Physarum can Compute Shortest Paths
SODA 2012

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paper available on my homepage

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Overview

- Nature can do amazing computations with little or slow hardware: bird flocking, human vision, fish swarms, ants, the slime mold physarum.
- The Physarum Experiment and the Proposed Mathematical Model, 10 minutes
- Interlude: basic facts about electrical networks, 30 minutes
- The Analysis of Physarum, 80 minutes
- Electrical Networks and Approximate Maximum Flows, 60 minutes
The Physarum Computer

Physarum, a slime mold, single cell, several nuclei builds evolving networks
Nakagaki, Yamada, Tóth, Nature 2000

show video
2008 Ig Nobel Prize

For achievements that first make people LAUGH then make them THINK

COGNITIVE SCIENCE PRIZE: Toshiyuki Nakagaki, Ryo Kobayashi, Atsushi Tero, Ágotá Tóth for discovering that slime molds can solve puzzles.

Mathematical Model (Tero et al.)

- $G = (V, E)$ undirected graph
- each edge $e$ has a positive length $L_e$ (fixed) and a positive diameter $D_e(t)$ (dynamic)
- send one unit of current (flow) from $s_0$ to $s_1$ in an electrical network where resistance of $e$ equals
  \[ R_e(t) = \frac{L_e}{D_e(t)}. \]
- $Q_e(t)$ is resulting flow across $e$ at time $t$
- **Dynamics:**
  \[ \dot{D}_e(t) = \frac{dD_e(t)}{dt} = |Q_e(t)| - D_e(t). \]

- 1 and 3 links

Tero et al., J. of Theoretical Biology, 553 – 564, 2007
Mathematical Model II: The Node Potentials

- **electrical flows are driven by node potentials**
- \( Q_e = \frac{D_e(p_u - p_v)}{L_e} \) is flow on edge \( \{ u, v \} \) from \( u \) to \( v \)
- flow conservation gives \( n \) equations, one for each vertex \( u \)

\[
\sum_{e=\{ u, v \} \in E} \frac{D_e(p_u - p_v)}{L_e} = b_u
\]

- \( b_{s_0} = 1 = -b_{s_1} \) and \( b_u = 0 \), otherwise
- together with \( p_{s_1} = 0 \), the above defines the \( p_v \)'s uniquely
- can be computed by solving a linear system
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Computer Experiments (Discrete Time)

initialize potentials

while true do
    update diameters: $D_e(t + 1) = D_e(t) + \epsilon(|Q_e(t)| - D_e(t))$
    recompute potentials
end while

In simulations, the system converges (Miyaji/Ohnishi 07/08)

- $e$ on shortest $s_0$-$s_1$ path: $D_e$ converges to 1
- $e$ not on shortest path: $D_e$ converges to 0

Miyaji/Ohnishi ran simulations only on small graphs

We ran experiments on thousands of graphs of size up to 50,000 vertices and 200,000 edges. Confirmed their findings.
The Questions

Does system convergence for all (!!!) initial conditions?

How fast is the convergence?

Details of the convergence process?

Beyond shortest paths?

Inspiration for distributed algorithms?
Convergence against Shortest Path

**Theorem (Convergence)**

*Dynamics converge against shortest path, i.e.,*

\[ D_e \to 1 \text{ for edges on shortest source-sink path and } D_e \to 0 \text{ otherwise}. \]

This assumes that shortest path is unique; otherwise converge against the set of flows of value 1 using only shortest source-sink paths.

Miyaji/Onishi previously proved convergence for planar graphs with source and sink on the same face.
A Single Link (Miyaji/Ohnishi)

\[ e \text{ has length } L \text{ and diameter } D \]

\[ Q = 1 \]

\[ \dot{D} = 1 - D \]

\[ D = 1 + (D(0) - 1)e^{-t} \rightarrow 1 \]

Resistance of \( e \) converges to \( L \).
Thus \( p_{s_0} - p_{s_1} \) converges to \( L \)
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Analysis: Two Parallel Links (Miyaji/Ohnishi)

\[ e_i \text{ has length } L_i, \quad L_1 < L_2, \text{ and diameter } D_i \]

\[ V = L_2 \ln D_2 - L_1 \ln D_1 \]

\[ \frac{d}{dt} L_i \ln D_i = L_i \frac{\dot{D}_i}{D_i} = L_i \left( \frac{D_i}{L_i} \right) \Delta - D_i = \Delta - L_i \]

\[ \dot{V} = L_1 - L_2 \]

\[ V(t) = V(0) + (L_1 - L_2)t \]
Analysis: Two Parallel Links (Miyaji/Ohnishi)

$e_i$ has length $L_i$, $L_1 < L_2$, and diameter $D_i$

$$V = L_2 \ln D_2 - L_1 \ln D_1$$

$$V(t) = V(0) + (L_1 - L_2)t$$

$V(t)$ goes to minus infinity and hence either $D_1$ unbounded or $D_2$ goes to zero; the former is impossible

$$D_2 \to 0 \Rightarrow Q_2 \to 0 \Rightarrow Q_1 \to 1 \Rightarrow D_1 \to 1 \Rightarrow p_S - p_t \to L_1$$
Parallel Links (Miyaji/Ohnishi 07)

parallel links with lengths $L_1 < L_2 < \ldots < L_k$

$D_1 \rightarrow 1$, $D_2, \ldots, D_k \rightarrow 0$

$p_{s_0} - p_{s_1} \rightarrow L_1$

but $D_2, \ldots, D_{k-1}$ do not necessarily converge monotonically
What did Evolution Optimize?

Evolution optimized dynamics so as to achieve a global objective. Which? (Lyapunov Function)

First idea: the energy of the flow $\sum_e Q_e \Delta e$ decreases over time

not true, even for parallel links

**Theorem**

For the case of parallel links: $\sum_i Q_i L_i$, $\sum_i D_i L_i / \sum_i D_i$, and $(p_s - p_t) \sum_i D_i L_i$ decrease over time

computer experiment: the obvious generalization (replace $i$ by $e$) to general graphs do not work
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computer experiment: the obvious generalization (replace \( i \) by \( e \)) to general graphs do not work
A not so Obvious Generalization

\[ \sum_i D_i L_i \over \sum_i D_i \Rightarrow \sum_e D_e L_e \]

value of min \( s_0 - s_1 \) cut with \( \text{cap}_e = D_e \)
What did Evolution Optimize?

Computer experiment:

\[ V := \sum_e D_e L_e \]

de epends on value of min \( s_0 - s_1 \) cut with \( \text{cap}_e = D_e \)

decreases

**Theorem (Lyapunov Function)**

\[ V + \left( \sum_{e \in \delta(\{s_0\})} D_e - 1 \right)^2 \]

decreases.

Derivative of \( V \) (essentially) satisfies

\[ \dot{V} \leq -c \cdot \sum_e (D_e - |Q_e|)^2. \]

Proof uses min-cut-max-flow and . . .
What did Evolution Optimize?

Computer experiment:

\[ V := \frac{\sum_e D_e L_e}{\text{value of min } s_0-s_1 \text{ cut with } \text{cap}_e = D_e} \text{ decreases} \]

**Theorem (Lyapunov Function)**

\[ V + \left( \sum_{e \in \delta(s_0)} D_e - 1 \right)^2 \text{ decreases.} \]

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Proof uses min-cut-max-flow and . . .
\[ V \text{ decreases and stays positive } \Rightarrow \dot{V} \to 0 \]
\[ \dot{V} \leq -c \cdot \sum_e (D_e - |Q_e|)^2 \]
\[ |D_e - |Q_e|| \text{ goes to zero for all } e \]
\[ Q_e = (D_e/L_e)\Delta_e \text{ and hence } \Delta_e \approx L_e \text{ for } Q_e \text{ and } t \text{ large} \]
\[ \Delta_{s_0s_1} \text{ converges to length of some source-sink path} \]
\[ \Delta_{s_0s_1} \text{ converges to length of shortest path} \]
Convergence against Shortest Path

Corollary (Convergence)

*Dynamics converge against shortest path, i.e.,*

\[ D_e \rightarrow 1 \text{ for edges on shortest s-t path and } D_e \rightarrow 0 \text{ otherwise.} \]

this assumes that shortest path is unique; otherwise, . . .

Miyaji/Onishi previously proved convergence for planar graphs.
Equilibria (Miyaji/Ohnishi 07)

Simplifying Assumption: no two $s_0$-$s_1$ paths have same length

**Equilibrium:** $D_e = |Q_e|$ for all $e$

The equilibria are precisely the $s_0$-$s_1$ paths, i.e.

\[
D_e = \begin{cases} 
1 & e \in P \\
0 & e \notin P 
\end{cases}
\]

for some source-sink path $P$.

- potential drop $\Delta_e$ along an edge $e$ satisfies $Q_e = D_e \Delta_e / L_e$.
- thus $\Delta_e = L_e$ in equilibrium for $e$ with $D_e > 0$
- thus $\Delta = L_P$ for any source-sink path $P$ with $D_e > 0$ for all $e \in P$. 
Elements of the Proof I

\[
C = \text{value of min } s_0-s_1 \text{ cut, if } \text{cap}_e = D_e
\]

By the Min-Cut-Max-Flow Theorem, there is a flow \( f \) of value 1 such that \( f_e \leq D_e/C \) for all \( e \).

By Thompson’s principle:

\[
\sum_e R_e Q_e^2 \leq \sum_e R_e f_e^2 \leq \frac{1}{C^2} \sum_e R_e D_e^2.
\]

\( C \rightarrow 1 \) since

- for every cut \( S \): \( \hat{C}_S \geq 1 - C_S \)
- for \( S = \{ s_0 \} \): \( \hat{C}_S = 1 - C_S \)
Elements of the Proof II

\[ V := \frac{\sum_e D_e L_e}{C} \]

where \( C = \) value of min \( s_0-s_1 \) cut with \( \text{cap}_e = D_e \).

\[ \dot{V} = \sum_e R_e |Q_e| \frac{D_e}{C} - \sum_e R_e \left( \frac{D_e}{C} \right)^2 \]

\( \leq \frac{1}{2} \left( 2 \sum_e R_e |Q_e| \frac{D_e}{C} - \sum_e R_e \left( \frac{D_e}{C} \right)^2 - \sum_e R_e Q_e^2 \right) \]

by prev. slide

\[ = -\frac{1}{2} \sum_e R_e \left( \frac{D_e}{C} - |Q_e| \right)^2 \leq -\frac{L_{\min}}{4} \sum_e \left( \frac{D_e}{C} - |Q_e| \right)^2 \]
Elements of the Proof III

\[ C \to 1 \text{ and } \sum_e \left( \frac{D_e}{C} - |Q_e| \right)^2 \to 0 \text{ imply} \]

\[ |D_e - |Q_e|| \to 0 \quad \text{for all } e. \]

There is always a path \( P \) with \( Q_e \geq 1/m \) for all \( e \in P \). For such \( e \),

\[ \Delta_e = L_e(1 \pm \epsilon) \quad \text{since} \quad Q_e = D_e \Delta_e/L_e. \]

Thus \( \Delta = (1 \pm \epsilon)L_P \) for some \( P \) always.

Thus \( \Delta \to L_P \) for some fixed path \( P \).
Elements of the Proof IV

\[ \Delta \to L_P \text{ for some fixed path } P. \]

Let \( P^* \) be the shortest source-sink path and assume \( P \neq P^* \).

\[
\frac{d}{dt} \sum_{e \in P^*} L_e \ln D_e = \sum_{e \in P^*} L_e \frac{|Q_e| - D_e}{D_e} = \sum_{e \in P^*} L_e \frac{D_e|\Delta_e|/L_e - D_e}{D_e} \\
\geq \sum_{e \in P^*} L_e \frac{D_e\Delta_e/L_e - D_e}{D_e} = \Delta - L_{P^*} \to L_P - L_{P^*}. 
\]

Thus \( \sum_{e \in P^*} L_e \ln D_e \) is unbounded, a contradiction.
Elements of the Proof V

\[ \Delta \rightarrow L_{P^*}. \]

Consider \( e \notin P^* \) and assume \( \neg(Q_e \rightarrow 0) \), say \( Q_e(t) \geq \delta > 0 \) for arbitrarily large \( t \).

For any such \( t \), there is a path \( P \) through \( e \) with \( Q_P > \delta/n^n \). Then \( \Delta \approx L_P \), a contradiction.

\( Q_e \rightarrow 1 \) for \( e \in P^* \) since \( Q_e \rightarrow 0 \) for \( e \notin P^* \).
Stable Topology (Miyaji/Ohnishi)

How fast is the convergence?

Definition: An edge $e = (u, v)$ stabilizes if for all $\varepsilon > 0$ either

- $p_u(T) \geq p_v(T) - \varepsilon$ for all large $T$ or
- $p_v(T) \geq p_u(T) - \varepsilon$ for all large $T$.
- $|p_v(T) - p_u(T)| \leq \varepsilon$ for all large $T$.

slightly more general than Miyaji/Ohnishi

Definition: A network stabilizes if all edges stabilize
A Path with Fixed Potential Difference

- assume \( p_a \) and \( p_b \) are fixed
- \( L(P) \) length of path from \( a \) to \( b \).
- define \( f = (p_a - p_b)/L(P) \) and assume \( f < 1 \)
- then for all edges of \( p \): \( D \) decays like \( \exp((f - 1)t) \)
- \( p_v \) converges to \( p_b + (p_a - p_b) \cdot \text{dist}(v, b)/L(P) \)
Stable Topology III

**Theorem**

*If network stabilizes, network converges as defined next.*

- decompose *undirected* $G$ into paths: $P_0 =$ shortest $s$-$t$ path

- for $v \in P_0$: $p_v \rightarrow \text{dist}(v, t)$

- for $e \in P_0$: $D_e \rightarrow 1$

- assume $P_0, \ldots, P_{i-1}$ are defined. Then
  - $P_i$ has endpoints $a$ and $b$ on $P_0 \cup \ldots \cup P_{i-1}$
  - internal nodes and edges are fresh
  - maximizes $f_i := (p_a - p_b)/L(P_i)$ this is less than one

- for $v \in P_i$: $p_v \rightarrow p_b + (p_a - p_b)\text{dist}(v, b)/L(P_i)$

- for $e \in P_i$: $D_e \rightarrow 0$, exponentially with rate $f_i - 1$.

- direct edges in $P_i$ in direction from from $a$ to $b$
Open Problems

Do networks stabilize?

If so, after what time?

More generally, how long does it take for the dynamics to converge?
Wheatstone Graph

- simplest graph where flow directions are not clear
- direction of flow on e ????
- potentials evolve non-monotonically; run NonMonotone
- state space is cyclic; run TwoChanges

Theorem
Wheatstone network stabilizes
Wheatstone Graph

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Theorem
Wheatstone network stabilizes
Wheatstone: Middle Edge Stabilizes

- \( R_i = L_i / D_i \) = resistance of edge \( i \).
- \( x_a = R_a / (R_a + R_c) \), similarly for \( b \).
- potential drop on edge \( a \) is \( x_a \Delta \).
- if \( x_a < x_b \), direction of \( e \) is RL
  - if \( x_a > x_b \), direction of \( e \) is LR
- \( x_a^* = L_a / (L_a + L_c) \), similarly for \( b \).

\[\text{assume } x_a^* \leq x_b^*\]
- observe: direction of flow on \( e \) does not depend on \( D_e \)
Wheatstone: Middle Edge Stabilizes

- split $[0, 1]$ into

  $$S = [0, x_a^*]$$
  $$M = [x_a^*, x_b^*]$$
  $$L = [x_b^*, 1]$$

- consider evolution of $(x_a, x_b)$

- in $S \times S$, both grow:

- in $M \times M$, $x_a$ decreases and $x_b$ grows

- in $L \times L$, both shrink

$$x_a = \frac{R_a}{R_a + R_c}$$
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- in \(S \times S\), both grow:

- in \(M \times M\), \(x_a\) decreases and \(x_b\) grows

- in \(L \times L\), both shrink

what if system stays in \(S \times S\) or \(L \times L\)?

\[ x_b \]

\[ x_a \]

\[ S \]
\[ M \]
\[ L \]

\[ RL \]
\[ LR \]
The Transportation Problem

- undirected graph $G = (V, E)$
- $b : V \rightarrow \mathbb{R}$ such that $\sum_{v} b_v = 0$
- $v$ supplies flow $b_v$ if $b_v > 0$
- $v$ extracts flow $|b_v|$ if $b_v < 0$
- find a cheapest flow where cost of sending $x$ units across an edge of length $L$ is $Lx$

Dynamics of Physarum solves transportation problem.

$D_e$’s converge against a mincost solution of transportation problem.

proof requires a non-degeneracy assumption
Open Problems and Related Work

Open Problems

- show: flow directions stabilize
- show: convergence is exponential
- remove degeneracy assumptions
- Physarum apparently can do more, e.g., network design. Prove it.
- inspiration for the design of distributed algorithms

Related Work

Ito/Johansson/Nakagaki/Tero: Convergence Properties for the Physarum Solver, Jan. 2011, change $|Q_e|$ into $Q_e$ and prove convergence for all graphs; do not claim biological significance
Network Design: Science 2010
Natural Computation

- Humans and Animals are not Turing Machines
  Part of their computational capabilities is based on their bodies
  Other Models of Computation are Relevant

- Suggestions for distributed algorithms

- CS methods can help analyzing such systems, do not leave it to physicists and biologists