Basic Facts about Electrical Networks

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Kirchoff’s Laws

- Let $G = (V, E)$ be an undirected graph; fix an orientation of each edge.
- $R_e =$ resistance of edge $e = (u, v)$
- $b_u =$ external current provided (extracted) at $u$; $\sum_u b_u = 0$.
- $Q_e =$ current through $e$ in the direction from $u$ to $v$ (might be negative)
- Current Law: $\sum_{e=(u,v)} Q_e - \sum_{e=(v,u)} Q_e = b_u$ for every $u$
- Ohm’s Law: If $R_e$ is the resistance of $e = (u, v)$, then
  \[ Q_e = \Delta_e / R_e \]
  where $\Delta_e$ is the potential difference between $u$ and $v$.
- Voltage Law: Potential differences sum to zero around any cycle and hence can assign potentials $\rho_u$ to the vertices
Superposition Principle

Additivity of Solutions

- Assume $b = b^{(1)} + b^{(2)}$ and $b^{(i)}$ legal
- Let $Q^{(i)}$ be an electrical flow for $b^{(i)}$. Then $Q^{(1)} + Q^{(2)}$ is electrical flow for $b$.
- Potentials also add.
Thompson’s Principle

Electrical Flows are Optimal

Let $Q$ be the electrical flow satisfying the demand vector $b$ and let $f$ be any flow satisfying it. Then

$$\mathcal{E}(Q) = \sum_e \Delta_e Q_e = \sum_e R_e Q_e^2 \leq \sum_e R_e f_e^2$$

Let $g = f - Q$. Then $g$ is a circulation and $f = g + Q$. Then

$$\sum_e R_e f_e^2 = \sum_e R_e (g_e^2 + 2g_e Q_e + Q_e^2) \geq \sum_e R_e Q_e^2 + \sum_e 2\Delta_e g_e.$$ 

The last term is zero since $g$ is a sum of circular flows and for any cycle the potential differences sum to zero.
Effective Resistance

Let $Q$ be an electrical flow of 1 from $s$ to $t$.

The effective resistance of the network is the potential difference $\Delta$ between $s$ and $t$.

This is also the energy of the flow.

\[ E(Q) = \Delta \cdot 1 = \Delta. \]

\[ E(Q) = \sum_e \Delta_e Q_e = \sum_P \sum_{e \in P} \Delta_e Q_P = \Delta \sum_P Q_P = \Delta. \]
Computing the Currents

Let $p_u$ be the (unknown) potential of node $u$. For any edge $e = (u, v)$ the current from $u$ to $v$ is $(p_u - p_v)/R_{uv}$. The net-current at $u$ is equal to $b_u$:

$$\sum_{v \in \delta(u)} \frac{p_u - p_v}{R_{uv}} = b_u.$$ 

or equivalently

$$\left(\sum_{v \in \delta(u)} \frac{1}{R_{uv}}\right) p_u + \sum_{v \in \delta(u)} \frac{-1}{R_{uv}} p_v = b_u.$$ 

Coefficient matrix is symmetric diagonally dominant (SDD).
Methods for Solving SSD-system $Ax = b$

- Cholesky factorization: $A = LL^T$ where $L$ is a lower triangular matrix.
- Gauss-Seidel Iteration: compute $x^{(k)}$ for $k = 1, 2, 3, \ldots$. For fixed $k$, compute $x_i^{(k+1)}$ for $i = 1, 2, 3, \ldots$:

$$
x_i^{(k+1)} = \frac{1}{a_{ii}} \left( b_i - \sum_{j > i} a_{ij}x_j^{(k)} - \sum_{j < i} a_{ij}x_j^{(k+1)} \right)
$$

- Preconditioning: solve $BAx = Bb$ for a suitable $B$.
- Recursive Preconditioning + Partial Cholesky + Chebyshev Iteration (Spielman/Teng, Koutsis/Miller/Peng): $\tilde{O}(m \log n \log \frac{1}{\epsilon})$. 
Kirchhoff’s Spanning Tree Theorem

- Assume $b_s = 1 = -b_t$ and $b_v = 0$ otherwise (Superposition)
- for a spanning tree $T$: $c(T) = \prod_e 1/R_e$
- $N = \sum_T c(T)$
- for an edge $e = (a, b)$: $S(a, b) =$ all spanning trees that contain $a$ and $b$ (in this order) on path from $s$ to $t$.

$$N(a, b) = \sum_{T \in S(a,b)} c(T).$$

$$Q_{(a,b)} = \frac{N(a, b) - N(b, a)}{N}$$

$Q$ is an electrical flow of value 1 from $s$ to $t$. 
Current Law Holds

For simplicity, multiply all currents by $N$.

$$Q^T_{(a,b)} = \begin{cases} 
    c(T) & \text{if } T \text{ contains } s \ldots ab \ldots t \\
    -c(T) & \text{if } T \text{ contains } s \ldots ba \ldots t \\
    0 & \text{otherwise.}
\end{cases}$$

Then

$$Q_{ab} = \sum_T Q^T_{ab}$$

$T$ induces a current of $c(T)$ from $s$ to $t$ along its path from $s$ to $t$. So total current is $N$ as desired and flow conservation holds.
Voltage Law Holds

*thicket* = spanning forests with two components $F_0$ and $F_1$ such that $s_i \in F_i$. $F = F_0 \cup F_1$.

$$Q^F_{(a,b)} = \begin{cases} 
Q^{F \cup ab}_{(a,b)} & \text{if } F \cup ab \text{ is a spanning tree} \\
0 & \text{otherwise.}
\end{cases}$$

Then $Q_e = \sum_F Q^F_e$. Let $C$ be a cycle.

$$\sum_{e \in C} d(e, C) \Delta_e = \sum_{e \in C} d(e, C) R_e Q_e = \sum_{e \in C} d(e, C) R_e \sum_F Q^F_e$$

$$= \sum_F \sum_{e \in C, e \text{ extends } F} d(e, C) R_e \prod_{e' \in F \cup e} 1 / R_{e'}$$

$$= \sum_F \prod_{e' \in F} 1 / R_{e'} \sum_{e \in C, e \text{ extends } F} d(e, C) = 0$$