1 A Master Theorem for Recurrences

Akra and Bazzi [AB98] recently gave a closed form solution for a large class of recurrences.

Theorem 1 *Let*

$$f(x) = \begin{cases} h(x) & \text{for } 1 \le x \le x_0 \\ a f(\frac{x}{b}) + g(x) & \text{for } x > x_0 \end{cases} \tag{1}$$

where

1. $a > 0$, $b > 1$, *and* $x_0 \ge b$ *are constants,*
2. $x \ge 1$ *is a real number,*
3. $d_1 \le h(x) \le d_2$ *for some positive constants* d_1 *and* d_2 *and all* x *with* $1 \le x \le x_0$, *and*
4. g *is a nonnegative function satisfying the* polynomial growth condition, *i.e., there are positive constants* c_1 *and* c_2 *such that*

   $$c_1 g(x) \le g(u) \le c_2 g(x) \quad \text{for all } x > x_0 \text{ and } u \in [x/b, x] \ .$$

5. *(technical condition)*

   $$I := \int_1^{x_0} \frac{g(u)}{u^{p+1}}\, du < \infty$$

Let p *be the unique real number for which* $a/b^p = 1$. *Then*

$$f(x) = \Theta\left(x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\, du\right)\right) \ .$$

Before we enter the proof, we give some examples.

- If $f(x) = 2f(x/2) + x$, then $p = 1$ and

  $$f(x) = \Theta\left(x(1 + \int_1^x \frac{u}{u^2}\, du)\right) = \Theta(x \log x) \ .$$

- If $f(x) = 3f(x/2) + x^2$, then $p = \log 3$ and

  $$f(x) = \Theta\left(x^{\log 3}(1 + \int_1^x \frac{u^2}{u^{1+\log 3}}\, du)\right) = \Theta(x^{\log 3}(1 + x^{2-\log 3})) = \Theta(x^2) \ .$$

- If $f(x) = f(x/2) + \log x$, then $p = 0$ and

  $$f(x) = \Theta\left(x^0(1 + \int_1^x \frac{\log u}{u^1}\, du)\right) = \Theta(1 + \log^2 x) = \Theta(\log^2 x) \ .$$

Proof: For any $x > x_0$, we have $x/b < x$ and $x/b > x_0/b \geq 1$. Thus $f(x)$ is well-defined. We need to prove

$$C \cdot x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\,du\right) \leq f(x) \leq D \cdot x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\,du\right) \tag{2}$$

for positive constants C and D. We prove the upper bound and leave the lower bound to exercise 1. The proof of the upper bound is by induction on the index of x, the smallest integer k such that $x/b^k \leq x_0$.

Assume first that $1 \leq x \leq x_0$. Then

$$f(x) = h(x) \leq d_2 \leq d_2 \frac{x^p}{\min(1, x_0^p)} \leq d_2 \frac{x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\,du\right)}{\min(1, x_0^p)} ,$$

since x^p assumes its minimum in $[1, x_0]$ at either $x = 1$ (if $p \geq 0$) or $x = x_0$ (if $p < 0$) and since the value of the integral is nonnegative. Thus the upper bound in equation (2) holds for any D for which $D \geq d_2/\min(1, x_0^p)$.

Assume next that $x > x_0$. The index of x/b is one smaller than the index of x and hence we may use the upper bound in equation (2) for x/b. We obtain:

$$\begin{aligned}
f(x) &= a f(\frac{x}{b}) + g(x) && \text{definition of } f\\
&\leq a \cdot D \cdot \left(\left(\frac{x}{b}\right)^p \left(1 + \int_1^{x/b} \frac{g(u)}{u^{p+1}}\,du\right)\right) + g(x) && \text{induction hypothesis}\\
&= D \cdot x^p \left(1 + \int_1^{x/b} \frac{g(u)}{u^{p+1}}\,du\right) + g(x) && \text{since } a/b^p = 1\\
&\leq D \cdot x^p \left(1 + \int_1^{x} \frac{g(u)}{u^{p+1}}\,du\right)
\end{aligned}$$

provided we can establish the inequality

$$g(x) \leq D \cdot x^p \int_{x/b}^{x} \frac{g(u)}{u^{p+1}}\,du \;.$$

The inequality is a simple consequence of the polynomial-growth assumption. Observe that $g(u) \geq c_1 g(x)$ for $x/b \leq u \leq x$ and $u^{p+1} \leq \max((x/b)^{p+1}, x^{p+1})$ and hence

$$\begin{aligned}
D \cdot x^p \int_{x/b}^{x} \frac{g(u)}{u^{p+1}}\,du &\geq D \cdot x^p (x - x/b) \frac{c_1 g(x)}{\max((x/b)^{p+1}, x^{p+1})}\\
&= D \cdot \frac{c_1(1 - 1/b)}{\max((1/b)^{p+1}, 1)} g(x)\\
&\geq g(x)
\end{aligned}$$

provided that $D \geq \max((1/b)^{p+1}, 1)/(c_1(1 - 1/b))$. This completes the induction step.

We satisfy both constraints on D by setting $D = \max(d_2, d_2/x_0^p, b/(c_1(b-1)), 1/(c_1(b-1)b^p))$. ∎

How does the value $I := x^p \int_1^x g(u)/u^{p+1}\,du$ depend on the relation of g and p? Inspection of a table of integrals[1] yields:

$$\begin{array}{lclclr}
\int_1^x \frac{1}{u^{1+\varepsilon}}\,du & = & -\left[\frac{1}{\varepsilon u^{\varepsilon}}\right]_1^x & = & \Theta(1) & \varepsilon > 0 \\
\int_1^x \frac{\ln^k u}{u}\,du & = & \left[\frac{\ln^{k+1} u}{k+1}\right]_1^x & = & \Theta(\ln^{k+1} x) & k \geq 0 \\
\int_1^x \frac{u^{\delta}\ln^k}{u}\,du & = & \Theta\left(\left[u^{\delta}\ln^k u\right]_1^x\right) & = & \Theta(x^{\delta}\ln^k x) & \delta > 0 \text{ and } k \geq 0
\end{array}$$

Thus

- $f(x) = \Theta(x^p)$, if $g(x) = O(x^{p-\varepsilon})$ for some $\varepsilon > 0$,
- $f(x) = \Theta(g(x)\log^{k+1} x)$ if $g(x) = \Theta(x^p \log^k)$ for some $k \geq 0$, and
- $f(x) = \Theta(g(x))$ if $g(x) = \Theta(x^{p+\delta}\log^k)$ for some $\delta > 0$ and $k \geq 0$.

It is instructive to inspect the value $I_\Delta := x^p \int_{x/b^i}^{x/b^{i-1}} g(u)/u^{p+1}\,du$. For $g(x) = x^p$, we have

$$I_\Delta = x^p \int_{x/b^i}^{x/b^{i-1}} \frac{g(u)}{u^{p+1}}\,du = x^p \int_{x/b^i}^{x/b^{i-1}} \frac{1}{u}\,du = x^p\left[\ln u\right]_{x/b^i}^{x/b^{i-1}} = x^p \ln b\ ,$$

i.e., all levels of the recursion contribute essentially the same amount to $f(x)$. If $g(x) = \omega(x^p)$, the outermost level ($i = 0$) contributes most, and if $g(x) = o(x^p)$, the bottommost level ($i = \log_b x$) contributes most.

Exercise 1 *Prove the lower bound of Theorem 1.*

Let us see a second proof. We first express $f(x)$ as a sum.

Lemma 1 *Let k be the minimal integer with $x/b^k \leq x_0$. Then*

$$f(x) = a^k f(x/b^k) + \sum_{0 \leq i < k} a^i g(x/b^i) = a^k h(x/b^k) + \sum_{0 \leq i \leq k} a^i g(x/b^i)\ .$$

Proof: Either by repeated substitution or by induction on k. ∎

Let us next study the two terms in the expression for $f(x)$. Since $1 \leq x/b^k \leq x_0$, we have $d_1 \leq h(x/b_k) \leq d_2$ and hence

$$a^k h(x/b^k) = \Theta(a^k) = \Theta(x^p)$$

[1] See for example, the Wikipedia entry on lists of integrals.

where the last equality can be established by taking logarithms on both sides and observing that $k\log a = (\log x)/(\log b)\log a$ and $p\log x = (\log a)/(\log b)\log x$.

We turn to the sum $S := \sum_{0\le i<k} a^i g(x/b^i)$ Define x^* as x/b^k. Then $x/b^i = x^* b^{k-i}$ and hence

$$S = a^k \sum_{0\le i<k} \frac{g(x^* b^{k-i})}{a^{k-i}} = x^p \sum_{1\le j\le k} \frac{g(x^* b^j)}{a^j} = x^p \sum_{1\le j\le k} \frac{g(x^* b^j)}{(b^p)^j} ==: x^p S'$$

where the second equality uses the substitution $j = k-i$ and the third equality uses $a = b^p$. Since sums are harder to evaluate than integrals, we want to turn the sum into a integral. Consider a single term $g(x^* b^j)/(b^j)^p$. We want it to be the value of an integral from $x^* b^{j-1}$ to $x^* b^j$. The length of this integral is $x^*(b^j - b^{j-1}) = x^* b^j (1-1/b)$. Thus

$$\frac{g(x^* b^j)}{(b^j)^p} = \frac{(x^*)^p g(x^* b^j)}{(x^* b^j)^p} = (x^*)^p \frac{b}{b-1} \int_{x^* b^{j-1}}^{x^* b^j} \frac{g(x^* b^j)}{(x^* b^j)^p \cdot (x^* b^j)}\, du = (x^*)^p \frac{b}{b-1} \int_{x^* b^{j-1}}^{x^* b^j} \frac{g(x^* b^j)}{(x^* b^j)^{p+1}}\, du\ .$$

With

$$h(u) := \frac{g(x^* b^j)}{(x^* b^j)^{p+1}} \quad \text{for } x^* b^{j-1} < u \le x^* b^j \text{, we have} \quad S' = (x^*)^p \frac{b}{b-1} \int_{x^*}^{x} h(u)\, du\ .$$

For $x^* b^{j-1} < u \le x^* b^j$, we have

$$\frac{h(u)}{g(u)/u^{p+1}} = \frac{g(x^* b^j)}{g(u)} \cdot \frac{(x^* b^j)^{p+1}}{u^{p+1}} = \Theta(1)\cdot\Theta(1) = \Theta(1)\ .$$

For the first fraction, this follows from the polynomial growth property of g, and for the second fraction, this follows from the polynomial growth property of x^{1+p}. Thus

$$S' = (x^*)^p \frac{b}{b-1} \int_{x^*}^{x} h(u)\, du = \Theta\left(\int_{x^*}^{x} \frac{g(u)}{u^{p+1}}\, du \right)\ .$$

and we completed the alternative proof of the AB-theorem. We should remark that the alternative proof does not generalize.

We come to extensions. In Theorem 1, $f(x)$ is defined as a times the value of f at x/b. More generally, we can define $f(x)$ as a linear combination of smaller values.

$$f(x) = \begin{cases} h(x) & \text{for } 1 \le x \le x_0 \\ \sum_{i=1}^{k} a_i f(\frac{x}{b_i}) + g(x) & \text{for } x > x_0 \end{cases} \tag{3}$$

where k is an integer constant and $a_i > 0$ and $b_i > 1$ are real constants. In order to make f well defined, we require $x_0 \ge \max_i b_i$. Then $1 \le x/b_i < x$ for all $x \ge x_0$ and all i. We define p as the unique real number for which $\sum_i a_i (1/b_i)^p = 1$.

Theorem 2 (Akra-Bazzi) *Under the assumptions stated above,*

$$f(x) = \Theta\left(x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\, du \right) \right)\ .$$

Proof: The proof is analogous to the proof of Theorem 1 and delegated to the exercises. It again uses induction on the index of x, the smallest integer k such that $x/(\min_i b_i)^k \leq x_0$. ▮

Exercise 2 *Prove Theorem 2*

Although Theorem 2 handles a broad class of recurrences, it does not cover a common form of recurrence arising in the analysis of algorithms. For example, in the recurrence for the running time of Karatsuba's algorithm, we reduced $T_K(n)$ to $T_K(\lceil n/2 \rceil + 1)$. The following extension of Theorem 2 deals with these variations. Consider

$$f(x) = \begin{cases} h(x) & \text{for } 1 \leq x \leq x_0 \\ \sum_{i=1}^{k} a_i f(\frac{x}{b_i} + h_i(x)) + g(x) & \text{for } x > x_0 \end{cases} \tag{4}$$

where k is an integer constant and $a_i > 0$ and $b_i > 1$ are real constants. The h_i are functions with $|h_i(x)| \leq x/(\log^{1+\varepsilon} x)$ for some $\varepsilon > 0$ and all $x > x_0$. In order to make f well defined, we require $1 \leq x/b_i + h_i(x) < x/b$ for all i and $x > x_0$ and some $b > 1$. As before, we define p as the unique real number for which $\sum_i a_i (1/b_i)^p = 1$.

Theorem 3 (Leighton) *Under the assumptions stated above and some more technical conditions on x_0 (see [Lei])*

$$f(x) = \Theta\left(x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\, du\right)\right) .$$

Proof: See [Lei] ▮

For the Karatsuba recurrence we would use $k = 3$, $a_i = 1$, $b_i = 2$ and $h_i(x) = \lceil x/2 \rceil + 1 - x/2$ for $1 \leq i \leq k$.

References

[AB98] M. Akra and L. Bazzi. On the solution of linear recurrence equations. *Computational Optimization and Applications*, 10(2):195–210, 1998.

[Lei] T. Leighton. Notes on better master theorems for divide and conquer recurrences. `http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.39.1636`.