Approximate Maximum Flow in Undirected Networks by Christiano, Kelner, Madry, Spielmann, Teng (STOC 2011)

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The Result

- $G = (V, E)$ undirected graph, s source, t sink.
- $u : E \to \mathbb{R}_{\geq 0}$, edge capacities
- $\epsilon > 0$

Can compute $(1 - \epsilon)$-approximate maximum flow in time $\widetilde{O}(mn^{1/3}\epsilon^{-11/3})$.

- approximate minimum cut in similar time bound
- previous best: $\widetilde{O}(m\sqrt{n}\epsilon^{-1})$ by Goldberg and Rao (98)
- uses electrical flows

Conversion to Integral Capacities

- $B = \max_{P \text{ is } s\text{-}t \text{ path}} \min_{e \in P} u_e$
- max bottleneck path in time $O(m + n \log n)$
- $B \le$ max flow $\le mB$.
- replace u_e by $\min(u_e, mB)$.
- removing all edges of capacity less than $\epsilon B/(2m)$ changes max-flow by at most $\epsilon B/2$.
- replace u_e by $\left\lfloor \frac{u_e}{\epsilon B/2m} \right\rfloor$

integral capacities in $[1, 2m^2/\epsilon]$

A High-Level View of the Algorithm

$F^* =$ value of maximum flow.

Do binary search on $[1, 2m^2/\epsilon]$. Let F be the current value of the search.

Have a subroutine Flow(F) which

- either finds a flow of value F that almost satisfies the capacity constraints or fails.
- if $F \le F^*$, it is guaranteed to return a flow.

Subroutine is realized via a low-level subroutine flow(F, w), which we discuss first. Here, w is a weight function on the edges.

Electrical Flows and Capacities

Resistances can simulate capacities

Let Q^* be a maximum flow. Orient edges in the direction of the flow, sort the graph topologically, and set

$$p_v = \text{number of nodes after } v \text{ in ordering.}$$

For $e = (u, v)$, set

$$R_e = Q^*_e / \Delta_e.$$

Then Q^* is the resulting flow.

An Observation

Set $R_e = 1/u_e^2$ and let $F \le F^*$. Let Q be an electrical flow of value F. Then

$$\sum_e (Q_e/u_e)^2 = \sum_e R_e Q_e^2 \le \sum_e R_e (Q_e^*)^2 = \sum_e (Q_e^*/u_e)^2 \le m.$$

Define the congestion of e as

$$\text{cong}_e := Q_e/u_e.$$

Then,

$$\frac{1}{m} \sum_e \text{cong}_e^2 \le 1 \quad \text{and} \quad \max_e \text{cong}_e \le \sqrt{m}.$$

The Subroutine flow(F, w)

1. Set $R_e = (w_e + \epsilon W/m)/u_e^2$, where $W = \sum_e w_e$.
2. Let Q be an electrical flow of value F.
3. If $\sum_e R_e Q_e^2 > (1+\epsilon)W$ declare failure.
4. return Q.

Properties

If $F \le F^*$, flow does not fail

If flow succeeds,

$$\sum_e \frac{w_e}{W} \mathrm{cong}_e \le 1 + \epsilon \quad \text{and} \quad \max_e \mathrm{cong}_e \le \rho := \sqrt{\frac{(1+\epsilon)m}{\epsilon}}.$$

Proof

Set $R_e = (w_e + \epsilon W/m)/u_e^2$, where $W = \sum_e w_e$. Let Q be an electrical flow of value F. If $F \le F^*$ then

$$\sum_e R_e Q_e^2 \le \sum_e R_e (Q_e^*)^2 = \sum_e (w_e + \frac{\epsilon W}{m}) \left(\frac{Q_e^*}{u_e}\right)^2 \le (1+\epsilon) W.$$

If

$$\sum_e (w_e + \frac{\epsilon W}{m}) \left(\frac{Q_e}{u_e}\right)^2 = \sum_e R_e Q_e^2 \le (1+\epsilon) W$$

then

$$\sum_e \frac{w_e}{W} \mathrm{cong}_e^2 \le 1 + \epsilon \quad \text{and} \quad \max_e \mathrm{cong}_e \le \sqrt{\frac{(1+\epsilon)m}{\epsilon}}.$$

From average squared congestion to average congestion

$$\begin{aligned}\sum_e w_e \text{cong}_e &= \sum_e w_e^{1/2} \cdot w_e^{1/2} \text{cong}_e \\ &\le \left(\sum_e w_e\right)^{1/2} \cdot \left(\sum_e w_e \text{cong}_e^2\right)^{1/2} \\ &\le W^{1/2}((1+\epsilon)W)^{1/2} \le (1+\epsilon)W.\end{aligned}$$

From flow to Flow

Flow(F)

set $w_e^{(1)} = 1$ for all e;

for $i = 1 \to T$ **do** $\{T = O(m^{1/2}\epsilon^{-5/2}$ suffices$\}$

$Q^{(i)} = \text{flow}(F, w)$; {if call fails, fail}

$\text{cong}_e^{(i)} = Q_e^{(i)}/u_e$ for all e

$w_e^{(i+1)} = w_e^{(i)}(1 + \epsilon\text{cong}_e^{(i)}/\rho)$ for all e;

end for

return

$$Q := \frac{1}{T} \sum_{1 \le i \le T} Q^{(i)}$$

Properties of Flow

Q is a flow of value F and if $F \le F^*$, Q exists.

$$Q_e = \frac{1}{T} \sum_{1 \le i \le T} Q_e^{(i)} = \frac{1}{T} \sum_{1 \le i \le T} u_e \cdot \mathrm{cong}_e^{(i)} = u_e \cdot \overline{\mathrm{cong}}_e$$

$$W^{(i+1)} = \sum_e w_e^{(i)} (1 + \epsilon \mathrm{cong}_e^{(i)} / \rho) \le (1 + \epsilon(1 + \epsilon)/\rho) W^{(i)}$$

$$W^{(T+1)} \le \exp(((1 + \epsilon)\epsilon/\rho)T) \cdot m$$

Properties of Flow

$$w_e^{(i+1)} = w_e^{(i)}(1 + \epsilon \mathrm{cong}_e^{(i)}/\rho) \ge w^{(i)} \exp((1-\epsilon)\epsilon \mathrm{cong}_e^{(i)}/\rho)$$

$$w_e^{(T+1)} \ge \exp((1-\epsilon)\epsilon \overline{\mathrm{cong}}_e/\rho) T)$$

$$((1-\epsilon)\epsilon \overline{\mathrm{cong}}_e/\rho) T \le \ln m + (\epsilon(1+\epsilon)/\rho) T$$

$$\overline{\mathrm{cong}}_e \le \frac{\rho \ln m}{(1-\epsilon)\epsilon T} + \frac{1+\epsilon}{1-\epsilon} \le \frac{\epsilon}{(1-\epsilon)} + \frac{1+\epsilon}{1-\epsilon} \le 1 + 4\epsilon$$

$$\text{for } T = (\rho \ln m)/\epsilon^2 = \widetilde{O}(m^{1/2}\epsilon^{-5/2})$$

Putting it together

$\widetilde{O}(m^{1/2}\epsilon^{-5/2})$ iterations suffice.

In each iteration we need to solve a SSD system and do linear extra work. Thus an iteration runs in time $\widetilde{O}(m \log 1/\epsilon)$.

Total running time is $\widetilde{O}(m^{3/2}\epsilon^{-5/2})$.

But, I promised $\widetilde{O}(mn^{1/3}\epsilon^{-11/3})$. This is reached in two steps:

- step one reduces to $\widetilde{O}(m^{4/3}\epsilon^{-3})$, and
- step two reduces to $\widetilde{O}(mn^{1/3}\epsilon^{-11/3})$. (Karger (98) and Bencur/Karger (02))

The First Step

Let H be a huge number; actually $H = (m \ln m)^{1/3}/\epsilon$.

What does $\text{cong}_e \geq H$ imply?

$Q_e/u_e \geq H$ and hence $u_e \leq Q_e/H \leq F/H$. Thus u_e is tiny.

We can afford to delete ϵH edges with huge congestion without sacrificing the approximation guarantee.

Modification of flow: if flow succeeds, i.e., $\mathcal{E}(Q) \leq (1+\epsilon)W$, and there is an edge e with huge congestion, delete the edge and continue without the edge.

Observe, that change allows us to replace ρ by H in the analysis.

Deleting Huge Edges

If flow succeeds, we have $\mathcal{E}(Q) \le (1+\epsilon)W$.

If e has huge congestion,

$$r_e Q_e^2 \ge \frac{\epsilon W}{m}\left(\frac{Q_e}{u_e}\right)^2 \ge \frac{\epsilon H^2}{(1+\epsilon)m}(1+\epsilon)W \ge \frac{\epsilon H^2}{(1+\epsilon)m}\mathcal{E}(Q).$$

Let $\beta = \epsilon H^2/((1+\epsilon)m)$. If e has huge congestion, e accounts for a β fraction of the energy of the flow.

Deletion of a huge edge forces the energy of the flow to increase by a factor $1/(1-\beta)$.

We have an upper bound on the final energy, namely $(1+\epsilon)W^{(T+1)}$. It is not too hard, to derive a lower bound on the energy of the first flow. Putting things together, we obtain a bound on the number of huge edges.

Deleting a Huge Edge II

Deletion of a huge edge increases the energy of the flow by a factor $1/(1-\beta)$.

Let p be the electrical potentials for flow of value $1/R_{\text{eff}}$. Then $p_s = 1$ and $p_t = 0$. Energy of this flow is equal to $1/R_{\text{eff}}$.

$$\frac{1}{R'_{\text{eff}}} = \inf_{\substack{q_s=1 \\ q;q_t=0}} \sum_{uv \in E \setminus e} \frac{(q_u - q_v)^2}{r_{uv}} \le \sum_{uv \in E \setminus e} \frac{(p_u - p_v)^2}{r_{uv}}$$

$$= \sum_{uv \in E} \frac{(p_u - p_v)^2}{r_{uv}} - \Delta_e^2 / r_e \le (1-\beta)\frac{1}{R_{\text{eff}}}$$

Thus, $\mathcal{E}(Q') = F^2 R'_{\text{eff}} \ge \frac{1}{1-\beta} F^2 R_{\text{eff}} = \mathcal{E}(Q)$.