 + 2.4 \rfloor$. For a given matrix, find such a rounding if it exists. Does such a rounding always exist?

   Hint: Set up a flow problem with two special vertices s and t, one vertex for each row and one vertex for each column.

Have fun with the solutions.