Physarum can Compute Shortest Paths
SODA 2012

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joint work with Vincenzo Bonifaci and Girish Varma

paper available on my homepage

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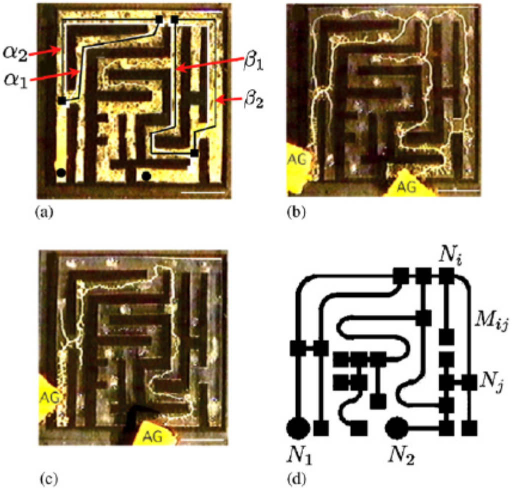
Overview

- Nature can do amazing computations with little or slow hardware: bird flocking, human vision, fish swarms, ants, the slime mold physarum.
- The Physarum Experiment and the Proposed Mathematical Model, 10 minutes
- Interlude: basic facts about electrical networks, 30 minutes
- The Analysis of Physarum, 80 minutes
- Electrical Networks and Approximate Maximum Flows, 60 minutes

The Physarum Computer



Physarum, a slime mold,

single cell, several nuclei

builds evolving networks

Nakagaki, Yamada, Tóth, Nature 2000

show video

2008 Ig Nobel Prize

For achievements that first make people LAUGH
then make them THINK

COGNITIVE SCIENCE PRIZE: Toshiyuki Nakagaki, Ryo Kobayashi, Atsushi Tero, Ágotá Tóth
for discovering that slime molds can solve puzzles.

REFERENCE: "Intelligence: Maze-Solving by an Amoeboid Organism," Toshiyuki Nakagaki, Hiroyasu Yamada, and Ágota Tóth, Nature, vol. 407, September 2000, p. 470.

Mathematical Model (Tero et al.)

- $G = (V, E)$ undirected graph
- each edge e has a positive length L_e (fixed) and a positive diameter $D_e(t)$ (dynamic)
- send one unit of current (flow) from s_0 to s_1 in an electrical network where resistance of e equals

$$R_e(t) = L_e / D_e(t).$$

- $Q_e(t)$ is resulting flow across e at time t
- Dynamics:

$$\dot{D}_e(t) = \frac{dD_e(t)}{dt} = |Q_e(t)| - D_e(t).$$

- 1 and 3 links

Tero et al., J. of Theoretical Biology, 553 – 564, 2007

Mathematical Model II: The Node Potentials

- electrical flows are driven by node potentials
- $Q_e = D_e(p_u - p_v)/L_e$ is flow on edge $\{u, v\}$ from u to v
- flow conservation gives n equations, one for each vertex u

$$\sum_{e=\{u,v\}\in E} D_e(p_u - p_v)/L_e = b_u$$

- $b_{s_0} = 1 = -b_{s_1}$ and $b_u = 0$, otherwise
- together with $p_{s_1} = 0$, the above defines the p_v's uniquely
- can be computed by solving a linear system

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Computer Experiments (Discrete Time)

initialize potentials
while true **do**
update diameters: $D_e(t+1) = D_e(t) + \epsilon(|Q_e(t)| - D_e(t))$
recompute potentials
end while

In simulations, the system converges (Miyaji/Ohnishi 07/08)

- e on shortest s_0-s_1 path: D_e converges to 1
- e not on shortest path: D_e converges to 0

Miyaji/Ohnishi ran simulations only on small graphs

We ran experiments on thousands of graphs of size up to 50,000 vertices and 200,000 edges. Confirmed their findings.

The Questions

Does system convergence for all (!!!) initial conditions?

How fast is the convergence?

Details of the convergence process?

Beyond shortest paths?

Inspiration for distributed algorithms?

Convergence against Shortest Path

Theorem (Convergence)

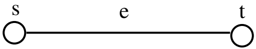
Dynamics converge against shortest path, i.e.,

$D_e \to 1$ *for edges on shortest source-sink path and* $D_e \to 0$ *otherwise.*

this assumes that shortest path is unique; otherwise converge against the set of flows of value 1 using only shortest source-sink paths

Miyaji/Onishi previously proved convergence for planar graphs with source and sink on the same face

A Single Link (Miyaji/Ohnishi)



e has length *L* and diameter *D*

$$Q = 1$$
$$\dot{D} = 1 - D$$
$$D = 1 + (D(0) - 1)e^{-t} \to 1$$

Resistance of *e* converges to *L*.

Thus $p_{s_0} - p_{s_1}$ converges to *L*

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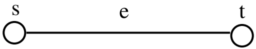

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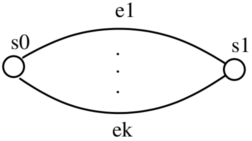
Analysis: Two Parallel Links (Miyaji/Ohnishi)



e_i has length L_i, $L_1 < L_2$, and diameter D_i

$$V = L_2 \ln D_2 - L_1 \ln D_1$$

$$\frac{d}{dt} L_i \ln D_i = L_i \frac{\dot{D}_i}{D_i} = L_i \frac{(D_i/L_i)\Delta - D_i}{D_i} = \Delta - L_i$$

$$\dot{V} = L_1 - L_2$$

$$V(t) = V(0) + (L_1 - L_2)t$$

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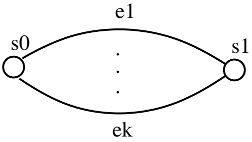

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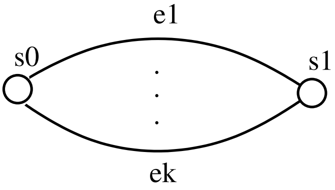
$$V = L_2 \ln D_2 - L_1 \ln D_1$$

$$V(t) = V(0) + (L_1 - L_2)t$$

$V(t)$ goes to minus infinity and hence either D_1 unbounded or D_2 goes to zero; the former is impossible

$$D_2 \to 0 \Rightarrow Q_2 \to 0 \Rightarrow Q_1 \to 1 \Rightarrow D_1 \to 1 \Rightarrow p_s - p_t \to L_1$$

Parallel Links (Miyaji/Ohnishi 07)



parallel links with lengths $L_1 < L_2 < \ldots < L_k$

$D_1 \to 1$, $D_2, \ldots, D_k \to 0$

$p_{s_0} - p_{s_1} \to L_1$

but $D_2, \ldots, D_{k-1}$ do not necessarily converge monotonically

What did Evolution Optimize?

Evolution optimized dynamics so as to achieve a global objective.
Which? (Lyapunov Function)

First idea: the energy of the flow $\sum_e Q_e \Delta_e$ decreases over time

not true, even for parallel links

Theorem

For the case of parallel links: $\sum_i Q_i L_i$, $\sum_i D_i L_i / \sum_i D_i$, and $(p_s - p_t) \sum_i D_i L_i$ decrease over time

computer experiment: the obvious generalization (replace i by e) to general graphs do not work

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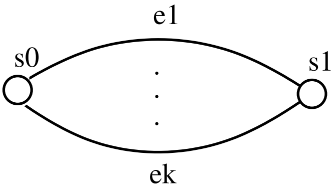
A not so Obvious Generalization



$$\frac{\sum_i D_i L_i}{\sum_i D_i} \quad \Rightarrow \quad \frac{\sum_e D_e L_e}{\text{value of min } s_0\text{-}s_1 \text{ cut with } cap_e = D_e}$$

What did Evolution Optimize?

Computer experiment:

$$V := \frac{\sum_e D_e L_e}{\text{value of min } s_0\text{-}s_1 \text{ cut with } cap_e = D_e} \quad \text{decreases}$$

Theorem (Lyapunov Function)

$$V + \left(\sum_{e \in \delta(\{s_0\})} D_e - 1\right)^2 \quad \textit{decreases.}$$

Derivative of V (essentially) satisfies

$$\dot{V} \leq -c \cdot \sum_e (D_e - |Q_e|)^2.$$

Proof uses min-cut-max-flow and ...

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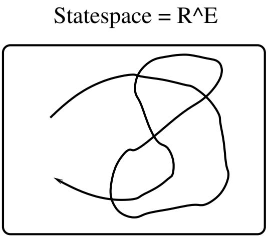
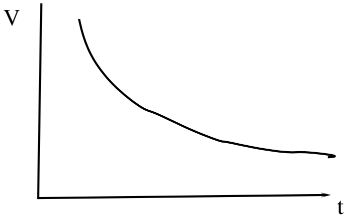




- V decreases and stays positive $\Rightarrow \dot{V} \to 0$
- $\dot{V} \le -c \cdot \sum_e (D_e - |Q_e|)^2$
- $|D_e - |Q_e||$ goes to zero for all e
- $Q_e = (D_e/L_e)\Delta_e$ and hence $\Delta_e \approx L_e$ for Q_e and t large
- $\Delta_{s_0 s_1}$ converges to length of some source-sink path
- $\Delta_{s_0 s_1}$ converges to length of shortest path
- ...

Convergence against Shortest Path

Corollary (Convergence)

Dynamics converge against shortest path, i.e.,

$D_e \to 1$ *for edges on shortest* s-t *path and* $D_e \to 0$ *otherwise.*

this assumes that shortest path is unique; otherwise, ...

Miyaji/Onishi previously proved convergence for planar graphs.

Equilibria (Miyaji/Ohnishi 07)

Simplifying Assumption: no two s_0-s_1 paths have same length

Equilibrium: $D_e = |Q_e|$ for all e

The equilibria are precisely the s_0-s_1 paths, i.e.

$$D_e = \begin{cases} 1 & e \in P \\ 0 & e \notin P \end{cases}$$

for some source-sink path P.

- potential drop Δ_e along an edge e satisfies $Q_e = D_e \Delta_e / L_e$.
- thus $\Delta_e = L_e$ in equilibrium for e with $D_e > 0$
- thus $\Delta = L_P$ for any source-sink path P with $D_e > 0$ for all $e \in P$.

Elements of the Proof I

C = value of min s_0-s_1 cut, if $cap_e = D_e$

By the Min-Cut-Max-Flow Theorem, there is a flow f of value 1 such that $f_e \le D_e/C$ for all e.

By Thompson's principle:

$$\sum_e R_e Q_e^2 \le \sum_e R_e f_e^2 \le \frac{1}{C^2} \sum_e R_e D_e^2.$$

$C \to 1$ since

- for every cut S: $\dot{C}_S \ge 1 - C_S$
- for $S = \{ s_0 \}$: $\dot{C}_S = 1 - C_S$

Elements of the Proof II

$$V := \frac{\sum_e D_e L_e}{C}$$

where C = value of min s_0-s_1 cut with $cap_e = D_e$.

$$\begin{aligned}
\dot{V} &= \sum_e R_e |Q_e| \frac{D_e}{C} - \sum_e R_e \left(\frac{D_e}{C}\right)^2 \quad \text{by calculation} \\
&\le \frac{1}{2}\left(2\sum_e R_e |Q_e| \frac{D_e}{C} - \sum_e R_e \left(\frac{D_e}{C}\right)^2 - \sum_e R_e Q_e^2\right) \text{ by prev. slide} \\
&= \frac{-1}{2}\sum_e R_e \left(\frac{D_e}{C} - |Q_e|\right)^2 \le \frac{-L_{\min}}{4}\sum_e \left(\frac{D_e}{C} - |Q_e|\right)^2
\end{aligned}$$

Elements of the Proof III

$C \to 1$ and $\sum_e \left(\frac{D_e}{C} - |Q_e|\right)^2 \to 0$ imply

$$|D_e - |Q_e|| \to 0 \quad \text{for all } e.$$

There is always a path P with $Q_e \geq 1/m$ for all $e \in P$. For such e,

$$\Delta_e = L_e(1 \pm \epsilon) \quad \text{since} \quad Q_e = D_e \Delta_e / L_e.$$

Thus $\Delta = (1 \pm \epsilon) L_P$ for some P always.

Thus $\Delta \to L_P$ for some fixed path P.

Elements of the Proof IV

$\Delta \to L_P$ for some fixed path P.

Let P^* be the shortest source-sink path and assume $P \neq P^*$.

$$\frac{d}{dt} \sum_{e \in P^*} L_e \ln D_e = \sum_{e \in P^*} L_e \frac{|Q_e| - D_e}{D_e} = \sum_{e \in P^*} L_e \frac{D_e |\Delta_e| / L_e - D_e}{D_e}$$

$$\geq \sum_{e \in P^*} L_e \frac{D_e \Delta_e / L_e - D_e}{D_e} = \Delta - L_{P^*} \to L_P - L_{P^*}.$$

Thus $\sum_{e \in P^*} L_e \ln D_e$ is unbounded, a contradiction.

Elements of the Proof V

$\Delta \to L_{P^*}$.

Consider $e \notin P^*$ and assume $\neg(Q_e \to 0)$, say $Q_e(t) \geq \delta > 0$ for arbitrarily large t.

For any such t, there is a path P through e with $Q_P > \delta/n^n$. Then $\Delta \approx L_P$, a contradiction.

$Q_e \to 1$ for $e \in P^*$ since $Q_e \to 0$ for $e \notin P^*$.

Stable Topology (Miyaji/Ohnishi)

How fast is the convergence?

Definition: An edge $e = (u, v)$ stabilizes if for all $\varepsilon > 0$ either

- $p_u(T) \geq p_v(T) - \varepsilon$ for all large T or
- $p_v(T) \geq p_u(T) - \varepsilon$ for all large T.
- $|p_v(T) - p_u(T)| \leq \varepsilon$ for all large T.
- slightly more general than Miyaji/Ohnishi

Definition: A network stabilizes if all edges stabilize

A Path with Fixed Potential Difference

- assume p_a and p_b are fixed
- $L(P)$ length of path from a to b.
- define $f = (p_a - p_b)/L(P)$ and assume $f < 1$
- then for all edges of p: D decays like $\exp((f-1)t)$
- p_v converges to $p_b + (p_a - p_b)dist(v,b)/L(P)$

Stable Topology III

Theorem

If network stabilizes, network converges as defined next.

- decompose *undirected* G into paths: P_0 = shortest s-t path
- for $v \in P_0$: $p_v \to dist(v,t)$ for $e \in P_0$: $D_e \to 1$
- assume $P_0, \ldots, P_{i-1}$ are defined. Then
  - P_i has endpoints a and b on $P_0 \cup \ldots \cup P_{i-1}$
  - internal nodes and edges are fresh
  - maximizes $f_i := (p_a - p_b)/L(P_i)$ this is less than one
  - for $v \in P_i$: $p_v \to p_b + (p_a - p_b)dist(v,b)/L(P_i)$
  - for $e \in P_i$: $D_e \to 0$, exponentially with rate $f_i - 1$.
  - direct edges in P_i in direction from from a to b

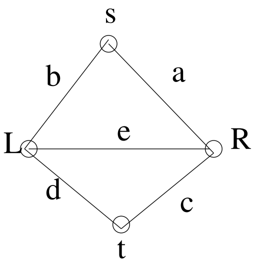
Open Problems

Do networks stabilize?

If so, after what time?

More generally, how long does it take for the dynamics to converge?

Wheatstone Graph



- simplest graph where flow directions are not clear
- direction of flow on e ????
- potentials evolve non-monotonically; run NonMonotone
- state space is cyclic; run TwoChanges

Theorem

Wheatstone network stabilizes

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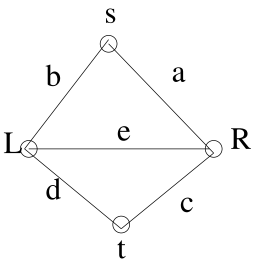

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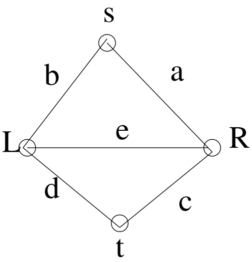
Wheatstone: Middle Edge Stabilizes



- $R_i = L_i/D_i$ = resistance of edge i.
- $x_a = R_a/(R_a + R_c)$, similarly for b.
- potential drop on edge a is $x_a\Delta$.
- if $x_a < x_b$, direction of e is RL
  if $x_a > x_b$, direction of e is LR
- $x_a^* = L_a/(L_a + L_c)$, similarly for b.

  assume $x_a^* \le x_b^*$
- observe: direction of flow on e does not depend on D_e

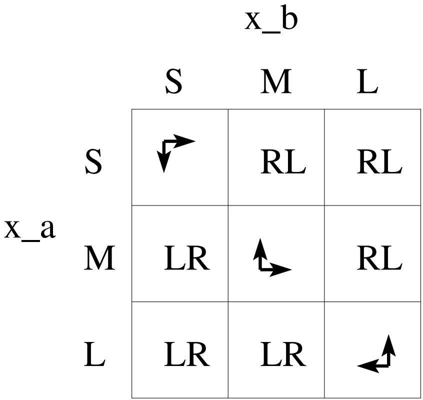
Wheatstone: Middle Edge Stabilizes

$x_a = R_a/(R_a + R_c)$

- split $[0, 1]$ into

$$S = [0, x_a^*)$$
$$M = [x_a^*, x_b^*]$$
$$L = [x_b^*, 1]$$

- consider evolution of (x_a, x_b)
- in $S \times S$, both grow:
- in $M \times M$, x_a decreases and x_b grows
- in $L \times L$, both shrink



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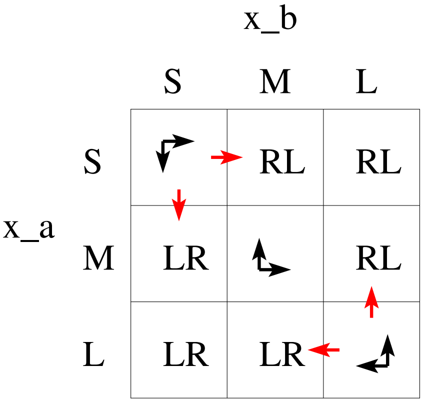

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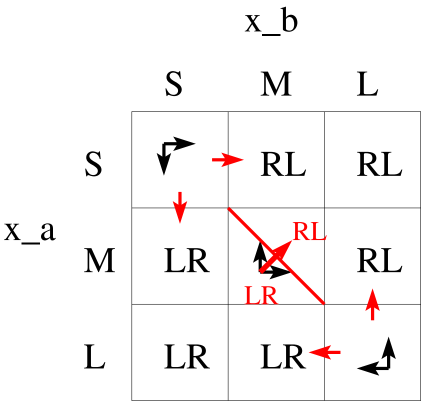

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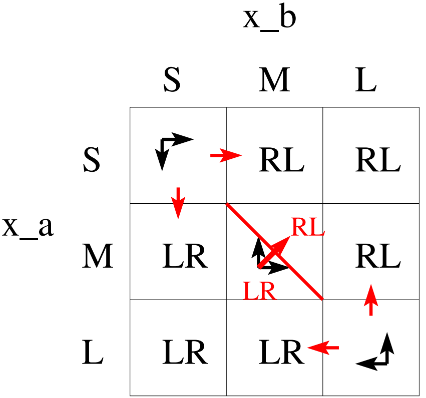


what if system stays in $S \times S$ or $L \times L$?

The Transportation Problem

- undirected graph $G = (V, E)$
- $b : V \to \mathbb{R}$ such that $\sum_v b_v = 0$
- v supplies flow b_v if $b_v > 0$
- v extracts flow $|b_v|$ if $b_v < 0$
- find a cheapest flow where cost of sending x units across an edge of length L is Lx

Dynamics of Physarum solves transportation problem.

D_e's converge against a mincost solution of transportation problem.

proof requires a non-degeneracy assumption

Open Problems and Related Work

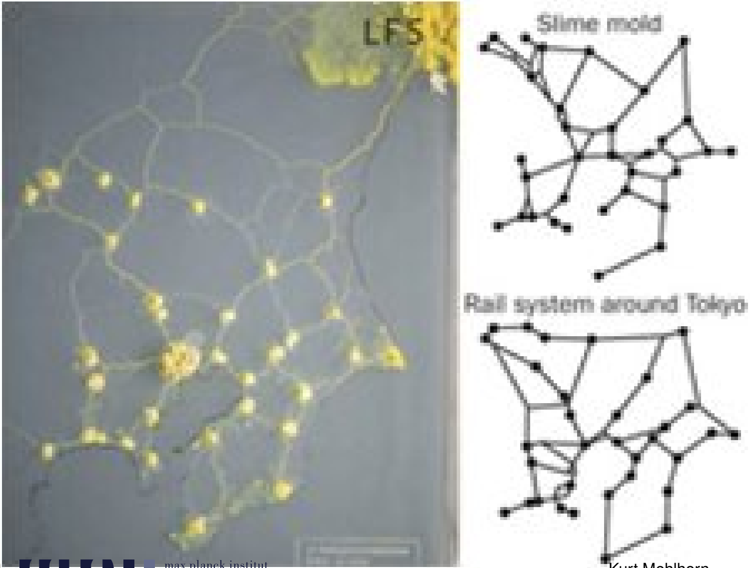
Open Problems

- show: flow directions stabilize
- show: convergence is exponential
- remove degeneracy assumptions
- Physarum apparently can do more, e.g., network design. Prove it.
- inspiration for the design of distributed algorithms

Related Work

Ito/Johansson/Nakagaki/Tero: Convergence Properties for the Physarum Solver, Jan. 2011, change $|Q_e|$ into Q_e and prove convergence for all graphs; do not claim biological significance

Network Design: Science 2010



Natural Computation

- Humans and Animals are not Turing Machines

  Part of their computational capabilities is based on their bodies

  Other Models of Computation are Relevant

- Suggestions for distributed algorithms

- CS methods can help analyzing such systems, do not leave it to physicists and biologists